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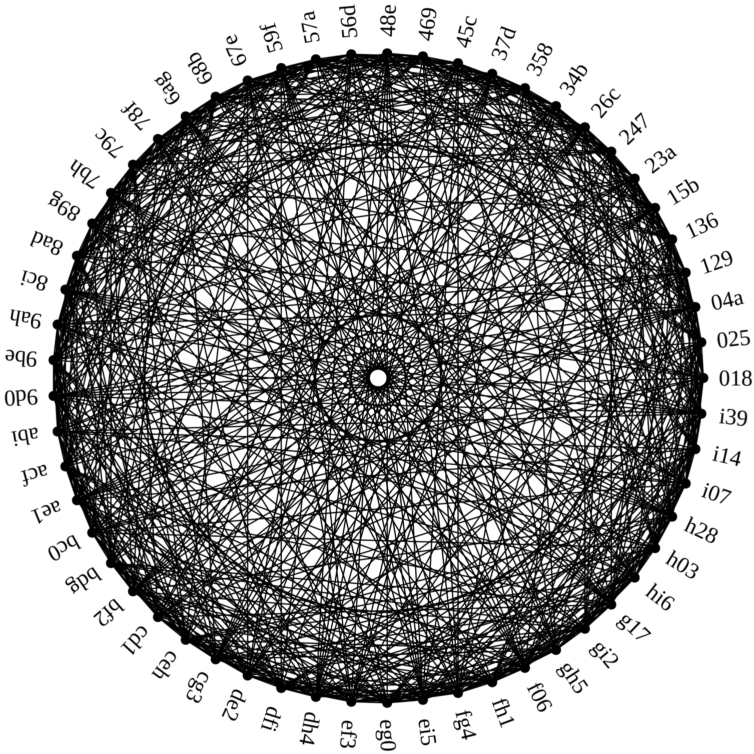
M500 210



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, send a stamped, addressed envelope to Diana Maxwell, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail.

What's in a theorem?

Sebastian Hayes

What is a *theorem*? Something that requires proof as opposed to an axiom, a postulate, or a definition—these, taken together, are the four Euclidian categories. Heath, the best translator of Euclid, speaks of 'propositions' rather than 'theorems' and in some ways this is better since it has the sense, 'Hey! What do you think of this?' A surprising number of the 'theorems' in Euclid are in fact constructions; the very first (Book 1. Prop. 1) is 'On a given finite straight line to construct an equilateral triangle'.

A theorem must have some generality: it is not the same as a 'result'. Strictly, to be classed as a theorem an assertion must actually have been proved—otherwise it is a conjecture. But usage is not consistent here: for some reason Fermat's claims always seem to have been rated as 'theorems' although they were often inspired guesses, while we still talk of Goldbach's conjecture ('Every even number greater than is the sum of two odd primes').

On what grounds do we consider one theorem better than another? After some ponderings I tentatively came up with the following desiderata. A good theorem should ideally at once be *true, simple, basic, unobvious, illuminating, suggestive, beautiful and readily applicable.*

(1) True Is this essential? Even if untrue a 'theorem' can be very worthwhile if it fulfils some of the other categories, in particular if it is *suggestive* (of new lines of research). Suppose, *pace* Wiles, Fermat's Last Theorem turned out to be false for some very large power This would hardly matter because it has given rise to such interesting and important mathematics over the centuries.

There are theorems which, though false, deserve to be true (e.g. Ramanujan's formula for the distribution of the primes) while there are in modern mathematics plenty of apparently true theorems that are so nonsensical they deserve to be false (e.g. Banach's two sphere theorem).

(2) Simple I mean simple to state not simple to prove—the question of whether there exists a simple proof for a given theorem (or even a proof at all) is a different issue altogether. Rather nicely the theorems of pure mathematics that have given the most trouble are the simplest to state (Fermat's Last, Four Colour, etc.).

(3) Basic For example: Angle at centre = twice angle at circumference; G.M. $\leq$ A.M.

The requirement of being basic conflicts with many other criteria.

(4) Unobvious. Is everything obvious? Nothing? Hardy considered that the term should be banned from mathematics but this approach, typical of a modern author, is unhelpful to say the least. If we did not take certain things as 'obvious', we would not be able to live, certainly not think. It is fatuous to introduce the proposition $A = A$ into a mathematical system as a *theorem*—but one contemporary writer (I forget who) has done just this.

The statement $1+1 = 2$ is best viewed as a *definition*. Alternatively we can view it as a recipe for constructing the natural numbers: take a block of something solid, add it to a similar one and carry on as long as you want making copies as you go. Certain basic numerical and geometric notions (nearness, farness, on, under &c.) are built into us: we have them, they are givens, so why not admit it? These notions/perceptions must above all not be presented as theorems—statements that require and can receive proof because there is nothing more basic on which we can ground them.

On the other hand, Euclid is absolutely right to introduce $a \times b = b \times a$ as a theorem instead of taking it on board as one of the axioms for fields as the moderns do. Viewed as a statement about the real world it is by no means obvious and is, as a matter of fact, not that easy to prove—Euclid has to appeal to seven or eight earlier theorems. A typical chieftain with control over ten villages, each of which was able to provide seven young men as warriors, would probably have been surprised to be told by his shaman that his strength was no less and no greater than that of a rival who only controlled seven villages each able to provide ten young men.

The combination *Basic* + *Unobvious* + *True* makes for a very impressive theorem. An amazing amount of Euclidian geometry can be established on the basis of Ptolemy's theorem (rules for adding sines and cosines, half angle formulae &c., even Pythagoras; see Eli Maor, *Trigonometric Delights*, Ch. 6). But the theorem 'The product of the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the products of the opposite sides' seems at first sight implausible and for that matter hardly worth stating.

The Taniyama–Shimura conjecture ('Every elliptic equation is linked to an equivalent modular form') was so unobvious as to appear quite fantastic to most mathematicians at the time it was first proposed. It must in some sense be basic since it was by proving this proposition that Wiles established Fermat's Last Theorem by a roundabout and incomprehensible route.

(5) Illuminating A computer, fed with a few logical axioms, can churn out countless derivations—but how many of them will be worth reading? Fermat scarcely ever gave any proofs and was occasionally wrong but he had the knack of throwing light on all sorts of areas of number theory. Can a theorem be basic without being illuminating? Yes. Pythagoras' theorem is a case in point: it doesn't really give you any new insight into the subject while Euclid's (or Eudoxus') brilliant treatment of proportion and similarity in Books V and VI makes the whole subject come alive and has inspired not only professional mathematicians but countless architects and painters as well.

(6) Suggestive I mean suggestive of further development. Certain theorems close doors rather than open them. Indeed certain theorems are designed to do just this, like the theorem which states that there does not exist any algebraic formula capable of producing all the primes. I sort of feel that a destructive theorem shouldn't be rated as highly as a constructive one: we don't put Genghis Khan in the same league as Napoleon although he is certainly as important historically.

(7) Beautiful Is beauty in the eye of the beholder? Are there any guidelines? 'Simplicity' and 'orderliness' are often invoked. Someone said of the theory of differential equations, 'This is not mathematics, it's stamp collecting'. (But the applied mathematician cannot afford to ignore the real world and Nature is not usually tidy.) Simplicity is especially prized when it is not expected. Mathematicians at the time gave up the search for a sum to the reciprocals of the squares until Euler fished out of his hat

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \;=\; \frac{\pi}{6}.$$

But richness of texture has its appeal also. If you can get incredibly complicated expressions boiling down to some very simple sum or product the resulting theorem has a baroque beauty as in so many of Ramanujan's discoveries (see M500 **202** pp 7–8). Symmetry is attractive but not if it is overdone. Once again it is particularly effective if it appears where one does not expect it as in

$$\tan\frac{(n+1)\,\alpha}{2} \;=\; \frac{\sin\alpha + \sin 2\alpha + \cdots + \sin n\alpha}{\cos\alpha + \cos 2\alpha + \cdots + \cos n\alpha}.$$

Baudelaire considered that there must be an element of *strangeness* in beauty; the strangeness shocks or at least attracts attention while orderliness reassures. De Moivre's formula and $e^{\pi i} + 1 = 0$ were surpassing strange when they were first unleashed on the world but it is well nigh impossible to startle a pure mathematician these days. The most surprising mathematical achievement in today's world would be to discover something important that the man in the street can actually understand.

(8) Readily applicable Alas, it is on this score that so many aspiring mathematical belles get disqualified. Leibnitz's

$$\frac{\pi}{4} \;=\; 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

is a pretty enough result but useless for calculating π since it converges so slowly. The same goes for Wilson's crisp $(p-1)! \equiv -1 \pmod p$ as a test for primality.

How does one discover a theorem of interest? There seems to be no lack of inventive minds amongst the contributors to this magazine, so it would be interesting to hear some of them tell us how they arrived at their discoveries. As far as I can make out, there seem to be three main procedures, which I baptise *the inductive method*, *the deductive method* and *the exploratory or playful method.*

By 'inductive method' I do not mean mathematical induction as such but simply the standard scientific method of scanning a mass of data and trying to discern some underlying structure or pattern. One then investigates as to whether the pattern keeps on recurring and if it does, one attempts to show that this feature is bound to persist (but this is part of proving, not discovery). Most theorems in number theory are discovered this way: I doubt if anyone ever deduced from first principles that $F_{2n} = F_n^2 + F_{n-1}^2$ —almost certainly Lucas and others noted that $5^2+3^2 = 34$ and $13^2 + 21^2 = 610$ and went on from there. Even such a prestigious theorem as the Prime Number Theorem originated in the scanning of data (allegedly, Gauss, as a teenager, conjectured that $\pi(x)$ varied with $x/(\log x)$ after examining a published list of primes).

The 'natural' movement of the human mind is from the particular to the general, the concrete to the abstract, which is why the inductive approach comes much more easily to most people. By concentrating on the logical aspects modern mathematicians have certainly tightened up the subject but at the cost of completely alienating the general public. This is far from being a good thing even for mathematics itself: it means that people for whom aesthetic considerations are uppermost avoid mathematics like the plague—the exact reverse of the situation that prevailed during the Renaissance and baroque eras.

The deductive method usually proceeds either by generalizing some known result, or by applying it to a particular case with unexpected consequences. This magazine has printed several articles in the last year or so which are generalizations of the Fibonacci sequence and one gathers from M500 **205** that Dennis Morris is currently involved in generalizing the hyperbolic functions, themselves generalizations of the trigonometric functions.

The starting point for an extension must itself already have some generality—a single numerical case is of little or no value. However, the starting point must not be too general: I don't think any contemporary pure mathematician ever sat down of an evening with the axioms of von Neumann set theory in order to see what new theorems he or she could deduce.

An example of the opposite process, particularizing, is Pascal considering the expansion of $(a+b)^n$ and setting $a = 1$, $b = 1$ thus showing that the total number of possible combinations of n objects taken r at a time is 2^n. (This includes the choice of not making a selection at all.) Setting some variable or variables at unity or at a multiple or submultiple of π seems to be a standard stratagem that has yielded a surprisingly rich harvest of theorems. One would like to hear of other 'tricks of the trade' but modern textbooks are surprisingly coy on the subject—I have yet to come across a chapter, let alone a whole book, entitled *How to Devise or Discover Interesting Theorems.*

The third method is not really a method at all: it is basically just messing about and seeing what comes up. *Homo sapiens* is, thank God, also *Homo ludens.* Leonhard Euler, undoubtedly the most prolific mathematician of all time (his works run to seventy-five large volumes), played around with mathematical formulæ as children play with toys (or did before the computer age). What would happen if we did this? Or this? And then that?

As far as I am concerned, mathematics is not an ensemble of watertight logical systems but more like a series of wildlife reserves where strange plants and animals can not only be observed but actually bred or grown from seed. It is notable that many of the most inventive mathematicians were amateurs, e.g. Leibnitz, Fermat. In contemporary theatre and above all 'painting' (conceptual art), originality is so much the order of the day that any sort of rubbish is acceptable provided you are doing something that nobody has done before. But for some reason in mathematics we have the opposite set-up: rigour has stifled elegance and inventiveness. Mathematics was once one of the 'humanities'. I am not quite sure what the humanities were, or were intended to be, but I assume the basic idea was that studying them did not just make you more learned but more 'human'. It would perhaps be going too far to claim that the humanistic approach which viewed mathematics as at once a science and an art and a philosophy of nature invariably produced a better type of person—one gathers that Newton was rather a nasty man—but there's no doubt in my mind that trying to turn someone into a logical machine is not likely to improve human nature. *Est in medio verum*—truth lies in the middle.

Solution 205.1 – Sphere in a cone

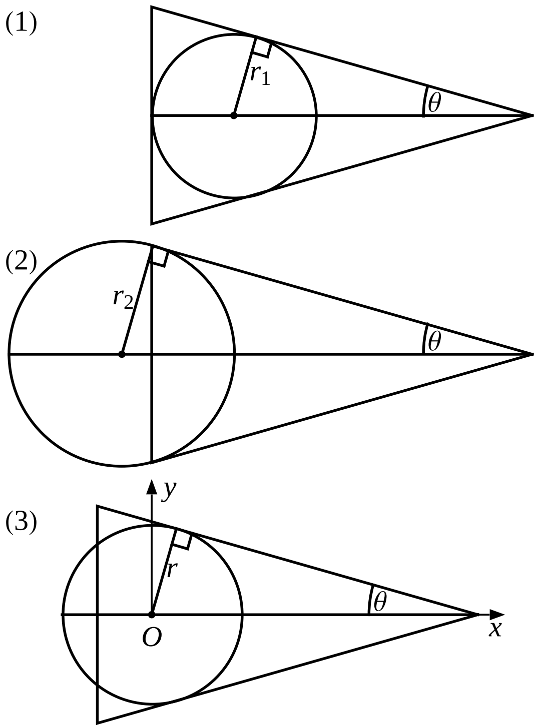
> Given a finite cone of fixed height 1 m and apex angle α, a sphere is inserted as far as possible into its open end. What is the maximum volume of that part of the sphere which is inside the cone?

Norman Graham

Let $\theta = \alpha/2$ and $c = \operatorname{cosec}\theta > 1$. Let V be the volume of the sphere S inside the cone C. The radius r of the sphere has two critical values.

(1) Let the radius of S be r_1 when S touches the plane P containing the rim of C. If $r < r_1$, S is entirely inside C, and V increases continuously as r increases from 0 to r_1.

(2) Let the radius of S be r_2 when S touches the rim of C. If $r > r_2$, S rests upon the rim of C. As r decreases from ∞ to r_2, S 'bulges' further into C, so V increases continuously.



From diagram (2), $r_2 \cos\theta = \tan\theta$. Therefore

$$r_2 = \frac{\sin\theta}{\cos^2\theta} = \frac{\sin\theta}{1-\sin^2\theta} = \frac{c}{c^2-1}.$$

It follows that V_m, the maximum value of V, occurs when r is at some point R in the range $[r_1, r_2]$. For a given value, take the origin O at the centre of S and the x-axis through the apex of C (diagram (3)). Then P is the plane $x = cr - 1$ and

$$V = \int_{cr-1}^{r} \pi y^2 dx = \pi \int_{cr-1}^{r} (r^2 - x^2) dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{cr-1}^{r}.$$

Therefore

$$\begin{aligned}\frac{V}{\pi} &= \frac{2r^3}{3} - r^2(cr-1) + \frac{(cr-1)^3}{3}\\ &= \frac{r^3}{3}\left(c^3-3c+2\right) - r^2(c^2-1) + rc - \frac{1}{3}.\end{aligned}$$

When V/π is a maximum, $(1/\pi)dV/dr = 0$. So R satisfies

$$(c-1)^2(c+2)R^2 - 2(c^2-1)R + c \;=\; 0.$$

Therefore

$$\begin{aligned}R &= \frac{1}{(c-1)^2(c+2)}\left(c^2-1 \pm \sqrt{(c^2-1)^2 - c(c-1)^2(c+2)}\right)\\ &= \frac{c+1\pm 1}{(c-1)(c+2)} \;=\; \frac{1}{c-1} \;\text{ or }\; \frac{c}{(c-1)(c+2)}.\end{aligned}$$

But $1/(c-1)$ is not a valid solution since it is greater than r_2. Hence the solution required is $R = c/((c-1)(c+2))$, which is in the range $[r_1, r_2]$. Inserting this value of r in the equation for V/π gives the answer after some algebra:

$$V_m \;=\; \frac{4\pi}{3(c-1)(c+2)^2}, \quad \text{where} \quad c \;=\; \frac{1}{\sin\alpha/2}.$$

Additional comments. The volume of the cone is

$$V_C \;=\; \frac{\pi}{3}\tan^2\theta \;=\; \frac{\pi}{3(c^2-1)}$$

and the volume of the optimum sphere is

$$V_S \;=\; \frac{4\pi R^3}{3} \;=\; \frac{4\pi c^3}{3(c-1)^3(c+2)^3}.$$

Hence

$$\frac{V_m}{V_C} \;=\; 1 - \left(\frac{c}{c+2}\right)^2 \quad \text{and} \quad \frac{V_m}{V_S} \;=\; \frac{(c-1)^2(c+2)}{c^3}.$$

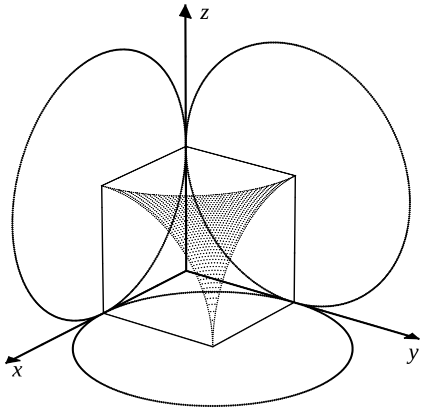
As α increases from 0 to π, c decreases from ∞ to 1, V_m/V_C increases from 0 to 8/9 and V_m/V_S decreases from 1 to 0.

Example: $\alpha = 60°$, $\theta = 30°$, $c = 2$, $r_1 = 1/3$, $r_2 = 2/3$, $R = 1/2$, $V_C = \pi/9$, $V_m = \pi/12$, $V_S = \pi/6$. Hence $V_m/V_c = 3/4$, $V_m/V_S = 1/2$ and the centre of S lies in the plane P.

Solution 200.4 – Circle in a box

> What is the locus of the centre of a unit-radius circle placed such that the circumference touches the positive (x, y)-plane, the positive (x, z)-plane and the positive (y, z)-plane?

Recall that in M500 **209** Steve Moon found the boundary of the locus, three unit-radius quarter-circles joining $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$, and reasoned that the locus itself must be a part of the sphere $x^2 + y^2 + z^2 = 2$. Dick Boardman and I (ADF) are happy with the boundary but we are a little concerned about the amount of arm waving used in the deduction of the final shape of the locus.



As explained below, the answer is obviously correct—but a proof that can only be described as truly rigorous eludes us.

Dick Boardman

I have strong numerical evidence for the following but cannot prove it: *The locus of the centre of a coin in the corner of a box is a sphere, centre at the corner, with a radius of* $\sqrt{2}$ *times the radius of the coin.* The evidence is provided by an algorithm and a computer program, which I describe here in the hope that someone can convert it into a proof.

The edges of the box meeting at the corner may be treated as a set of axes with origin at the corner. Choose three lengths a, b, c along the x, y and z axes. These three points define a triangle in a plane. The in-circle of that triangle will touch the sides of the box as a coin would.

The lengths of the sides of the triangle are found using Pythagoras' theorem. The area is found from Heron's formula. The radius of the in-circle is $2 \times \text{area}/\text{perimeter}$.

To find the centre of the in-circle I need some results from vector analysis. A vector has three coordinates $\mathbf{v} = (x, y, z)$ and a unit vector is one where $\sqrt{x^2 + y^2 + z^2} = 1$. A line through a point whose vector is $\mathbf{a}$ in direction $\mathbf{b}$ is $\mathbf{a} + t\mathbf{b}$, where t is a parameter. A vector in the direction of the angle bisector between unit vectors $\mathbf{u_1}$ and $\mathbf{u_2}$ is $t(\mathbf{u_1} + \mathbf{u_2})$.

The in-centre is well known to be the intersection of the angle bisectors of the triangle. To compute the coordinates of the in-centre I carry out the following steps.

Compute unit vectors along the sides of the triangle.
Compute the lines which are the angle bisectors of the triangle.
Compute the point which is the intersection of those angle bisectors.
Compute the distance of this point from the origin.
Compute the ratio of this distance to the radius of the in-circle.

I wrote a computer program which carries out these operations and ran it for a large set of values. In all cases it gave a final answer of $\sqrt{2}$ to within the accuracy of the system.

Reference: C. E. Weatherburn, *Elementary Vector Analysis*.

Problem 210.1 – Determinant

Compute

$$\begin{vmatrix} 4 & a+b+c+d & a^2+b^2+c^2+d^2 & a^3+b^3+c^3+d^3 \\ a+b+c+d & a^2+b^2+c^2+d^2 & a^3+b^3+c^3+d^3 & a^4+b^4+c^4+d^4 \\ a^2+b^2+c^2+d^2 & a^3+b^3+c^3+d^3 & a^4+b^4+c^4+d^4 & a^5+b^5+c^5+d^5 \\ a^3+b^3+c^3+d^3 & a^4+b^4+c^4+d^4 & a^5+b^5+c^5+d^5 & a^6+b^6+c^6+d^6 \end{vmatrix}.$$

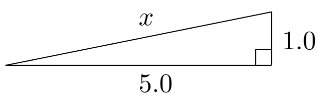
Problem 210.2 – Cosecs

Show that

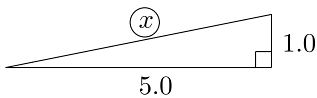
$$\operatorname{cosec} 10^\circ + \operatorname{cosec} 50^\circ - \operatorname{cosec} 70^\circ \;=\; 6.$$

Are there other interesting identities of the same kind?

GCSE question. Find x in this diagram:



Candidate's answer. Here:



MAXIMA: a free symbolic algebra package

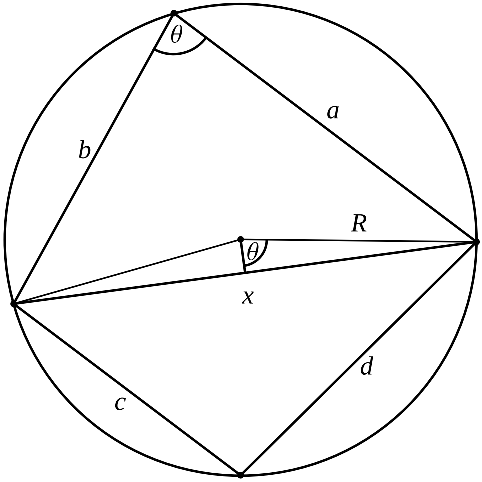
Dick Boardman

Many people who earn their living using mathematics, people like system designers, engineers and statisticians, use a symbolic algebra package. These products are very good but also very expensive. Recently, I came across a free package which does a lot of what the professional ones do. I can recommend it to anyone who likes solving puzzles involving algebra, calculus or large numbers. The package is called MAXIMA. It was originally written at MIT under the name MACSYMA and funded by the American government on condition that it was made freely available. However, everybody's baby is nobody's baby, and the package never received the publicity and support it deserved. Some generous academics have adapted it to run under WINDOWS and made it available over the Internet.

An applied mathematician attacking a new problem must first express the problem in mathematical terms, choosing variables, deciding what simplifications and assumptions can be made, and setting up equations. He/she then solves the equations, interprets the solution in terms of the original problem and presents the results as clearly as possible.

A symbolic algebra system helps with the second and possibly the third stages. As an example, I will show how MAXIMA can be used to solve a recent M500 puzzle.

Recall the cyclic quadrilateral (M500 Problem 203.4).



In M500 **208**, Ted Gore began by showing that

$$x^2 \;=\; a^2 + b^2 - 2ab\cos\theta \;=\; c^2 + d^2 + 2cd\cos\theta$$

and that $x = 2R\sin\theta$.

For convenience, replace x^2 by a variable called xsq, $\cos\theta$ by ct, $(\sin\theta)^2$ by st2, and the square of R, the radius of the circumcircle, by RR. In what follows, lines beginning with labels like '`(%i1)`' and ending with ';' are the inputs to MAXIMA and lines beginning with labels like '`(%o1)`' are responses.

`(%i1)` e1: xsq = a*a + b*b − 2*a*b*ct;

`(%o1)` $\text{xsq} \;=\; -2\,a\,b\;\text{ct} + b^2 + a^2$

[We are cheating. In reality MAXIMA uses the standard character set and a fixed-pitch font. So the above would look like this.

```
                              2     2
(%o1)   xsq = - 2 a b ct + b   + a
```

Although acceptable on a screen, it looks pretty ghastly on paper. And it gets worse. So for your benefit we have translated all MAXIMA output to 'proper' mathematics. — ADF]

`(%i2)` e2: xsq = c*c + d*d + 2*c*d*ct;

`(%o2)` $\text{xsq} \;=\; d^2 + 2\,c\;\text{ct}\;d + c^2$

`(%i3)` e3: solve([e1, e2], [ct, xsq]);

`(%o3)` $$\left[\left[\text{ct} \;=\; \frac{-d^2 - c^2 + b^2 + a^2}{2cd + 2ab},\;\; \text{xsq} \;=\; \frac{ab(d^2 + c^2) + b^2cd + a^2cd}{cd + ab}\right]\right]$$

`(%i4)` e4: part(part(e3, 1), 1);

`(%o4)` $$\text{ct} \;=\; \frac{-d^2 - c^2 + b^2 + a^2}{2cd + 2ab}$$

`(%i5)` e5: part(e4, 2);

`(%o5)` $$\frac{-d^2 - c^2 + b^2 + a^2}{2cd + 2ab}$$

(%i6) st2: factor(1 − e5*e5);

(%o6) $-\dfrac{(d-c-b-a)(d-c+b+a)(d+c-b+a)(d+c+b-a)}{4(cd+ab)^2}$

(%i8) e6: part(part(e3, 1), 2);

(%o8) $\text{xsq} = \dfrac{ab(d^2+c^2)+b^2cd+a^2cd}{cd+ab}$

(%i10) xsq: part(e6, 2);

(%o10) $\dfrac{ab(d^2+c^2)+b^2cd+a^2cd}{cd+ab}$

(%i12) RR: xsq/4/st2;

(%o12) $-\dfrac{(cd+ab)(ab(d^2+c^2)+b^2cd+a^2cd)}{(d-c-b-a)(d-c+b+a)(d+c-b+a)(d+c+b-a)}$

(%i13) factor(RR);

(%o13) $-\dfrac{(ad+bc)(bd+ac)(cd+ab)}{(d-c-b-a)(d-c+b+a)(d+c-b+a)(d+c+b-a)}$

The square root of this quantity is the circumradius. Note the minus sign at the beginning. This reflects the fact that the sum of any three sides must exceed the fourth side so that $d-c-b-a$ must be negative. Ted Gore showed that

$$(\text{area})^2 = \left(\frac{ab+cd}{2}\right)^2 (\sin\theta)^2;$$

that is,

$$(\text{area})^2 = \frac{(a+b+c-d)(d-c+b+a)(d+c-b+a)(d+c+b-a)}{16}.$$

Note the similarity with Heron's formula for the area of a triangle.

These results show that the area and circumradius of a cyclic quadrilateral depend only on the sides and not on the order of the sides round the cyclic quadrilateral.

Solution 207.2 – Parts of a partition

A partition of n can be represented as a vector $(a_1, a_2, \ldots, a_n)$, where the a_i are defined by

$$n = a_1 + 2a_2 + 3a_3 + \cdots + na_n.$$

What is the maximum possible number of non-zero elements a_i in a partition of n? How many partitions of n use this maximum number?

Steve Moon

To maximize the non-zero elements a_i in a partition of n, start with the premise that all $a_i = 1$. Consider the case where $a_i = 1$ fills the element spaces from the left without gaps, thus.

$$n \;\rightarrow\; (1, 1, \ldots, 1, 0, 0, \ldots, 0)$$

with the 1s stopping at the k^{th} element. Then

$$n \;=\; a_1 + 2a_2 + \cdots + ka_k \;=\; 1 + 2 + \cdots + k,$$

and n is the k^{th} triangular number. There are at most k non-zero elements for triangular numbers $n_k = n$, and this partition can occur only once. (Any $a_i > 1$ introduces at least one more zero.)

For n between two triangular numbers, $n_k < n < n_{k+1}$, the partition is of the form

$$(1 + (n - n_k), 1, 1, \ldots, 1, 0, 0, \ldots, 0)$$

with the ones stopping at the k^{th} element. Here, $a_0 = 1 + (n - n_k)$ and this partition, which maximizes non-zero a_i, can occur only once. When $n = n_k$, the next a_{k+1} position can be filled: $a_0 = a_{k+1} = 1$.

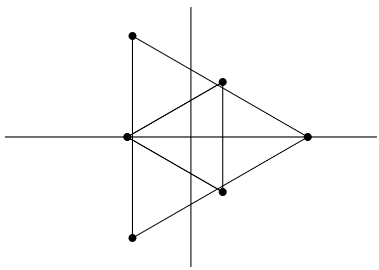
Hence in a partition of n the maximum number of non-zero a_i is given by k, where n_k is the triangular number such that $n_k \le n < n_{k+1}$. This partition occurs once only.

Solution 207.4 – Sextic

Solve $500x^6 - 13000x^3 = 77613$.

Steve Moon

The six roots of the equation are shown opposite. The one on the right is $3.141592653695\cdots \approx \pi$ (!).



A revolutionary view of numbers, a revolutionary view of groups, the unification of mathematics, and the unification of physics

Dennis Morris

The conventional view of numbers is that there are the real numbers and that attached to them in some way to do with the polynomial $x^2 + 1 = 0$ are the complex numbers. The conventional view is no more than this. Research that I have recently done has driven me to a very different view, which I now expound.

We begin with the standard form Cayley tables† of the first few abelian groups:

$$C_1 = \begin{bmatrix} A \end{bmatrix}, \quad C_2 = \begin{bmatrix} A & B \\ B & A \end{bmatrix}, \quad C_3 = \begin{bmatrix} A & B & C \\ C & A & B \\ B & C & A \end{bmatrix}.$$

We replace the elements of the group with real numbers:

$$C_1 = \begin{bmatrix} 5 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 6 & 77 & 0 \\ 0 & 6 & 77 \\ 77 & 0 & 6 \end{bmatrix}.$$

The first of these is an example of a 1-dimensional number—a real number. The second is an example of a 2-dimensional number, and the third is an example of a 3-dimensional number. Clearly, we can repeat this procedure for all abelian groups.

We replace the numbers with real variables and proclaim the arrays to be matrices:

$$C_1 = \begin{bmatrix} a \end{bmatrix}, \quad C_2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad C_3 = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}.$$

These matrix forms (with restrictions upon the allowed values of the variables that are of no importance) are all algebraic fields.

What is it about the real numbers and the complex numbers that deserves the appellation 'number'? Why, it is because they form an algebraic field. Thus, the 2-dimensional and the 3-dimensional numbers are just as much numbers as are the real numbers. We exponentiate the 2-dimensional numbers:

$$\exp\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix}\right) = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{bmatrix} = \begin{bmatrix} x & y \\ y & x \end{bmatrix}.$$

Within the polar form, there are no restrictions‡. Comparing determinants gives $r = \sqrt{x^2 - y^2}$. Thus we have the rotation matrix, trigonometric functions, and distance function of 2-dimensional hyperbolic space. Hence, within the algebra of the 2-dimensional numbers is 2-dimensional hyperbolic space. Similarly, every such algebra has a space within it.

> We have elevated every abelian group to an algebraic field. Within every one of these algebraic fields, there is a space. This is the unification of groups, numbers, algebra, and geometry.

The algebras can be extended by the introduction of real 'space squeezing' parameters:

$$C_1 = \begin{bmatrix} a \end{bmatrix}, \quad C_2 = \begin{bmatrix} a & b \\ jb & a \end{bmatrix}, \quad C_3 = \begin{bmatrix} a & b & c \\ jc & a & kb \\ jb & \frac{j}{k}c & a \end{bmatrix}.$$

These are algebraic fields for all values of $j, k \neq 0$. When $j = -1$, the 2-dimensional algebra is the complex number algebra. The complex numbers are now seen to be a form of the 2-dimensional numbers and not an extension of the 1-dimensional real numbers. The exponentiation of the 2-dimensional algebra with $j = -1$ will throw out the 2-dimensional euclidean space. This is now seen to be a version of the 2-dimensional hyperbolic space.

Implications for theoretical physics

The standard theory of particle physics is written in the algebra of the complex numbers but has within it groups other than C_2. It seems to me that the theory should be written in the algebra appropriate to each particular group. My research has shown that if this were to be done, the existing standard theory would be affected only by the introduction of extra information. The existing theory would stand without a wrinkle. The space squeezing parameters scream out general relativity. Together with the application of the appropriate algebras to the standard theory, I suspect we have here not only the unification of mathematics but also the unification of physics.

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† The standard form of a Cayley table is with the identities on the leading diagonal.

‡ Which is why they are of no importance.

Quick problem. Compute $\dfrac{(0!)^2}{2!} + \dfrac{(1!)^2}{4!} + \dfrac{(2!)^2}{6!} + \cdots$.

Solution 207.1 – 25 points

Start with a 5×5 square array of unmarked points. (*) Mark any four unmarked points which are at the corners of a square. Repeat (*) as often as possible. How many times you can perform (*)? What about 'corners of a square' replaced by 'corners of a rectangle whose sides are parallel to the edges of the array'?

Ian Adamson

The question asks first for the number of squares in a 5×5 array. This is six; we have found such a configuration and $6 \cdot 4 = 24$. The number of rectangles with sides parallel to the edges is four by the following argument.

Let us suppose that an $n \times n$ array is maximally marked with $4r$ points each at the corner of one of r rectangles. Now assume that n is odd and partition the array into R, the $2n-1$ points on two adjacent edges and call S the remaining $(n-1)^2$ points.

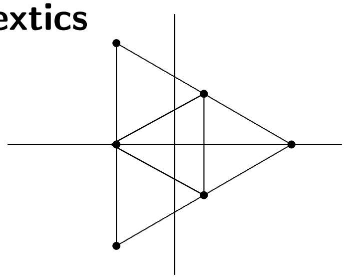
But first consider an $n \times (n-1)$ array (say T), that is without an edge (say U) containing n of the points in R. Any marked points in U may be replaced in pairs by points in T on lines perpendicular to U since each such line contains (odd) n points and only at most (even) $n-1$ points can be the corners of a rectangle.

Now any remaining marked points in R are similarly replaced by points in S. We still have the r rectangles but all their corners are in S. Is it possible that $4r = (n-1)^2 = |S|$ when r would equal its upper bound, $(n-1)^2/4$? It is easily seen that it is possible, for example 'unit' squares $(n-1)/2 \times (n-1)/2$. If we insist upon rectangles with the sense that their adjacent sides are unequal then it is clear that such rectangles ('overlapping') are possible.

Finally assume n is even but, as that's trivial, we claim the answer is $[n/2]^2$.

Problem 210.3 – Triangular sextics

Inspired by the solution of the equation at the bottom of page 13, find a characterization of those 6th degree polynomials whose roots lie *exactly* on the vertices and the side midpoints of an equilateral triangle.



Deal or no deal

Jeremy Humphries

There's an interesting programme with a mathematical and psychological basis on C4. There are 22 players, and each draws one of 22 sealed boxes, containing £0.01, £0.1, £0.5, £1, £5, £10, £50, £100, £250, £500, £750, £1000, £3000, £5000, £10,000, £15,000, £20,000, £35,000, £50,000, £75,000, £100,000, £250,000. Nobody knows what's in each box. One player is chosen at random to be the contestant C for the day. He takes his box to the chair, where the host is Noel Edmunds. His box remains sealed until the game ends. The other players will get a chance to be C on another day. Nobody is eliminated until they have played. When a player leaves a replacement is brought in.

Round 1: C nominates five boxes, which are opened to reveal the contents. There is then a phone call from 'The Banker', who is never seen, and who makes an offer for C's box. C can say 'deal' or 'no deal'. Round 2: As Round 1, except it's three boxes, not five. Subsequent rounds: As round 2.

If the play goes all the way to round 6 without C accepting a deal, then there are only two boxes left, C's box and one other. The Banker makes a final offer, which C can accept or reject. Sometimes the Banker will offer C the opportunity to swop boxes with the other player at this point. Then C accepts or rejects the swop (if offered), opens whichever box is then in his possession, and gets the contents.

The Banker's aim is to pay out as little as possible. Everybody else, including the host, wants C to get a hefty amount, or at least pretends so.

Example. Round 1. Opened: £0.01, £250, £750, £50,000, £100,000. Offer: £1,300. No deal. Round 2. Opened: £0.1 £100, £15,000. Offer: £2,900. No deal. Round 3. Opened: £500, £75,000, £250,000. Offer: £700. No deal. Round 4. Opened: £5, £10, £50. Offer: £4900. No deal. Round 5. Opened: £1000, £10,000, £20,000. Still in play: £0.5, £1, £3000, £5000, £35,000. Offer: £6500. Deal.

Is there a good strategy? The Banker's offer is always less than the expectation, which is the average of the remaining boxes. In the early rounds it is dramatically less, though in the later rounds the discrepancy is usually not so stark. Nevertheless, the mathematically sound strategy is never to take the offer. If you played many times you would win an average of £25,712.12 per play. However, when you play only once, is it sensible to go with the mathematics?

What's the next number?

ADF

Recall Diana Maxwell's problem on page 18 of M500 **207**. What's next in the sequence

$$\mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, ?, \ldots ?$$

Obviously the answer is 4. But why? From the blurb accompanying the original statement of the problem it is clear that we are looking at a feature of the English language. After a certain amount of experimenting with various kinds of word play one might deduce the following construction.

> To compute the nth element of the sequence, write the English name for n and count the letters to get another number. Repeat with the new number. Stop when stability is achieved.

Thus, for example,

$$42 \text{ (forty two)} \to 8 \text{ (eight)} \to 5 \text{ (five)} \to 4 \text{ (four)} \to 4,$$

and from 4 there is nowhere else to go. We suggest that the 4s continue for ever; perhaps someone can supply a *proof* that this is really so.

On the other hand, in Spanish we have

$$1 \text{ (uno)} \to 3 \text{ (tres)} \to 4 \text{ (cuatro)} \to 6 \text{ (seis)} \to 4 \to 6,$$

and now the sequence is in a loop consisting of $\{4, 6\}$. But 5 (cinco) stabilizes immediately. The sequence begins

> {4, 6}, {4, 6} (dos), {4, 6}, {4, 6}, {5}, {4, 6}, {5} (siete), {4, 6} (ocho), {5} (nueve), {4, 6} (diez), {4, 6} (once), {4, 6} (doce), {5} (trece), {5} (catorce), {4, 6} (quince), {5} (diez y seis), {4, 6} (diez y siete), {5} (diez y ocho), {4, 6} (diez y nueve), {4, 6} (viente), . . .

and at least for numbers of moderate size continues with $\{4, 6\}$ and $\{5\}$, not necessarily in any instantly recognizable pattern. However, we did detect a significantly increasing bias towards $\{4, 6\}$, which led us to ask in **207** an extremely interesting question: Is the number of occurrences of $\{5\}$ infinite?

Of course, you can play this game in other languages, and, indeed, we would be interested if you are prepared to do so and send us the results—especially in those cases where the thing does not degenerate into a trivial repetition of the same number. Meanwhile here is something completely different, this time from **Jeremy Humphries**:

1, 11, 21, 1211, 111221, 312211, ?, ?

Letters to the Editor

n-dimensional space–time

Dennis Morris's article, 'A revolutionary view of space' (M500 **207**) is fascinating and, as far as I can judge, important. But how are we supposed to take it—as pure mathematics or as physics? As far as I am aware, Banach spaces 'don't exist' and I don't think anyone wanting the State to spend money looking for them would get much of a hearing. Green, however, the co-inventor of string theory, apparently does believe in the existence of eight (or is it eleven?) extra dimensions we don't know are there and I believe some attempts have been made to detect them, so far without success. One would hope that Dennis Morris includes in his book suggestions of what experiments could be done to provide support for the theory of CN spaces—I don't mean this as a criticism.

There is also a conceptual/methodological problem. Dennis Morris says that 'the higher-dimensional CN spaces are "folded up" in the lower-dimensional CN spaces (string theory?).' Taken to its logical conclusion the whole lot must be 'enfolded' in a one-dimensional all-round space–time 'something' which is reminiscent of the 'original Tao' of ancient Chinese philosophy. We have here a two-tiered schema which seems to be cropping up time and again in cosmology and theoretical physics; what we can observe pre-exists in a more subtle and rarefied state which is not directly observable but which nonetheless 'contains' the former. Thus, some theorists, the so-called Brussels School, view the entire universe as a runaway fluctuation of a pre-existing quantum field.

The question is now, 'Why didn't this more fundamental entity just stay as it was?'—the modern equivalent of Leibnitz's by no means stupid question, 'Why is there something instead of nothing?' There is, for example, no strictly mathematical or physical reason why the wave function of quantum mechanics should ever collapse at all, and some physicists have seriously suggested that it is human (or maybe even feline) consciousness that triggers the change. This doesn't appeal to me much as it puts the clock right back to subjectivism. But the wave function *is* continually collapsing since otherwise no elementary particle would have a precise position at all, would be 'all over the place'—and there would be nothing recognizable around us at all.

In the days when people believed in a Creator God, they could of course always say, 'Such and such happened because God wanted it to.' Today we can't do that but we still seem to need the idea of natural laws governing physical behaviour. We should, then, maybe introduce a basic law which states that the wave function must always collapse eventually, and that, in the present context of CN spaces, lower-dimensional space(s) must always unfold into higher (with possibly an upper limit). This 'law' can perhaps be viewed as a generalization of the Law of Entropy though I'm not so sure about this—it depends whether one views, for example, our physical universe as being more, or less, ordered than what it emerged from.

Sebastian Hayes

Precision

I was reading a medical piece on blast injuries, and it said this:

> The positive pressure phase of the blast wave lasts only a few milliseconds, but close to an explosion it may rise to over 6894 kN/m squared (kilonewtons per metre squared).

I wondered how they came up with such a precise figure as 6894. Then I worked out that 6894 kilonewtons per metre squared (or kilopascals) is 1000 psi.

It reminds me of those recipes which tell you to take 397 grammes of some ingredient. What they mean is 'open a 14-ounce tin'.

Jeremy Humphries

What's next?

I was looking at 1, 2, 4, 8, 20 (connected with Conway soldiers) and while I was there thought I'd see what Sloane [www.research.att.com/~njas/sequences] had to say about Diana's seventeen 4s [page 18]. He lists 72 such sequences, mostly continuing with a 5 as next number. I submit the genuine next number is 19. One day I might try to prove it.

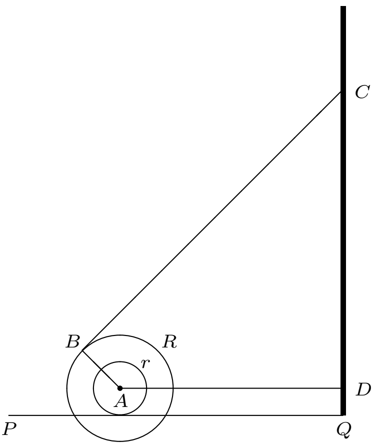
Eddie Kent

Problem 210.4 – Coal

There is a coal deposit that occurs underground in a plane inclined at angle θ to the horizontal. You make vertical holes at points A, B and C in a horizontal plane on the surface, detecting the coal seam at depths a, b and c respectively. If $AB = x$, $AC = y$ and $\angle BAC = \alpha$, what is θ?

Problem 210.5 – A monkey and a pole

Ken Greatrix



A monkey climbs a pole along DC to a height, y. It is tethered by a line to a drum of radius R which rolls on its axle of radius r along a flat bed PQ. During this rolling, the drum moves horizontally by a distance x. You may assume that the pole isn't high enough for the drum to collide with it, that the drum rolls back to its start position when the monkey climbs down again and that the line is always tangential to the drum. What is the relationship between x and y?

I already have a solution but it's iterative and clumsy. Is there a direct function that can be applied here? I have deduced that $ABCD$ is a cyclic quadrilateral, because ABC and ADC are right-angles. Ptolemy's theorem doesn't apply because there are too many unknowns.

Mathematics Revision Weekend 2006

The **32nd M500 Society Mathematics Revision Weekend** will be held at **Aston University**, **Birmingham** over **8–10 September 2006**.

The cost, including accommodation (with *en-suite* facilities) and all meals from bed and breakfast Friday to lunch Sunday is £195 – £230. The cost for non-residents is £100 (includes Saturday and Sunday lunch). M500 members get a discount of £10. For full details and an application form, see the Society's web page, www.m500.org.uk, or send a stamped, addressed envelope to

Jeremy Humphries, M500 Weekend 2006.

The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. Tutorial sessions start at 19.30 on the Friday and finish at 17.00 on the Sunday. We plan to present most OU mathematics courses.

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