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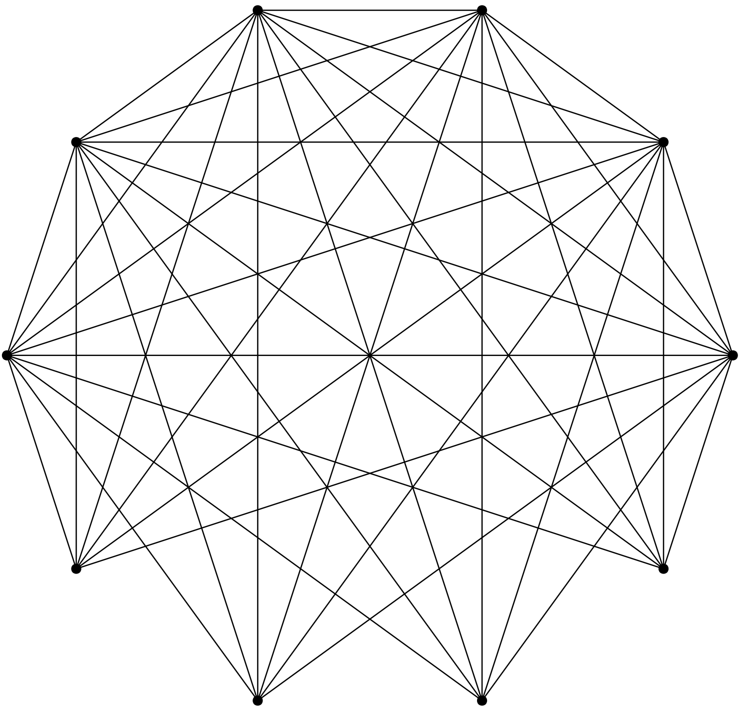
M500 293



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Sums of powers of digits

Tony Forbes

For positive integers k and n, define the function $D_k(n)$ by

$$D_k(n) \;=\; \sum_{d,\ d \text{ runs through the digits of } n} d^k.$$

So, for example, $D_6(43210) = 4^6 + 3^6 + 2^6 + 1 = 4890$.

Naturally one gets excited whenever $D_k(n) = n$, apart, of course, from the trivial case $n = 1$. On the other hand, G. H. Hardy's enthusiasm is more subdued. In *A Mathematician's Apology* he quotes $153 = 1 + 5^3 + 3^3$, $370 = 3^3 + 7^3$, $371 = 3^3 + 7^3 + 1$ and $407 = 4^3 + 7^3$ as the only solutions of $D_3(n) = n \geq 2$ and then goes on to question the mathematical significance of these discoveries. Hardy says, 'These are odd facts, very suitable for puzzle columns and likely to amuse amateurs, but there is nothing in them which appeals much to a mathematician. The proofs are neither difficult nor interesting—merely a little tiresome.' To continue with our amateurish amusement, here are two examples of a more substantial nature.

$$\begin{aligned} 32164049651 &= 3^{11} + 2^{11} + 1 + 6^{11} + 4^{11} + 4^{11} + 9^{11} + 6^{11} + 5^{11} + 1, \\ 564240140138 &= 5^{13} + 6^{13} + 4^{13} + 2^{13} + 4^{13} + 1 + 4^{13} + 1 + 3^{13} + 8^{13}. \end{aligned}$$

Exercise for reader: Find another n such that $D_{11}(n) = n$.

If n also has exactly k digits, it is called (amongst quite a lot of other things) a *Pluperfect Digital Invariant*, or, if that's too long for the mind to cope with, you can shorten it to PPDI. It is known that there are precisely 88 of them, which you can see at `https://www.deimel.org/rec_math/DI_3.htm`. Moreover, the number of digits in a PPDI must belong to

$$\begin{aligned} N \;=\; \{&1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 16, 17, 19, 20, 21, 23, 24, 25, 27, 29, \\ &31, 32, 33, 34, 35, 37, 38, 39\}. \end{aligned}$$

Hence the second example given above doesn't count because it has twelve digits, not thirteen.

Let us ignore this deviation and concentrate on numbers that satisfy

$$D_k(n) \;=\; n, \qquad \text{where } k \geq 1 \text{ and } n \geq 2 \text{ are integers,} \tag{$*$}$$

and with no further restrictions. I see that in M500 **163** I refer to them as *Recurring Digital Invariants of order k and cycle length* 1, abbreviated to

RDI$(k, 1)$. OEIS [`https://oeis.org/A003321`] calls them *Perfect Digital Invariants*, PDI. I find this slightly confusing because I have seen at least one source where the term PDI is reserved for n which satisfy $(*)$ but are not PPDIs. Perhaps I can resolve the ambiguity by calling them *Perfect But Not Necessarily Pluperfect Digital Invariants*, or PBNNPPDI, but I won't. In any case we might need to specify the exponent somewhere; so if the exponent is k, it seems sensible to adopt the notation PDI(k).

Clearly, when $k = 1$, the only solutions of $(*)$ are 2, 3, …, 9, and when $k = 2$ there are no solutions at all. For a complete list of the what is claimed to be the smallest PDI(k) for $k \leq 109$, see `https://oeis.org/A003321/b003321.txt`. And here are a couple more.

k	d	n	
44	43	68102095360217518611149183484609929555099 43	
85	83	1161109387870080752186374532232603914369577 9961208 573941316709607456134088705946891	[1]

[1] Older readers might remember this one from M500 **163**.

My interest in these things is not they exist, for, as Hardy says, 'Blah blah … ' (see above), but how one goes about finding them.

One useful observation is that the order of the digits doesn't really matter. Denote by n^{S} the number you get by sorting the digits of n. Thus, for example, $74000252^{\mathrm{S}} = 22457$. Now define the function $S_k(n)$ by $D_k(n)^{\mathrm{S}}$, and let us call a number that satisfies $S_k(n) = n^{\mathrm{S}} = n$ a PSDI(k) (with the sortedness being indicated by the letter S). We have seen that 153 is a PDI(3), but if we sort the digits, then 135 is a PSDI(3). It is clear (if it is not, then perhaps you should prove it; see Problem 293.6 on page 20) that n is a PSDI(k) if and only if $D_k(n)$ is a PDI(k). The point is (I think) that it might be much easier to find PSDI(k)s, and once we have found a PSDI(k) we can readily convert it to a PDI(k) if it isn't already one.

In M500 **163** I offered a very simple algorithm for finding PDIs. Here it is again, this time modified for PSDIs.

> Choose an exponent k and a starting number, n_0 with its digits sorted.
>
> Generate a sequence n_0, n_1, …, where $n_{i+1} = S_k(n_i)$, stopping when it goes into a loop.
>
> If the loop has length 1, report a PSDI(k).

This was good enough to find the entries in the table but probably not the most efficient way to pick up everything. To do the job properly, you might want to consult L. E. Deimel, Jr. and M. T. Jones, Finding Pluperfect Digital Invariants: Techniques, Results and Observations, *J. Recreat. Math.* **14** (1981), 97–108, in which they compute, amongst other things, the complete list of the 88 PPDIs. They show that, provided k is not too large, it is computationally feasible to do an exhaustive search for *all* PDI(k)s, with k digits. However, there is nothing in their methods that requires the number of digits to be the same as the exponent. Their techniques could be extended to look for unrestricted PDI(k)s.

We finish with a problem. Either

(i) show that there exists a PDI(k) > 1 for all sufficiently large k, or

(ii) show that there are infinitely many k for which no PDI(k) > 1 exists.

Solution 288.1 – Matrix powers

Given a_1, b_1, c_1, d_1, let M be a 2×2 matrix defined by

$$M^n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}, \quad n = 1, 2, \ldots.$$

Show that $b_n c_1 = b_1 c_n$ for $n = 1, 2, \ldots$.

Peter Fletcher

If

$$M^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

then clearly

$$M^1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

and

$$M^{n+1} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

so that obviously $M^{n+1}(2,2) = c_n b_1 + d_n d_1$.

We can also write

$$M^{n+1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

so that

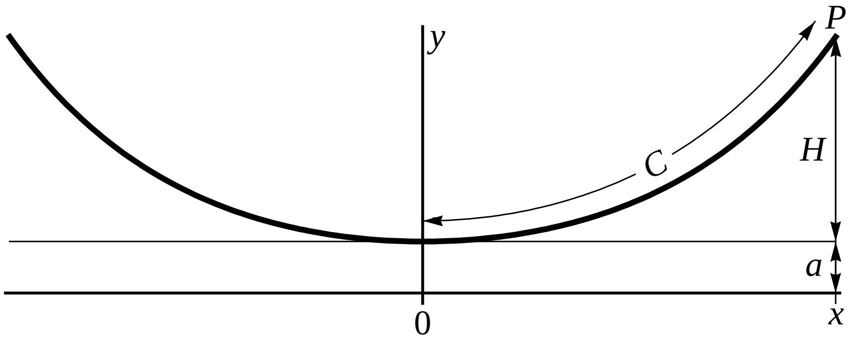
$$M^{n+1}(2,2) = c_1 b_n + d_1 d_n.$$

Therefore $b_n c_1 = b_1 c_n$ for $n = 1, 2, \ldots$.

Solution 288.9 – Chain

> A chain of length $2C$ is suspended between the tops of two vertical poles of height H, $0 \le H \le C$ and just grazes the ground at its centre. How far apart are the poles?
>
> A large company was and possibly still is offering $H = C = 40\,\mathrm{m}$, as a test for job applicants. Presumably, mathematical skills are irrelevant and one's employment prospects are determined only by how long it takes to discover that this special case is trivial.

Chris Pile



A hanging chain takes the form of a catenary. The equation of the curve is

$$y \;=\; a\cosh\frac{x}{a},$$

where a is the value when $x = 0$. The length of the curve from the minimum point to some point $P = (x, y)$ is given by

$$C = \int_0^x \sqrt{1 + \left(\frac{dy}{dt}\right)^2}\,dt \;=\; a\sinh\frac{x}{a}.$$

At ground level the top of the pole, P, is $H + a$ above the x-axis; i.e. $a\cosh(x/a) = H + a$. Therefore

$$\cosh\frac{x}{a} \;=\; \frac{H+a}{a}, \qquad \sinh\frac{x}{a} \;=\; \frac{C}{a},$$

and using $\cos^2(x/a) - \sinh^2(x/a) = 1$,

$$\left(\frac{H+a}{a}\right)^2 - \left(\frac{C}{a}\right)^2 \;=\; 1, \quad \text{and hence} \quad a \;=\; \frac{C^2 - H^2}{2H}.$$

Also $\cosh(x/a) + \sinh(x/a) = e^{x/a}$. Hence

$$e^{x/a} = \frac{H+a}{a} + \frac{C}{a} = \frac{H+C}{a} + 1.$$

Therefore $x = a\ln((H+C)/a+1)$, where $2x$ is the distance between the poles.

Example: The poles supporting my washing line in the garden are $7'\,6''$ high. Lowering the line until it just touches the ground in the middle, I measured the length as $44'\,6''$. Therefore

$$H = 7.5, \quad C = 22.25, \quad a = 29.25, \quad x = 20.52.$$

Therefore the distance between the poles is $2x = 41$ feet plus about half an inch. This was confirmed to be correct within the expected accuracy of the experiment.

Problem 293.1 – Ten thousand prisoners

A large jail houses 10000 prisoners. One day the governor decides to have some fun with them. He sets up 10000 boxes and puts the prisoners' names in them, one-to-one. The prisoners are gathered together. They are permitted to discuss a strategy for what follows, but once the game starts further communication is forbidden. (How the warders perform the almost impossible task of enforcing this rule is of no concern to us here.)

The game starts. Each prisoner in turn enters the room containing the boxes and acts as follows.

(1) He chooses a box and opens it.

(2) If his name appears, he leaves the room and the names are returned to the same boxes ready for the next prisoner (if any).

(3) If his name does not appear, he repeats the procedure from (1), unless it is the 9950th box he has opened in which case the game stops and the governor orders some terrible punishment for *all* the prisoners.

The game comes to a quiet end with no penalty if the 10000th prisoner's name appears in a box that he has opened.

What strategy should the prisoners adopt, and what is the probability of avoiding the terrible fate?

Solution 287.4 – Two games

> The two-player game PickABead uses an even number N of interlocking beads, numbered 1 to N. The first player puts the beads together in some order to form a necklace. The second player breaks the necklace in one place, then removes a bead from one end. The players then alternate removing beads from an end of their choice, until there are none left. The winning player is the one with the higher sum of bead numbers. What strategy can the second player use to ensure that the first player never wins?
>
> A second game HighBead is identical to PickABead, except that each player must always remove the end bead with the higher number. What is the minimum N (M, say) such that, by suitable construction of the necklace, the first player can always win. Is there a simple way of devising a winning construction for any $N \geq M$, ideally one that gives the highest possible winning margin?

Roger Thompson

In M500 **290**, I gave a solution to the PickABead problem, but hadn't made much progress with HighBead. I have now found a crude, but very simple algorithm for ensuring that the first player always wins for $N > 26$.

Let $K = 0.8\sqrt{(N)}$, rounded down to the nearest integer. Construct a necklace made up of the pairs

$$\{N, N/2\}, \{N-1, N/2-1\}, \{N-2, N/2-2\}, \ldots, \{N/2+K+1, K+1\}.$$

Now insert $\{A, A+N/2\}$, $(A = 1, 2, \ldots, K)$ between these pairs, starting at the left hand end, and approximately evenly spaced. For example, for $N = 28$ (for which $K = 4$), this gives the necklace

1,15,28,14,27,13,**2,16,**26,12,25,11,24,10,**3,17,**23,9,22,8,**4,18,**21,7,20,6,19,5

with the inserts shown in bold. The highest score that the second player can get is 197, breaking the necklace just before 24 or 10, giving the first player a score of 209. For $N > 400$, the first player's score is at least twice the second player's.

I have not found a way to generate a definitively optimal solution. The following table shows the best obtained by using an evolutionary algorithm, namely swapping pairs at random, continuing further if this produces an equal or lower highest score, and backtracking if no improvement appears after many trials. The figures in brackets show the number of different break points that give the score shown. In all cases, the best necklaces show the same pattern as above, namely single pairs $\{a, b\}(a < b)$ inserted into longer sequences of pairs $\{a_i, b_i\}(a_i > b_i)$.

N	Score	N	Score	N	Score	N	Score
10	28(3)	12	37(8)	14	49(2)	16	61(8)
18	76(2)	20	91(8)	22	109(2)	24	127(8)
26	145(8)	28	166(2)	30	187(12)	32	210(8)
34	234(8)	36	260(12)	38	287(8)	40	315(2)
42	343(14)	44	372(4)	46	403(2)	48	435(16)
50	469(8)	52	504(14)	54	540(14)	56	576(4)
58	614(12)	60	654(10)	62	695(2)	64	737(18)
66	779(4)	68	823(4)	70	867(12)	72	913(16)
74	960(16)	76	1008(14)	78	1057(8)	80	1109(8)
82	1159(4)	84	1212(14)	86	1267(2)	88	1321(8)
90	1376(10)	92	1434(2)	94	1493(4)	96	1551(16)
98	1612(4)	100	1674(16)				

Problem 293.2 – Graphs with integer eigenvalues

Tony Forbes

For $i = 1, 2, \ldots,$ define a graph G_i as follows.

Let $n_i = (i-1)^2 + 1$. The vertices of G_i are $1, 2, \ldots, n_i$. For the edges, write down the pairs $\{a, b\}$, $1 \le a < b \le n_i$ in lexicographical order and remove the last $i(i-1)/2$ items from the list. The remaining $n_i(n_i-1)/2 - i(i-1)/2$ pairs form the edges of G_i.

Prove that the adjacency matrix of G_i has integer eigenvalues, or find a counter-example. For the first few, we have the following.

i	n_i	edges	eigenvalues
1	1	0	0
2	2	0	$0, 0$
3	5	7	$3, -2, -1, 0, 0$
4	10	39	$8, -3, -1, -1, -1, -1, -1, 0, 0, 0$
5	17	126	$15, -4, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, 0, 0, 0, 0$
6	26	310	$24, -5, -1$ nineteen times$, 0, 0, 0, 0, 0$

Solution 241.5 – Diagonal elements

Let a, b, c and d be integers. Draw up an $\infty \times \infty$ table with a, b, c and d in the top left corner, as shown.

a	c	$\ldots$
b	d	$\ldots$
$\ldots$	$\ldots$	$\ldots$

Then fill the rest of the table according to the rules: (i) if three consecutive rows in the same column contain x, y and z in that order, then $z = x + y$; (ii) if three consecutive columns in the same row contain x, y and z in that order, then $z = x + y$. Obtain a formula for the diagonal elements of the table.

In particular, if $a = b = c = 0$ and $d = 1$, you should get the squares of the Fibonacci numbers: 0, 1, 1, 4, 9, 25, 64, 169, 441, 1156, 3025, 7921, ...

Peter Fletcher

Either by brute force or by using Maple, we can find the first ten diagonals of the table as follows.

n	n^{th} term
0	a
1	d
2	$a + (b + c) + d$
3	$a + 2(b + c) + 4d$
4	$4a + 6(b + c) + 9d$
5	$9a + 15(b + c) + 25d$
6	$25a + 40(b + c) + 64d$
7	$64a + 104(b + c) + 169d$
8	$169a + 273(b + c) + 441d$
9	$441a + 714(b + c) + 1156d$

The coefficients of a may be written

$$1,\ 0,\ 1,\ 1,\ 2^2,\ 3^2,\ 5^2,\ 8^2,\ 13^2,\ 21^2,$$

and those of $(b + c)$ may be written

$$0,\ 0,\ 1,\ 2,\ 2 \cdot 3,\ 3 \cdot 5,\ 5 \cdot 8,\ 8 \cdot 13,\ 13 \cdot 21,\ 21 \cdot 34.$$

The coefficients of d are the same as those of a, except that they are offset by one place to the left and the last one is 34^2.

The presence of Fibonacci numbers is clear. If $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$, then the first 10 Fibonacci numbers, F_0 to F_9, are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34. Then clearly we can write diagonal n, the $(n+1)^{\text{th}}$ diagonal (with a being the first one, 0), for $n \ge 1$, as

$$F_{n-1}^2 a + F_{n-1} F_n (b+c) + F_n^2 d.$$

To show that this works, the 3^{rd} diagonal, with $n = 2$, is

$$F_1^2 a + F_1 F_2 (b+c) + F_2^2 d \;=\; a + (b+c) + d$$

and the 10^{th}, with $n = 9$, is

$$\begin{aligned} F_8^2 a + F_8 F_9 (b+c) + F_9^2 \;&=\; 21^2 a + 21 \cdot 34 (b+c) + 34^2 d \\ &=\; 441a + 714(b+c) + 1156d. \end{aligned}$$

If $a = b = c = 0$ and $d = 1$, then obviously we have a sequence of squares of Fibonacci numbers: the coefficients of d are 0, 1, 1, 4, 9, 25, 64, 169, 441, 1156, ...

Problem 293.3 – Binomial coefficients cubed

Let $A_{j,k}$ denote the coefficient of $x^j y^k$ in $\big(1 + y(1+x)^2(1+xy)\big)^n$. Show that

$$A_{n,n} \;=\; \sum_{k=0}^{n} \binom{n}{k}^3.$$

This came from a book of test papers for university scholarship candidates, and whilst I was setting it up for M500 my eyes drifted to another problem on the same page. It looked interesting; so I put it in M500 as Problem 293.5 on page 18.

Problem 293.4 – Triangular numbers

Let $T_n = n(n+1)/2$, the nth triangular number. Show that

$$(T_n - T_{n-1})^2 \;=\; T_n + T_{n-1} \;=\; n^2$$

and hence that $\sum_{n=1}^{N} n^3 = T_N^2$.

Solution 273.1 – Hair

When you visit your hairdresser within 30 days of your previous appointment the cost is £30. Thereafter she adds a premium of $30\,d^{5/4}$ pence, where d is the number of further days you delay your next appointment. For example, if you leave it for 42 days, the cost will be £36.70035…, which we assume she will round to £36.70. Obviously, in her view the surcharge is justified as compensation for additional work created by excessive hair growth.

If you get your hair cut at regular finite intervals to maintain a neat and tidy appearance at minimum cost, how often should you go?

Peter Fletcher

The daily cost of haircuts in pence is

$$f(d) \;=\; \frac{3000+30d^{5/4}}{30+d}$$

where $d \geq 0$. We want to minimize $f(d)$, which we can do by solving $f'(d)=0$.

$$\begin{aligned} f'(d) \;&=\; \frac{(30+d)\cdot 30\cdot \frac{5}{4}d^{1/4} - \left(3000+30d^{5/4}\right)}{(30+d)^2} \\ &=\; \frac{30}{4}\left(\frac{(30+d)\cdot 5d^{1/4} - 4\left(100+d^{5/4}\right)}{(30+d)^2}\right) \\ &=\; \frac{15}{2}\left(\frac{d^{5/4}+150d^{1/4}-400}{(30+d)^2}\right). \end{aligned}$$

Now $f'(d)=0$ also when

$$g(d) \;=\; d^{5/4}+150d^{1/4}-400=0.$$

We can solve this equation using Newton's method, for which we shall need

$$g'(d) \;=\; \frac{5}{4}d^{1/4}+\frac{150}{4}d^{-3/4} \;=\; \frac{5}{4}\left(d^{1/4}+30d^{-3/4}\right).$$

Then

$$d_{n+1}=d_n-\frac{g\left(d_n\right)}{g'\left(d_n\right)}$$

and starting from $d_1 = 30$,

$$\begin{aligned} d_2 &= 26.3659210234, \\ d_3 &= 26.4243906216, \\ d_4 &= 26.4244089016, \\ d_5 &= 26.4244089016. \end{aligned}$$

The hairdresser will deal in whole days only: if we round d down and round f to the nearest penny, we find $f(26) = 85$. However, we also get 85 for any d between 20 and 35.

What this means is that the daily cost of a haircut is 85p if the interval between haircuts is anything between $30 + 20 = 50$ and $30 + 35 = 65$ days.

The hairdresser would presumably not work on Sundays, so a haircut every 56 or 63 days would satisfy the requirements of regularity and minimum cost.

Solution 217.4 – n^2

Where n is a positive integer, prove that $(n+1)^n - 1$ is divisible by n^2.

Peter Fletcher

We have

$$(1+n)^n = \sum_{k=0}^{n} \binom{n}{k} n^k.$$

Now since

$$\sum_{k=1}^{n} \binom{n}{k} n^k = \binom{n}{1} n + \sum_{k=2}^{n} \binom{n}{k} n^k = n^2 + \sum_{k=2}^{n} \binom{n}{k} n^k$$

is clearly divisible by n^2, it follows that

$$(1+n)^n - 1 = \binom{n}{0} + \sum_{k=1}^{n} \binom{n}{k} n^k - 1 = \sum_{k=1}^{n} \binom{n}{k} n^k$$

is also divisible by n^2.

Solution 289.4 – Squares

Find all solutions in positive integers d and n of

$$n^2 \equiv n \pmod{10^d}.$$

Or, if you prefer, find all numbers which are the last d digits of their squares.

Ted Gore

Let $n^2 = a10^d + n$, so that $n(n-1) = a10^d$. It's not easy to find solutions manually so I used a computer program to generate some solutions by finding pairs n and $n-1$ such that $n(n-1)$ is divisible by 10^d. The following tables give results for odd n and even n respectively.

d	n	a
1	5	2
2	25	6
3	625	390
5	90625	82128
6	890625	793212
7	2890625	835571
8	12890625	1661682

d	n	a
1	6	3
2	76	57
3	376	141
4	9376	8790
6	109376	11963
7	7109376	5054322
8	87109376	75880433

Example: Taking n even, for $d = 7$ we get $n^2 = 50543227109376$.

Each row in the odd table can be used to generate the next row. I give two examples.

In the row for $d = 5$ take the least significant digit of a and affix it to the front of the n value. This gives the n value for $d = 6$.

In the row for $d = 3$ the least significant digit of a is a zero, so take the last two digits and affix them to the front of the n value. This gives the n value for $d = 5$.

Let $k = 1$ for $d = 5$ and 2 for $d = 3$. Then $n_{d+k} = (a_d \bmod 10^k)10^d + n_d$. A slightly more complicated result applies for even n. Thus k is set as before and $n_{d+k} = [10^k - (a_d \bmod 10^k)]10^d + n_d$.

We need a proof that the rules will always generate another valid number. Let N be generated from n by the rules above and let $a = b10^k + c$, where $c = a \bmod 10^k$.

Taking the odd case we have $N = c10^d + n$ so that $N^2 = c^2 10^{2d} + n^2 + 2nc10^2$. Taking $n^2 = (b10^k + c)10^d + n$ we arrive at

$$N^2 = [c^2 10^{2d} + 2nc10^d + b10^{d+k}] + N.$$

The part in square brackets has at least $d + k$ trailing zeros which will not effect the $d + k$ digits of N.

For the even case, let $f = 10^k - c$. Then $N = f10^d + n$ and after some manipulation we get

$$N^2 = [f^2 10^{2d} + 2(n-1)f10^d + (b+1)10^{d+k}] + N.$$

Once again the part in square brackets has sufficient trailing zeros to avoid affecting the digits of N.

Solution 289.3 – Continued fractions

Prove the following continued fraction formulæ:

$$\frac{e+1}{e-1} = [2; 6, 10, 14, 18, \ldots], \qquad \frac{e^2+1}{e^2-1} = [1; 3, 5, 7, 9, \ldots].$$

Is there a nice expression that has continued fraction [1; 2, 3, 4, ...]?

Peter Fletcher

We have

$$\frac{e+1}{e-1} \approx 2.163\,953\,413\,739 = \frac{2\,163\,953\,413\,739}{1\,000\,000\,000\,000}.$$

Wikipedia (`https://en.wikipedia.org/wiki/Continued_fraction`) gives a straightforward tabular method of finding continuous fractions, which we can follow:

Step	Real number	Integer part	Fractional part	Note	Reciprocal
1	$\frac{2\,163\,953\,413\,739}{1\,000\,000\,000\,000}$	2	$\frac{163\,953\,413\,739}{1\,000\,000\,000\,000}$		$\frac{1\,000\,000\,000\,000}{163\,953\,413\,739}$
2	$\frac{1\,000\,000\,000\,000}{163\,953\,413\,739}$	6	$\frac{99\,293\,556\,595}{1\,000\,000\,000\,000}$	(i)	$\frac{1\,000\,000\,000\,000}{99\,293\,556\,595}$
3	$\frac{1\,000\,000\,000\,000}{99\,293\,556\,595}$	10	$\frac{71\,146\,953\,461}{1\,000\,000\,000\,000}$	(ii)	$\frac{1\,000\,000\,000\,000}{71\,146\,953\,461}$
4	$\frac{1\,000\,000\,000\,000}{71\,146\,953\,461}$	14	$\frac{55\,415\,606\,069}{1\,000\,000\,000\,000}$	(iii)	$\frac{1\,000\,000\,000\,000}{55\,415\,606\,069}$
5	$\frac{1\,000\,000\,000\,000}{55\,415\,606\,069}$	18	$\frac{45\,458\,146\,842}{1\,000\,000\,000\,000}$	(iv)	$\frac{1\,000\,000\,000\,000}{45\,458\,146\,842}$

Notes: (i) $$\frac{1\,000\,000\,000\,000}{163\,953\,413\,739} = 6.099\,293\,556\,595;$$

(ii) $$\frac{1\,000\,000\,000\,000}{99\,293\,556\,595} = 10.071\,146\,953\,461;$$

(iii) $$\frac{1\,000\,000\,000\,000}{71\,146\,953\,461} = 14.055\,415\,606\,069;$$

(iv) $$\frac{1\,000\,000\,000\,000}{55\,415\,606\,069} = 18.045\,458\,146\,842.$$

The last fraction in the above table evaluates to

$$\frac{1\,000\,000\,000\,000}{45\,458\,146\,842} = 21.998\,257\,066\,566,$$

but per Maple,

$$\frac{e+1}{e-1} = [2; 6, 10, 14, 18, 22, 26, \ldots, 106, 110, 114, \ldots],$$

suggesting that the sequence might continue *ad infinitum*.

Similarly,

$$\frac{e^2+1}{e^2-1} \approx 1.313\,035\,285\,499 = \frac{1\,313\,035\,285\,499}{1\,000\,000\,000\,000}.$$

Using the same tabular method again:

Step	Real number	Integer part	Fractional part	Note	Reciprocal
1	$\frac{1\,313\,035\,285\,499}{1\,000\,000\,000\,000}$	1	$\frac{313\,035\,285\,499}{1\,000\,000\,000\,000}$		$\frac{1\,000\,000\,000\,000}{313\,035\,285\,499}$
2	$\frac{1\,000\,000\,000\,000}{313\,035\,285\,499}$	3	$\frac{194\,528\,049\,469}{1\,000\,000\,000\,000}$	(v)	$\frac{1\,000\,000\,000\,000}{194\,528\,049\,469}$
3	$\frac{1\,000\,000\,000\,000}{194\,528\,049\,469}$	5	$\frac{140\,646\,825\,636}{1\,000\,000\,000\,000}$	(vi)	$\frac{1\,000\,000\,000\,000}{140\,646\,825\,636}$
4	$\frac{1\,000\,000\,000\,000}{140\,646\,825\,636}$	7	$\frac{110\,007\,605\,775}{1\,000\,000\,000\,000}$	(vii)	$\frac{1\,000\,000\,000\,000}{110\,007\,605\,775}$
5	$\frac{1\,000\,000\,000\,000}{110\,007\,605\,775}$	9	$\frac{90\,280\,557\,922}{1\,000\,000\,000\,000}$	(viii)	$\frac{1\,000\,000\,000\,000}{90\,280\,557\,922}$

Notes (v) $$\frac{1\,000\,000\,000\,000}{313\,035\,285\,499} = 3.194\,528\,049\,469;$$

$$\text{(vi)} \qquad \frac{1\,000\,000\,000\,000}{194\,528\,049\,469} = 5.140\,646\,825\,636;$$

$$\text{(vii)} \qquad \frac{1\,000\,000\,000\,000}{140\,646\,825\,636} = 7.110\,007\,605\,775;$$

$$\text{(viii)} \qquad \frac{1\,000\,000\,000\,000}{110\,007\,605\,775} = 9.090\,280\,557\,922.$$

The last fraction in the above table evaluates to

$$\frac{1\,000\,000\,000\,000}{90\,280\,557\,922} = 11.076\,581\,968\,667$$

and per Maple,

$$\frac{e^2+1}{e^2-1} = [1; 3, 5, 7, 9, 11, 13, \ldots, 53, 55, 57, \ldots]$$

again suggesting that the sequence might continue *ad infinitum*.

For the continuous sum, $[1; 2, 3, 4, \ldots]$, if we start at $8 + 1/9$, we find

$$8 + \frac{1}{9} = \frac{73}{9}; \quad 7 + \frac{9}{73} = \frac{520}{73}; \quad 6 + \frac{73}{520} = \frac{3\,193}{520}; \quad 5 + \frac{520}{3\,193} = \frac{16\,485}{3\,193};$$

$$4 + \frac{3\,193}{16\,485} = \frac{69\,133}{16\,485}; \quad 3 + \frac{16\,485}{69\,133} = \frac{223\,884}{69\,133}; \quad 2 + \frac{691\,33}{223\,884} = \frac{516\,901}{223\,884};$$

and

$$1 + \frac{223\,884}{516\,901} = \frac{740\,785}{516\,901} \approx 1.433\,127\,426\,722.$$

There is no obvious 'nice' expression which evaluates to this number, but one website that knows otherwise is 'The On-line Encyclopedia of Integer Sequences' `https://oeis.org/A060997`: apparently

$$\frac{I_0(2)}{I_1(2)} = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!n!}}{\displaystyle\sum_{n=0}^{\infty} \frac{n}{n!n!}} = \frac{1}{[0; 1, 2, 3, \ldots]} = [1; 2, 3, 4, \ldots],$$

where $I_\alpha(x)$ is the modified Bessel function of the first kind, e.g. `http://mathworld.wolfram.com/ModifiedBesselFunctionoftheFirstKind.html`.

Letters

The Brachistochrone Problem

Dear Eddie,

Many thanks for M500 **291**. Interesting article on the Brachistochrone Problem. Do you think that Bernoulli recognised Newton's authorship of the anonymous solution by his mathematical method, the style of his Latin, or (a dead giveaway, really) the fact that the letter came from England and there was only one man in England at that time who could have solved the problem?

Bernoulli didn't say 'The lion is known by his footprint'. He said tanquam ex ungue leonem, 'as if a lion [could be recognised] from its claw'. This was not an original phrase, as it had already been used by Laevinius Torrentius in 1587 to refer to identifying an author by his style. But it's a lot older than that, and Greek rather than Latin, ἐξ ὄνυχος τὸν λέοντα. The lyric poet Alcaeus of Mytilene (*c.* 625–20 – *c.* 580 BC) used it to claim that the famous Greek sculptor Phidias, on being shown a lion's claw, could make a statue of a lion perfectly in proportion to it.

Much later, in the early 19th century, ex ungue leonem became a catchphrase of the new science of comparative anatomy after Cuvier identified a fossil found in a gypsum mine in Montmartre as an early opossum simply by looking at its jawbone, then scraped away the gypsum to reveal two pelvic bones peculiar to marsupials to prove his assertion.

Ralph Hancock

Jeremy,

I was a 'B Student' (B0xxxxxx) in 1972.

The M500 entry by Sebastian Hayes immediately reminded me of the first experiment for the first TMA we had on S100 in which we had to verify Galileo's Circle Chord Theorem by rolling and timing a ball down a flat surface. So we were rolling it down the chord of a circle, as opposed to swinging it through the arc of a circle on the end of a pendulum.

I discovered over the years since then that very few people seem to have heard of Galileo's Circle Chord Theorem, which I would sum up as follows.

> A weight constrained (on a pendulum) to falling down the arc of a circle, does not keep constant time as the amplitude of the swing changes, but a weight constrained to fall down any chord

of a circle would keep constant time irrespective of the length of the chord.

Unfortunately, when hitting the sharp angle at the bottom, it would lose energy and would not roll up the other way very satisfactorily, so a long pendulum with a relatively short swing is probably the best approximation to a chord in a practical clock.

Colin Davies

Solution 245.4 – GCSE question

Compute

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}+\sqrt{k-1}} \quad \text{and} \quad \sum_{k=1}^{n} \frac{(-1)^k}{\sqrt{k}-\sqrt{k-1}}.$$

Peter Fletcher

The first sum is

$$\begin{aligned} S_1 = \sum_{k=1}^{n} \frac{1}{\sqrt{k}+\sqrt{k-1}} &= \sum_{k=1}^{n} \frac{\sqrt{k}-\sqrt{k-1}}{\left(\sqrt{k}+\sqrt{k-1}\right)\left(\sqrt{k}-\sqrt{k-1}\right)} \\ &= \sum_{k=1}^{n} \left(\sqrt{k}-\sqrt{k-1}\right) \end{aligned}$$

and similarly the second sum is

$$S_2 = \sum_{k=1}^{n} \frac{(-1)^k}{\sqrt{k}-\sqrt{k-1}} = \sum_{k=1}^{n} (-1)^k \left(\sqrt{k}+\sqrt{k-1}\right).$$

Writing out the first few and last two terms in S_1, we get

$$1-0, \quad \sqrt{2}-1, \quad \sqrt{3}-\sqrt{2}, \quad \ldots, \quad \sqrt{n-1}-\sqrt{n-2}, \quad \sqrt{n}-\sqrt{n-1}.$$

Adding these terms together, it is clear that most cancel out and we are left with $S_1 = \sqrt{n}$.

Doing the same with S_2, we find

$$-1-0, \ \sqrt{2}+1, \ -\sqrt{3}-\sqrt{2}, \quad \ldots, \ \pm\sqrt{n-1}\pm\sqrt{n-2}, \ \mp\sqrt{n}\mp\sqrt{n-1}.$$

We can see that most terms cancel again, but this time the last two terms depend on whether n is odd or even.

If n is odd, $S_2 = -\sqrt{n}$ and if n is even, $S_2 = \sqrt{n}$.

Solution 284.1 – Squares

> Show that if a positive integer N can be expressed as $N = a^2 + kb^2 = c^2 + kd^2$ with a, b, c, d, k integers, $a \neq c$ and $k \geq 2$, then N must be composite.

Peter Fletcher

Since

$$a^2 + kb^2 \;=\; c^2 + kd^2$$

it follows that

$$k\left(b^2 - d^2\right) \;=\; c^2 - a^2 \quad \text{and} \quad k \;=\; \frac{c^2 - a^2}{b^2 - d^2}.$$

Therefore

$$N \;=\; a^2 + \left(\frac{c^2 - a^2}{b^2 - d^2}\right) b^2 \;=\; \frac{a^2\left(b^2 - d^2\right) + b^2\left(c^2 - a^2\right)}{b^2 - d^2}$$

$$=\; \frac{b^2c^2 - a^2d^2}{b^2 - d^2} \;=\; \frac{(bc - ad)(bc + ad)}{(b - d)(b + d)}$$

and N is composite.

Problem 293.5 – Equilateral triangle

An equilateral triangle is inscribed in the ellipse

$$b^2x^2 + a^2y^2 \;=\; a^2b^2.$$

If its vertices are at (x_1, y_1), (x_2, y_2), (x_3, y_3) and its centroid is at (x, y), show that

$$3xy \;=\; x_1y_1 + x_2y_2 + x_3y_3.$$

This is quoted from *Mathematical Problem Papers*, 3rd edition, compiled and arranged by the Rev. E. M. Radford, M.A. (Cambridge, 1923). There is something here that I (TF) possibly don't understand. If you choose to submit a solution, perhaps you can explain why the ellipse is there.

A trillion swatches

Jeremy Humphries

I was listening to an episode of 'Behind the Scenes' on BBC Radio 4 recently (4 Dec 2019 actually), which was about the fashion industry, particularly concerning the undesirable environmental and social impact of cheap, throwaway clothes. They were interviewing a fashion designer and company executive, Amy Powney, who resists making clothes that damage people directly (slave-like labour, etc.) or indirectly (compromising the ability of the planet to support human life). So good for her, and may her efforts make an impact of the desirable kind.

However, I was given pause by something she said in the interview. "You have to go to Premiere Vision, that happens twice a year in Paris. I can only describe it as an aeroplane hangar. And I wouldn't even like to guess how many swatches are in that room—it's in the trillions."

Can this be true? I began to wonder how much space a trillion (10^{12}) of something needs. I looked round the M500 office for a suitable candidate and spotted a ream of A4 paper. That's 500 sheets, and it's a pack measuring roughly $20 \times 30 \times 5$ in centimetres. So it's 3000 cc, which means 6 cc per sheet. A trillion sheets is therefore 6 trillion cc. Since a cubic metre is a million cc, we see that a trillion sheets of A4 paper will take up 6 million cubic metres.

That would be a cube of side 182 metres, or 200 yards, packed solid. I don't really know what a fabric swatch in this context looks like, but I don't doubt that it's more voluminous than a piece of A4 paper.

If the fabric swatch had the volume of, say, only two sheets of A4 paper, which is likely an underestimate, then you could just about stack a trillion of them—12 million cubic metres—inside the largest building in the world. That's the Boeing factory at Everett, Washington, which weighs in, or rather volumes in, at a bit over 13 million cubic metres. But of course if you wanted to give the folks space to examine the merchandise, and do their deals, and take refreshment and so on, you would need a building the size of many Boeing factories to facilitate that. You'd still struggle at 100 times the Boeing factory size, I reckon. And if they had such a structure in Paris, then the Guinness World Records people would have noticed it.

We conclude therefore that while there may be a lot of fabric swatches at Premiere Vision in Paris, Ms Powney's guess of 'trillions' is more than somewhat on the optimistic side.

Problem 293.6 – Sums of powers of digits

Let n be a positive integer. Let n^{S} denote the number you get by sorting the decimal digits of n, and for positive integers n, k, define

$$D_k(n) \;=\; \sum_{d,\ d \text{ runs through the digits of } n} d^k.$$

Show that for any positive integers n and k,

$$n = n^{\mathrm{S}} = (D_k(n))^{\mathrm{S}} \quad \text{if and only if} \quad D_k(D_k(n)) = D_k(n).$$

Or find a counter-example.

For instance, $93084^{\mathrm{S}} = 3489$ and you can easily verify that $D_5(93084) = 93084$. To check the claim in this particular case, we have

$$3489 \;=\; 3489^{\mathrm{S}} \;=\; (D_5(3489))^{\mathrm{S}},$$

and this is consistent with

$$D_5(D_5(3489)) \;=\; D_5(3489) \;=\; 93084.$$

A billion kilowatts

Tony Forbes

Jeremy's trillion swatches (page 19) reminded me to enquire a little more closely into *High Hopes*, performed by Doris Day.†

The song is concerned with the activities of an ant and a ram, both intent on causing trouble. The ant tries to relocate a rubber tree and the ram wants to destroy a hydroelectric power station by punching a hole in the dam. Eventually the ram is successful and, as Doris sings, *Oops there goes a billion kilowatt dam!*

A billion kilowatt dam? Assuming 90 per cent efficiency, dropping

$$\frac{1000000000000\,\mathrm{J}}{0.9 \times (108\,\mathrm{m}) \times \left(9.81\,\mathrm{m/s^2}\right)} \;\approx\; 1050000 \text{ tonnes}$$

of water through 108 meters will generate a trillion joules. For a billion kilowatts, you would want to do that every *second*. Maybe you can set this up by damming the Zambezi at the top of Victoria Falls (height 108 m) and siting the turbines at the bottom. Unfortunately it won't work; 1.05 million tonnes per second is more than 960 times the river's average flow rate [`https://en.wikipedia.org/wiki/Victoria_Falls`]. Any other suggestions?

†You can hear it during the credits at the end of *Antz*.

Problem 293.7 – Numbers that do not contain E

Let $E(N)$ denote the number of positive integers $n \le N$ such that n does not have the letter E in its usual English representation. For example, $E(10) = 3$ (the relevant numbers being TWO, FOUR, SIX), and hopefully you will verify that $E(100) = E(1000) = 19$, $E(10000) = 79$ and $E(100000) = 399$.

Compute $\liminf_{N\to\infty} E(N)/N$ and $\limsup_{N\to\infty} E(N)/N$.

You can of course try this with other letters. With F, for example, there appears to be a nice pattern: $F(10) = 8$, $F(100) = 64$, $F(1000) = 512$, $F(10000) = 4096$ and $F(100000) = 32768$. And for the letter Z, the pattern is even nicer.

Problem 293.8 – Roots

Let a, b, c, d denote the roots of the quartic $x^4 - x^3 - 4x^2 + 4x + 1$. Determine the quartic that has roots $a^2 - 2$, $b^2 - 2$, $c^2 - 2$, $d^2 - 2$.

Problem 293.9 – Sin 105 degrees

What (if anything) is wrong with this argument? We have

$$\sin 45^\circ \;=\; \frac{1}{\sqrt{2}} \qquad \text{and} \qquad \sin 60^\circ \;=\; \frac{\sqrt{3}}{2}.$$

Hence

$$\sin 105^\circ \;=\; \sin 45^\circ + \sin 60^\circ \;=\; \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \;=\; \frac{1+\sqrt{3}}{2\sqrt{2}}.$$

Things you can't buy in shops – III

Following on from the lists that we printed in M500 **278** and M500 **289**, here are a few more useful, everyday items you might consider enquiring about when your peaceful browsing in a shop is interrupted by an enthusiastic salesperson:

Lavatory paper, soap, hand sanitizer, pasta, detergent, breakfast cereal, canned fish, cough medicine, canned grapefruit, other canned fruit, Spam, corned beef, other canned meat, aspirin, nappies, vitamin pills, bottled water, bread, butter, jam, honey, marmalade, eggs, lemon curd, flour, peanut butter, washing powder, rice, curry sauce, pasta sauce, sunflower oil, tomato ketchup, noodles, potatoes, ginger, raspberries, strawberries, blackberries, sausages, organic carrots, olive oil, tomato puree, fresh pork chops, packet soup, canned soup, fresh chicken, duck, table salt, cooking salt, antibacterial wipes, chickpeas, canned chicken curry, lemon juice, sugar, tea, potato crisps, yeast, facial tissues, paracetamol, nuts, kitchen tissues, baked beans, canned vegetables, a clinical thermometer, tampons, hair shampoo, mustard powder, lavatory cleaner, long-life milk, dried beans, lentils, baby food, disinfectant, olives, floor cleaner, diet coke, bleach, pet food, diet supplements, and last but not least, a Covid-19 antibody test kit.

Contents — M500 293 – April 2020

04.00

Front cover A graph with integer eigenvalues.