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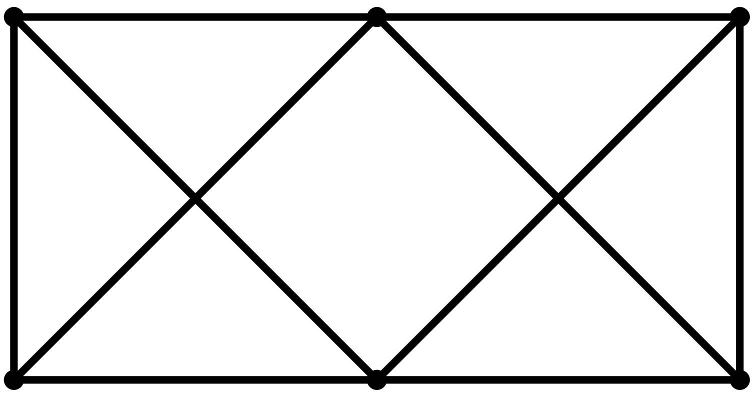

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Hypergeometric distribution and air quality

Tommy Moorhouse

In England and Wales there is an air quality objective (a target) that the hourly mean concentration of nitrogen dioxide at any relevant location should not exceed 200 units more than 18 times per year. We don't need to worry about the details.

The government's advisors have been considering whether emergency electricity generators powered by diesel will lead to the air quality target being breached. There are lots of planning applications for these generator sets around the country, and the current generation of diesel generators emit a lot of nitrogen oxides, relatively speaking. By nature, they will operate only when the demand for electrical power exceeds the grid capacity, and many potential operators are estimating that the sets will run for around 500 hours in any year or bidding for contracts on the basis of such an estimate (in fact there is another reason for choosing 500 hours but we don't need to go into that).

It is straightforward to use an atmospheric dispersion model to work out how many hours per year the concentration would exceed the objective at a given location if the plant were to operate all year round. Unfortunately it is hard to predict when the generator sets will operate. The government's advisors use a hypergeometric distribution to model the situation. The hypergeometric distribution gives the probability of finding m successes in a sample of k events from a population of n events with r successes as

$$p(m) \;=\; \frac{\binom{k}{m}\binom{n-k}{r-m}}{\binom{n}{r}}.$$

Assuming that the total running time is exactly 500 hours and assuming that these hours are randomly distributed over the year (probably not reflecting the likely demand for the generator power) calculate the maximum number of hours out of 8760 hours of operation that could exceed the target ('bad' hours or 'successes') but give a 95% chance that the number of 'bad' hours taken over 500 operating hours (the sample) will be 18 or fewer.

Reference

Diesel generator short term NO2 impact assessment, AQMAU-C1457-RP01, AQMAU 2016 (Environment Agency)

Solution 287.3 – Group determinant

Let G be a finite group of order n, label its elements G_1, G_2, $\ldots$, G_n, and associate with each element G_i a variable x_i, $i = 1$, 2, $\ldots$, n. The *group determinant* of G is the determinant of the matrix M whose element in row r, column c is x_k, where r, $c = 1, 2, \ldots, n$ and $G_k = G_r G_c^{-1}$. For example, the diagonal elements of M will be x_e, where G_e is the identity element of the group. (You might want to prove that this definition is sound.)

Now consider S_3, the group of permutations of $(1, 2, 3)$, with elements

$$(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1).$$

Construct the group determinant of S_3 and show that it factorizes as ABC^2, where A, B and C are expressions in x_1, x_2, $\ldots$, x_6.

Stuart Walmsley

Introduction

The general problem concerns a finite group of order n with elements G_0, G_1, $\ldots$, G_{n-1}. (It is convenient to give the identity the subscript 0.) A square table is formed of products $G_j G_k^{-1}$ so that the diagonal element is always the identity: $G_j G_j^{-1} = G_0$. From the properties of a group the n elements in a row (that is for a fixed value of j) are all distinct so that each element of the group appears once and only once. The same result applies to the elements of a column. In this way, the table is a Latin square: each of the n elements appears once in each row and once in each column.

Here the determinant of the same table is considered. The element G_j is formally replaced by g_j, which is considered to be an algebraic variable. This is called the group determinant: a homogeneous polynomial of degree n in the n variables g_j.

The objective is to expand the determinant of the symmetric group of degree 3, S_3, and to show that it has the factored form ABC^2. The group S_3 has order $3! = 6$.

An implicit solution has been given for the simpler case of a cyclic group in the answer I gave to Problem 261.6 (M500 **263**). It will be found that this result can be used in the present problem.

The group determinants of C_3 and C_2

The cyclic group C_3 is a subgroup of S_3. It can be constructed from three of the permutations of S_3 converted to the g notation used in the group determinant.

$$(123) \;\to\; G_0 \;\;\text{(the identity)}, \qquad (231) \;\to\; G_1, \qquad (312) \;\to\; G_2.$$

It is found that

$$\begin{array}{rclrclrcl} G_1^2 &=& G_2, & G_1^3 &=& G_0, & G_1^{-1} &=& G_2, \\ G_2^2 &=& G_1, & G_2^3 &=& G_0, & G_2^{-1} &=& G_1, \end{array}$$

so that the group determinant is

$$|C_3| \;=\; \det \begin{bmatrix} g_0 & g_2 & g_1 \\ g_1 & g_0 & g_2 \\ g_2 & g_1 & g_0 \end{bmatrix}.$$

The factorization of the determinant is completely determined by symmetry as explained in the solution to Problem 261.6.

It may be shown that if the product of two square matrices P and Q is R,

$$PQ \;=\; R,$$

then the corresponding determinants are related in a similar way:

$$|P||Q| \;=\; |R|.$$

Group theory shows that for a cyclic group, a matrix T can be found so that

$$TC_3T^{-1} \;=\; \Gamma$$

such that Γ is a diagonal matrix (that is, the only non-zero elements are along the principal diagonal: $\Gamma_{1,1}$, $\Gamma_{2,2}$, $\Gamma_{3,3}$). In this way the group determinant is just the product of these three factors:

$$|C_3| \;=\; \Gamma_{1,1}\Gamma_{2,2}\Gamma_{3,3}.$$

The elements of T take a simple form:

$$T_{j,k} \;=\; (1/\sqrt{3})\; e_{j,k},$$

where

$$e_{j,k} \;=\; \exp(2\pi i\; jk/3), \qquad j,k = 0,1,2, \quad jk = jk \pmod 3;$$

T is a unitary matrix and its inverse is given by

$$T^{-1}_{j,k} \;=\; T^*_{k,j}.$$

Then

$$\Gamma_{j,m} \;=\; \frac{1}{3}\sum_k \sum_l e_{j,k}\; g_{k,l}\; e^*_{l,m}.$$

The factor $g_{k,l}$ may be written $G_{k-l,0}$: $(k-l) \pmod 3$. Let $p = k - l$ so that $k = p + l$ and replace the sum over k by a sum over p:

$$\Gamma_{j,m} \;=\; \frac{1}{3}\sum_p \sum_l e_{j,p+l}\; g_{p,0}\; e^*_{l,m} \;=\; \frac{1}{3}\sum_p \sum_l e_{j,p}\; e_{j,l}\; g_{p,0}\; e^*_{l,m}.$$

The two sums may then be separated:

$$\Gamma_{j,m} \;=\; \frac{1}{3}\left(\sum_l e_{j,l}\; e^*_{l,m}\right)\left(\sum_p e_{j,p}\; g_{p,0}\right).$$

The first sum is

$$\sum_l e_{j,l}\; e^*_{l,m} \;=\; 3\,\delta_{j,m}, \qquad \delta_{j,m} = 1 \text{ if } j = m, \quad \delta = 0 \text{ if } j \neq m.$$

So that the resulting matrix is diagonal.

Writing out in full,

$$3\Gamma \;=\; \begin{bmatrix} e_0 & e_0 & e_0 \\ e_0 & e_1 & e_2 \\ e_0 & e_2 & e_1 \end{bmatrix} \begin{bmatrix} g_0 & g_2 & g_1 \\ g_1 & g_0 & g_2 \\ g_2 & g_1 & g_0 \end{bmatrix} \begin{bmatrix} e_0 & e_0 & e_0 \\ e_0 & e_2 & e_1 \\ e_0 & e_1 & e_2 \end{bmatrix},$$

giving

$$\Gamma \;=\; \begin{bmatrix} e_0g_0 + e_0g_1 + e_0g_2 & 0 & 0 \\ 0 & e_0g_0 + e_1g_1 + e_2g_2 & 0 \\ 0 & 0 & e_0g_0 + e_2g_1 + e_1g_2 \end{bmatrix},$$

The group determinant is then (remembering that $e_0 = 1$)

$$|C_3| \;=\; (g_0 + g_1 + g_2)(g_0 + e_1g_1 + e_2g_2)(g_0 + e_2g_1 + e_1g_2).$$

The group determinant of the cyclic group C_2 is also used in the solution of the present problem. It consists of two elements G_0 and G_1 with defining relations

$$G_1^2 \;=\; G_0, \qquad G_1^{-1} \;=\; G_1$$

and group determinant

$$|C_2| \;=\; \det \begin{bmatrix} g_0 & g_1 \\ g_1 & g_0 \end{bmatrix},$$

which may be directly expanded and factorized

$$|C_2| \;=\; (g_0^2 - g_1^2) \;=\; (g_0 + g_1)(g_0 - g_1).$$

The group determinant of S_3

The elements of S_3 given as permutations of (123) are converted to the G_j notation in the following way.

$$\begin{array}{cccccc} (123) & (231) & (312) & (132) & (321) & (213) \\ G_0 \text{ (identity)} & G_1 & G_2 & G_3 & G_4 & G_5 \end{array}$$

It is found that

$$\begin{array}{rclrclrcl} G_1^2 &=& G_2, & G_1^3 &=& G_0, & G_1^{-1} &=& G_2, \\ G_2^2 &=& G_1, & G_2^3 &=& G_0, & G_2^{-1} &=& G_1, \\ G_3^2 &=& G_0, & & & & G_3^{-1} &=& G_0, \\ G_4^2 &=& G_0, & & & & G_4^{-1} &=& G_0, \\ G_5^2 &=& G_0, & & & & G_5^{-1} &=& G_0. \end{array}$$

Also

$$G_4G_1 \;=\; G_1^2G_4 \;=\; G_2G_4.$$

From these relations the group determinant may be constructed.

$$|S_3| \;=\; \det \begin{bmatrix} g_0 & g_2 & g_1 & g_3 & g_4 & g_5 \\ g_1 & g_0 & g_2 & g_4 & g_5 & g_3 \\ g_2 & g_1 & g_0 & g_5 & g_3 & g_4 \\ g_3 & g_4 & g_5 & g_0 & g_2 & g_1 \\ g_4 & g_5 & g_3 & g_1 & g_0 & g_2 \\ g_5 & g_3 & g_4 & g_2 & g_1 & g_0 \end{bmatrix}.$$

It is seen that the determinant contains the determinant of its cyclic subgroup C_3, which is in fact repeated twice along the principal diagonal. The remaining off-diagonal regions have the same pattern with the second and third columns interchanged. A partial diagonalization can be achieved using an extended form of the matrix T used to factorize the group determinant of C_3. It has the form

$$\begin{bmatrix} T & \mathbf{0} \\ \mathbf{0} & T \end{bmatrix},$$

that is, T is repeated along the main diagonal and there are zeroes elsewhere. The result is

$$|S_3| = \det \begin{bmatrix} e_0g_0 + e_0g_1 + e_0g_2 & 0 & 0 & e_0g_3 + e_0g_4 + e_0g_5 & 0 & 0 \\ 0 & e_0g_0 + e_1g_1 + e_2g_2 & 0 & 0 & 0 & e_0g_3 + e_1g_4 + e_2g_5 \\ 0 & 0 & e_0g_0 + e_2g_1 + e_1g_2 & 0 & e_0g_3 + e_2g_4 + e_1g_5 & 0 \\ e_0g_3 + e_0g_4 + e_0g_5 & 0 & 0 & e_0g_0 + e_0g_1 + e_0g_2 & 0 & 0 \\ 0 & 0 & e_0g_3 + e_1g_4 + e_2g_5 & 0 & e_0g_0 + e_1g_1 + e_2g_2 & 0 \\ 0 & e_0g_3 + e_2g_4 + e_1g_5 & 0 & 0 & 0 & e_0g_0 + e_2g_1 + e_1g_2 \end{bmatrix}.$$

This is the product of three quadratic determinants:

$$\begin{aligned} |S_3| = {} & \det \begin{bmatrix} e_0g_0 + e_0g_1 + e_0g_2 & e_0g_3 + e_0g_4 + e_0g_5 \\ e_0g_3 + e_0g_4 + e_0g_5 & e_0g_0 + e_0g_1 + e_0g_2 \end{bmatrix} \\ & \det \begin{bmatrix} e_0g_0 + e_1g_1 + e_2g_2 & e_0g_3 + e_1g_4 + e_2g_5 \\ e_0g_3 + e_2g_4 + e_1g_5 & e_0g_0 + e_2g_1 + e_1g_2 \end{bmatrix} \\ & \det \begin{bmatrix} e_0g_0 + e_2g_1 + e_1g_2 & e_0g_3 + e_2g_4 + e_1g_5 \\ e_0g_3 + e_1g_4 + e_2g_5 & e_0g_0 + e_1g_1 + e_2g_2 \end{bmatrix}. \end{aligned}$$

The first factor is equivalent to the group determinant of C_2 and can be resolved into two linear factors. The second and third factors are the same and remain quadratic. In this way (and noting that $e_0 = 1$):

$$\begin{aligned} |S_3| = {} & \big((g_0 + g_1 + g_2) + (g_3 + g_4 + g_5)\big)\big((g_0 + g_1 + g_2) - (g_3 + g_4 + g_5)\big) \\ & \big((g_0 + e_1g_1 + e_2g_2)(g_0 + e_2g_1 + e_1g_2) \\ & \quad - (g_3 + e_1g_4 + e_2g_5)(g_3 + e_2g_4 + e_1g_5)\big)^2. \end{aligned}$$

The quadratic terms can be alternatively written by expanding the brackets and noting that $e_1 + e_2 = -1$.

$$\begin{aligned} |S_3| &= \big((g_0+g_1+g_2)+(g_3+g_4+g_5)\big)\big((g_0+g_1+g_2)-(g_3+g_4+g_5)\big) \\ &\quad \Big(\big((g_0^2+g_1^2+g_2^2)-(g_0g_1+g_0g_2+g_1g_2)\big) \\ &\qquad -\big((g_3^2+g_4^2+g_5^2)-(g_3g_4+g_3g_5+g_4g_5)\big)\Big)^2. \end{aligned}$$

This shows that the group determinant has the form ABC^2, as demanded by the problem; A and B are both linear and C is quadratic. This is a general property of the factorized form of group determinants. If a factor is linear it occurs once, if quadratic twice, and so on.

TF: The determinants for a few more groups, computed with a little help from MATHEMATICA (C_5 is omitted because it takes up too much space).

$$\begin{aligned}
C_2:&\ (x_1-x_2)(x_1+x_2);\\
C_3:&\ (x_1+x_2+x_3)(x_1^2-x_1x_2+x_2^2-x_1x_3-x_2x_3+x_3^2);\\
C_4:&\ (x_1-x_2+x_3-x_4)(x_1+x_2+x_3+x_4)\\
&\ (x_1^2+x_2^2-2x_1x_3+x_3^2-2x_2x_4+x_4^2);\\
C_6:&\ (x_1-x_2+x_3-x_4+x_5-x_6)(x_1+x_2+x_3+x_4+x_5+x_6)\\
&\ (x_1^2-x_1x_2+x_2^2-x_1x_3-x_2x_3+x_3^2+2x_1x_4-x_2x_4-x_3x_4+x_4^2\\
&\quad -x_1x_5+2x_2x_5-x_3x_5-x_4x_5+x_5^2-x_1x_6-x_2x_6+2x_3x_6\\
&\quad -x_4x_6-x_5x_6+x_6^2)\\
&\ (x_1^2+x_1x_2+x_2^2-x_1x_3+x_2x_3+x_3^2\\
&\quad -2x_1x_4-x_2x_4+x_3x_4+x_4^2-x_1x_5-2x_2x_5-x_3x_5+x_4x_5\\
&\quad +x_5^2+x_1x_6-x_2x_6-2x_3x_6-x_4x_6+x_5x_6+x_6^2);\\
C_2\times C_2:&\ (x_1+x_2-x_3-x_4)(x_1-x_2+x_3-x_4)(x_1-x_2-x_3+x_4)\\
&\ (x_1+x_2+x_3+x_4);\\
D_4:&\ (x_1+x_2+x_3+x_4-x_5-x_6-x_7-x_8)\\
&\ (x_1-x_2+x_3-x_4+x_5-x_6+x_7-x_8)\\
&\ (x_1-x_2+x_3-x_4-x_5+x_6-x_7+x_8)\\
&\ (x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8)\\
&\ (x_1^2+x_2^2-2x_1x_3+x_3^2-2x_2x_4+x_4^2-x_5^2-x_6^2\\
&\quad +2x_5x_7-x_7^2+2x_6x_8-x_8^2)^2.
\end{aligned}$$

Solution 288.4 – Two dice

> You go to the casino to play a game involving two dice and the redistribution of wealth. The casino's die has its faces marked 2,3,6,6,6,7, and yours is marked 1,1,5,5,9,9. At each turn the dice are thrown and the winner is whoever's die shows the higher number. The odds are 1 : 1 and the loser pays the winner the agreed stake.
>
> As usual, the game is biased in favour of the casino, as you can verify by examining all 36 possible outcomes (or otherwise). After a while you detect this bias and you refuse to play any more unless the game is made fairer.
>
> You suggest throwing each die twice and taking the average of the two scores. For example, if the casino throws (6,7) and you throw (1,9), you lose because 6.5 is greater than 5. The casino, whose ultimate aim is to ruin you, agrees on the additional condition that it wins if the averages are the same—as would happen if the scores were (7,7) and (5,9). You agree and the game continues under the new rules. Was this wise?

Peter Fletcher

By a simple process of counting possible outcomes from rolling each die twice, we can construct Tables 1 and 2. The probabilities in each table add up to 1.

Table 1: My dice rolls

Outcomes		Average	Probability
1	1	1	4/36
1	5	3	8/36
1	9	5	8/36
5	5	5	4/36
5	9	7	8/36
9	9	9	4/36

Using the probabilities in Tables 1 and 2, we can construct Table 3, where we calculate the probabilities of winning or losing, depending on the rolls of the dice. Again, the probabilities add up to 1.

Table 2: The casino's dice rolls

Outcomes		Average	Probability
2	2	2	1/36
2	3	2.5	2/36
2	6	4	6/36
2	7	4.5	2/36
3	3	3	1/36
3	6	4.5	6/36
3	7	5	2/36
6	6	6	9/36
6	7	6.5	6/36
7	7	7	1/36

Table 3: Win or lose?

My average	Casino's	Winner	Probability
1	>1	Casino	$\frac{4}{36} \cdot 1 = \frac{1}{9}$
3	<3	Me	$\frac{8}{36} \cdot \frac{3}{36} = \frac{1}{54}$
3	>3	Casino	$\frac{8}{36} \cdot \frac{33}{36} = \frac{11}{54}$
5	<5	Me	$\frac{12}{36} \cdot \frac{18}{36} = \frac{1}{6}$
5	≥ 5	Casino	$\frac{12}{36} \cdot \frac{18}{36} = \frac{1}{6}$
7	<7	Me	$\frac{8}{36} \cdot \frac{35}{36} = \frac{35}{162}$
7	$=7$	Casino	$\frac{8}{36} \cdot \frac{1}{36} = \frac{1}{162}$
9	<9	Me	$\frac{4}{36} \cdot 1 = \frac{1}{9}$

From Table 3, we can calculate the probability of my winning: it is

$$\frac{1}{54} + \frac{1}{6} + \frac{35}{162} + \frac{1}{9} \ = \ \frac{83}{162}.$$

Since this is greater than 1/2, the game is now biased in my favour and the casino was unwise to agree to the new rules.

A gravitational estimation

Colin Aldridge

The M500 Society visited Isaac Newton's birthplace in November, where about a dozen of us learnt a lot about the background to his life. It was a thoroughly enjoyable and informative afternoon.

Amongst the exhibits was a gravitational lamp where a weight gradually falls under gravity and drives a dynamo which lights two LEDs. The light is enough to read by, just, and this gravity light is now being used in areas which are very poor and 'off grid'. The red loop drives the pulley system which allows the 10 kg weight to be lifted easily.

That set me wondering along the lines of the Tritium puzzle posed in M500 **286** The weight is about 10 kg and falls through about 1 m over a 30 minute period. So what is the wattage of each bulb?

Problem 292.1 – Angle trisection

Whilst idly browsing volume 14 (1992) of *The Mathematical Intelligencer* I (TF) found this interesting formula in Ian Stewart's review of *A Budget of Trisections* by Underwood Dudley. Stewart advises, 'Don't set this one to high-school students.' So please do not read any further if you are still at school. Let

$$D \;=\; 10\cos(\theta/2)(6-\cos\theta)\sqrt{29-4\cos\theta}$$

and suppose $-120^\circ \le \theta \le 120^\circ$. Show that

$$\tan\frac{\theta}{3} \;=\; \frac{\sin\theta}{6-\cos\theta}\cdot\frac{1608-676\cos\theta+68\cos^2\theta-D}{168+100\cos\theta-68\cos^2\theta+D}.$$

Well, nearly. Near enough maybe to cause excitement amongst persons whose business it is to trisect angles.

Cycling sequences

Tommy Moorhouse

We will define a sequence of integers by choosing two relatively prime integers $m > 1$ and $n > m$ and a 'seed' integer a_0. Set

$$a_k = a_{k-1} + m$$

if n does not divide a_{k-1} and

$$a_k = \frac{a_{k-1}}{n}$$

otherwise.

As a simple example take $m = 2$, $n = 3$. With seed $a_0 = 1$ we have $a_1 = 3$, $a_2 = 1$ and the sequence loops through these values thereafter. The seed $a_0 = 2$ gives $a_1 = 4$, $a_2 = 6$, $a_3 = 2$ and again the sequence cycles through these values. We write

$$1 \to 3 \to 1,$$

and

$$2 \to 4 \to 6 \to 2.$$

In this case we say that the sequence cycles. The sequence

$$5 \to 7 \to 9 \to 3 \to 1 \to 3$$

will be said to end in a cycle. A more interesting example uses $m = 3$, $n = 11$ and you might like to try different seeds to see what happens.

Prove that all such sequences end in a cycle or find a counter-example.

Problem 292.2 – Positive definite matrix

A matrix $M(d)$ is constructed with diagonal elements d and real entries elsewhere. Prove that $M(d)$ is positive definite for all sufficiently large real d. Or find a counter-example.

Problem 292.3 – Half

Find a graph that has eigenvalue $1/2$. Or prove that no such graph exists.

How to do mathematical research

Tony Forbes

Continuing from the last time I (then slightly anonymously) wrote about this subject [M500 **264** 17] and the reply by J. J. Reynolds [M500 **267** 20], and being prompted by requests from a number of M500 Society members, I offer, slightly less facetiously, the following advice to anyone who is interested in starting to do mathematical research.

(i) *Get a computer.* Choose one with the fastest clock speed and with as many cores as you can afford. You can buy a refurbished 4-core ex-office computer quite cheaply.

(ii) Get a symbolic mathematics package, such as MATHEMATICA.

(iii) Get a C compiler, such as GCC (GNU Compiler Collection).

(iv) Learn how to use (i), (ii) and (iii).

And for when you want to get your results published.

(v) Get LaTeX and learn how to use it. This is conveniently bundled as MiKTeX. Also you might also want to consider WINEDT, assuming your computer has an appropriate operating system. WINEDT costs a few tens of dollars but is not vital, and I do know people who seem to manage without it; however, it does make the creation of LaTeX documents a lot easier.

(vi) If you have another proprietary word processor installed on your system, *get rid of it.*

(vii) What to do when your research has reached a brick wall? Write the paper. Omit the theorem you can't prove (or state it as a conjecture) and submit the paper anyway. A lot can happen during the time it takes for the journal to respond.

I have also been asked if I can offer any useful help with debugging computer programs. Here, all I can do is recycle the advice given to students by computer science lecturers.

(viii) Get yourself a rubber duck. Talk to it. Carefully explain to your rubber duck the purpose of each line of your code. Tell it what your code actually does, what it is supposed to do, and perhaps discuss better ways of achieving the same effect.

Please note that the above represents my own experience. When I signed up to do an OU PhD my main entry qualification was that I had already achieved (i)–(v). And I did (vi) on the day when I got that dreaded message, 'Unable to open document . . . '.

A triangle, an octahedron and a sphere

Tommy Moorhouse

Consider a unit-sided equilateral triangle T defined by its vertices ABC. It is a curious fact that coordinates (x, y, z) can be assigned to any point inside the triangle such that

$$x + y + z = 1.$$

To do this define the x coordinate to be the length of a line segment starting on AB running parallel to AC and meeting the point. Similarly the y coordinate is measured along a line starting on CA and running parallel to CB to meet the point. Finally the z coordinate uses the line through the point, starting on BC and running parallel to BA. The proof that $x + y + z = 1$ is extremely simple and requires no algebra – have a try.

We can map the triangle into the surface of a unit sphere using

$$(x, y, z) \to (X, Y, Z) = (\sqrt{x}, \sqrt{y}, \sqrt{z}).$$

By construction $X^2 + Y^2 + Z^2 = 1$ and the image can be considered as an equilateral triangle on the sphere, bounded by arcs of great circles meeting at right angles, with an area an eighth of that of the sphere. There are eight ways to choose the signs of the square roots (i.e. $(+, +, +)$, $(-, +, +)$ etc.) and we can interpret this as mapping the triangle to different faces of an octahedron.

The inverse mapping, from the spherical triangle to T, is given by

$$\pi : (X, Y, Z) \to (x, y, z) = (X^2, Y^2, Z^2).$$

If you know some elementary differential geometry try this problem. On T the coordinate differentials satisfy $dx + dy + dz = 0$. Use this together with the tangent vector map

$$\pi_* V_X = \frac{d}{dt} \pi(X + tV)|_{t=0}$$

to show that the tangent vectors to the sphere in $\mathbb{R}^3$ are orthogonal to the position vector of their point of application.

Problem 292.4 – Primes

Given $t \geq 3$, show that there exist primes $p_1 < p_2 < \cdots < p_t$ such that $p_1 + p_2 > p_t$. Or find a counter-example. For instance, when $t = 10$ we can satisfy the condition with 37, 41, 43, 47, 53, 59, 61, 67, 71, 73.

Solution 219.3 – Circumcircle

A triangle has sides which are the three roots of the cubic

$$x^3 - ax^2 + bx - c.$$

Show that its circumcircle has radius

$$\frac{c}{\sqrt{4a^2b - 8ac - a^4}}.$$

Peter Fletcher

Let the roots of the cubic be x_1, x_2 and x_3 so that

$$\begin{aligned} a &= x_1 + x_2 + x_3, \\ b &= x_1x_2 + x_1x_3 + x_2x_3, \\ c &= x_1x_2x_3. \end{aligned}$$

From the sine rule we know that the circumradius R is $x_1/\bigl(2\sin\alpha\bigr)$, where α is the angle opposite x_1.

We also know from the cosine rule that

$$x_1^2 = x_2^2 + x_3^2 - 2x_2x_3\cos(\alpha)$$

so that

$$\cos(\alpha) = \frac{x_2^2 + x_3^2 - x_1^2}{2x_2x_3}$$

and

$$\begin{aligned} \sin(\alpha) &= \sqrt{1 - \left(\frac{x_2^2 + x_3^2 - x_1^2}{2x_2x_3}\right)^2} \\ &= \frac{\sqrt{4x_2^2x_3^2 - x_1^4 - x_2^4 - x_3^4 - 2x_2^2x_3^2 + 2x_1^2x_2^2 + 2x_1^2x_3^2}}{2x_2x_3}. \end{aligned}$$

Therefore

$$R = \frac{x_1x_2x_3}{\sqrt{2x_1^2x_2^2 + 2x_1^2x_3^2 + 2x_2^2x_3^2 - x_1^4 - x_2^4 - x_3^4}}.$$

We now need to write this in terms of a, b and c. The numerator of R is obviously c.

Consider

$$\begin{aligned} a^2 &= x_1^2 + x_2^2 + x_3^2 + 2(x_1x_2 + x_1x_3 + x_2x_3) \\ &= x_1^2 + x_2^2 + x_3^2 + 2b. \end{aligned}$$

Then

$$a^2 - 2b \;=\; x_1^2 + x_2^2 + x_3^2$$

and

$$a^4 - 4a^2b + 4b^2 \;=\; x_1^4 + x_2^4 + x_3^4 + 2\bigl(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2\bigr)$$

so that

$$4a^2b - a^4 - 4b^2 \;=\; -\left(x_1^4 + x_2^4 + x_3^4\right) - 2\bigl(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2\bigr). \qquad \text{(i)}$$

Now consider

$$b^2 \;=\; x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + 2\bigl(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2\bigr).$$

From this we can write

$$4\bigl(b^2 - 2x_1x_2x_3(x_1 + x_2 + x_3)\bigr) \;=\; 4\bigl(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2\bigr)$$

or

$$4b^2 - 8ac \;=\; 4\bigl(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2\bigr).$$

Adding this equation to Eqn. (i) gives

$$4a^2b - a^4 - 8ac \;=\; -\left(x_1^4 + x_2^4 + x_3^4\right) + 2\bigl(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2\bigr)$$

and

$$R \;=\; \frac{c}{\sqrt{4a^2b - 8ac - a^4}}.$$

Problem 292.5 – Integral

Let n be a positive integer. Show that

$$\int_0^\infty \frac{t^x - t^y}{(1+t)^{x+y}}\,\frac{dt}{t^n} \;=\; 0 \quad \text{for all } x, y > n - 1$$

if and only if $n = 1$. So, if you define a two-variable function by

$$\beta_n(x,y) \;=\; \int_0^\infty \frac{t^x}{(1+t)^{x+y}}\,\frac{dt}{t^n},$$

then β_n is symmetrical in x and y only when $n = 1$ in which case $\beta_1(x,y)$ is the standard beta function.

Solution 289.1 – Odd divisor sum

Let

$$S(m,n) \;=\; \frac{A(m,n)}{n^2} \;=\; \frac{1}{n^2}\sum_{k=n+1}^{2^m n} q(k), \qquad m,n = 1,2,\ldots,$$

where the function $q(k)$ is defined by

$$q(k) \;=\; \begin{cases} k & \text{if } k \text{ is odd,} \\ q(k/2) & \text{otherwise;} \end{cases}$$

i.e. $q(k)$ is the largest odd divisor of k. (i) Compute $S(1,1)$ and show that

$$S(m+1,1) \;=\; 4S(m,1)+1.$$

(ii) Show that in general $S(m,n)$ is an integer that is (somewhat amazingly (at least in the opinion of me (TF))) independent of n, and hence obtain a closed formula for $S(m,n)$.

Ted Gore

Consider $A(3,5) = \sum_{k=6}^{40} q(k)$. This is the equivalent in the definition of S of the pair $m = 3$, $n = 5$. Moreover, $A(3,5)$ is the sum of 35 terms which can be divided into 3 subsets.

Subset 1 is the first 5 odd numbers; subset 2 is the first 10 odd numbers; subset 3 is the first 20 odd numbers. Now $\sum_{i=1}^{r}(2i-1) = r^2$ so that

$$A(3,5) \;=\; 5^2+10^2+20^2 \;=\; 5^2\cdot 21.$$

Hence $S(3,5) \;=\; \dfrac{A(3,5)}{5^2} \;=\; 21.$

Generalizing this, we can see that m is the number of subsets and the sizes of the subsets are n, $2n$, $4n$, $\ldots$:

$$A(m,n) \;=\; n^2\sum_{i=0}^{m-1}4^i \;=\; n^2\,\frac{4^m-1}{3}, \qquad S(m,n) \;=\; \frac{4^m-1}{3}.$$

Now

$$\frac{S(m+1,n)}{S(m,n)} \;=\; \frac{4^{m+1}-1}{4^m-1} \;=\; \frac{4(4^m-1)+3}{4^m-1} \;=\; 4+\frac{1}{S(m,n)},$$

so that $S(m+1,n) = 4S(m,n)+1$.

M500 Mathematics Revision Weekend 2020

The forty-sixth M500 Revision Weekend will be held at

Kents Hill Park Training and Conference Centre,

Milton Keynes, MK7 6BZ

from Friday 15th to Sunday 17th May 2020.

The standard cost, including accommodation (with en suite facilities) and all meals from dinner on Friday evening to lunch on Sunday is £275 for single occupancy, or £240 per person for two students sharing in either a double or twin bedded room. The standard cost for non-residents, including Saturday and Sunday lunch, is £160.

Members may make a reservation with a £25 deposit, with the balance payable at the end of February. Non-members must pay in full at the time of application and all applications received after 28th February 2020 must be paid in full before the booking is confirmed. Members will be entitled to a discount of £15 for all applications received before 14th April 2020. The Late Booking Fee for applications received after 14th April 2020 is £20, with no membership discount applicable.

There is free on-site parking for those travelling by private transport. For full details and an application form, see the Society's web site:

`www.m500.org.uk`.

The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. We expect to offer tutorials for most undergraduate and postgraduate mathematics OU modules, subject to the availability of tutors and sufficient applications.

Answer to M500 291 End of year quiz: films starring Doris Day (1922–2019) → characters played by her; (AIP ↦ EDJ, April In Paris, Ethel 'Dynamite' Jackson), (BRJ ↦ KW, Billy Rose's Jumbo, Kitty Wonder), (BTLOTSM ↦ MW, By The Light Of The Silvery Moon, Marjorie Winfield), (C ↦ PF, Caprice, Patricia Foster), (CJ ↦ MJC, Calamity Jane, Martha Jane Canary), (DND ↦ JH, Do Not Disturb, Janet Harper), (IAGF ↦ JA, It's A Great Feeling, Judy Adams), (IHTJ ↦ JO, It Happened To Jane, Jane Osgood), (ISYIMD ↦ GLK, I'll See You In My Dreams, Grace LeBoy Kahn), (J ↦ JB, Julie, Julie Benton), (LCB ↦ CT, Lover Come Back, Carol Templeton), (LM ↦ CW, Lucky Me, Candy Williams), (LMOLM ↦ RE, Love Me Or Leave Me, Ruth Etting), (LOB ↦ MH, Lullaby Of Broadway, Melinda Howard), (MDIY ↦ MG, My Dream Is Yours, Martha Gibson), (ML ↦ KP, Midnight Lace, Kit Preston), (MOD ↦ EWA, Move Over Darling, Ellen Wagstaff Arden), (OMB ↦ MW, On Moonlight Bay, Marjorie Winfield), (PDETD ↦ KRM, Please Don't Eat The Daisies, Kate Robinson Mackay), (PT ↦ JM, Pillow Talk, Jan Morrow), (ROTHS ↦ GG, Romance On The High Seas, Georgia Garrett), (S ↦ DD, Starlift, herself), (SMNF ↦ JK, Send Me No Flowers, Judy Kimball), (SW ↦ LR, Storm Warning, Lucy Rice), (TBOJ ↦ JM, The Ballad Of Josie, Josie Minick), (TFT ↦ NC, Tea For Two, Nanette Carter), (TGBB ↦ JN, The Glass Bottom Boat, Jennifer Nelson), (TMWKTM ↦ JCM, The Man Who Knew Too Much, Josephine Conway McKenna), (TP ↦ ES, Teacher's Pet, Erica Stone), (TPG ↦ KBW, The Pajama Game, Katherine 'Babe' Williams), (TTOIA ↦ BB, The Thrill Of It All, Beverly Boyer), (TTOL ↦ IP, The Tunnel Of Love, Isolde Poole), (TTOM ↦ CT, That Touch Of Mink, Cathy Timberlake), (TWPS ↦ JW, The West Point Story, Jan Wilson), (TWT ↦ AA, The Winning Team, Aimee Alexander), (WSYGE ↦ AM, With Six You Get Eggroll, Abby McClure), (WWYWTLWO ↦ MG, Where Were You When The Lights Went Out?, Margaret Garrison), (YAH ↦ LT, Young At Heart, Laurie Tuttle), (YMWAH ↦ JJ, Young Man With A Horn, Jo Jordan).

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Problem 292.6 – Tracks

This is like Problem 257.4 – Tracks (which was answered by Reinhardt Messerschmidt in M500 **259**) but the final question is different. Your MP3 player has tracks, T_0, T_1, ..., T_n of lengths t_0, t_1, ..., t_n respectively. The device selects tracks at random and plays them in full. The probability of track T_i getting selected is proportional to t_i. What is the expected minimum playing time to hear each track at least once?

Front cover Graph with 6 vertices and 10 edges.