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The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

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Getting dressed again

Robin Marks

In M500 **199**, ADF asks for a proof that $S(a,b) = (a+b)!/(a!b!) = S(b,a)$, where $S(a,b)$ is the number of ways of splicing a sequence of b things into a sequence of a things.

Suppose we have b objects of type B and a objects of type A. In the specific example given in M500 **199**, let {left sock, right sock, left shoe, right shoe} be a set of type A objects. Let {bra, panties, dress} be a set of type B objects. If we write A and B to denote the types, we can write down each ordered sequence of the $a+b$ objects, for example $AAAABBB$, $AAABABB$.

Altogether there are $\binom{a+b}{b} = \dfrac{(a+b)!}{a!\,b!}$ different sequences of these objects. Let us define a function $P(a,b)$ as follows:

$$P(a,b) \;=\; \begin{cases} \dfrac{(a+b)!}{a!\,b!} & \text{if} \quad a,b \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The letter P stands for Pascal; when $a \geq 0$ and $b \geq 0$ the values of $P(a,b)$ make up Pascal's triangle. For example the 1 at the top of the triangle is $P(0,0)$.

Assuming a and b are non-negative integers, the expression $(a+b)!/(a!\,b!)$ can be broken into two parts as follows:

$$P(a,b) = \frac{(a+b)!}{a!\,b!} = (a+b)\frac{(a+b-1))!}{a!\,b!} = a\frac{(a+b-1)!}{a!\,b!} + b\frac{(a+b-1)!}{a!\,b!}.$$

Note that the first term disappears if $a=0$ and the second term disappears if $b=0$. If neither term disappears, we get

$$P(a,b) \;=\; \frac{(a+b-1)!}{(a-1)!\,b!} + \frac{(a+b-1))!}{a!\,(b-1)!} \;=\; P(a-1,b) + P(a,b-1).$$

For example, when $a=4$ and $b=3$ we get

$$P(4,3) = \frac{7!}{4!\,3!} = 35 = P(3,3) + P(4,2) = \frac{6!}{3!\,3!} + \frac{6!}{4!\,2!} = 20 + 15.$$

Furthermore, $P(a-1,b)$ and $P(a,b-1)$ can each in turn be decomposed into two parts. The process of decomposition of $P(a,b)$ continues thus:

$$\begin{aligned}
P(a,b) &= P(a-1,b)+P(a,b-1) \\
&= P(a-2,b)+2P(a-1,b-1)+P(a,b-2) \\
&= P(a-3,b)+3P(a-2,b-1)+3P(a-1,b-2)+P(a,b-3) \\
&= P(a-4,b)+4P(a-3,b-1)+6P(a-2,b-2) \\
&\quad + 4P(a-1,b-3)+P(a,b-4)
\end{aligned}$$

and so on. After $a+b$ decomposition steps we will have an expression which contains $P(a-a,b-b)$ in one of its terms. We cannot go any further because $P(0,0)$ cannot be decomposed. Note that the coefficient of each term can easily be calculated from the negative numbers appearing inside the brackets of the function P. For example, the coefficient of $P(a-3,b-1)$ is $P(3,1)=4$ and the coefficient of $P(a-2,b-2)$ is $P(2,2)=6$. Hence the equations above can be written more succinctly as

$$P(a,b)=\sum_{k=0}^{1}P(k,1-k)P(a-k,b-(1-k))=\sum_{k=0}^{2}P(k,2-k)P(a-k,b-(2-k))$$

and so on. This can be written even more concisely as

$$P(a,b)=\sum_{k=0}^{m}P(k,m-k)P(a-k,b-(m-k)),\quad 0\le m\le a+b. \tag{1}$$

If we extend the summation limits of (1) a little we can get an attractive pattern. For example when $a=4$ and $b=3$ we get the $4+3+1=8$ rows shown below; the top row is generated when $m=0$, the bottom row when $m=a+b$.

0 1 **0** 7 **0** 21 **1** 35 **0** 35 **0** 21 **0** 7 **0** 1

0 1 **0** 6 **1** 15 **1** 20 **0** 15 **0** 6 **0** 1

0 1 **1** 5 **2** 10 **1** 10 **0** 5 **0** 1

1 1 **3** 4 **3** 6 **1** 4 **0** 1

1 0 **4** 1 **6** 3 **4** 3 **1** 1

1 0 **5** 0 **10** 1 **10** 2 **5** 1 **1** 0

1 0 **6** 0 **15** 0 **20** 1 **15** 1 **6** 0 **1** 0

1 0 **7** 0 **21** 0 **35** 0 **35** 1 **21** 0 **7** 0 **1** 0

Each number in bold should be multiplied by the number on its right, then each row summed. The positive bold numbers form Pascal's triangle. The positive light-faced numbers form an inverted Pascal's triangle. From the top row downwards we get equal sums 35, 15+20, 5+2·10+10, 1+3·4+3·6+4.

To interpret this look, for example, at the numbers on the third row, $5+2\cdot 10+10$. The first pair of numbers is the number of ways of arranging AA, $P(2,0)$, times the number of ways of arranging $AABBB$, $P(2,3)$. The second pair of numbers is the number of ways of arranging AB, $P(1,1)$, times the number of ways of arranging $AAABB$, $P(3,2)$. The third pair of numbers is the number of ways of arranging BB, $P(0,2)$, times the number of ways of arranging $AAAAB$, $P(4,1)$. If we add these up we get the total number of sequences of any two items together with the other five items; that is, we get the total number of sequences of all seven items.

We don't really want those zero values. Each zero occurs when P has a negative argument. To prevent the first argument in $P(a-k, b-(m-k))$ being negative, we need $k \leq a$. To prevent the second argument being negative we need $k \leq m-b$. We therefore change the summation limits of (1):

$$P(a,b) \;=\; \sum_{k=m-b}^{a} P(k, m-k)P(a-k, b-(m-k)), \tag{2}$$

provided $0 \leq m \leq a+b$. This is our general result. To agree with ADF's formula, we want the lower limit of the summation to be 1, so we need $1 = m-b$; hence $m = b+1$. Substituting $b+1$ for m in (2) we get

$$\begin{aligned} P(a,b) \;&=\; \sum_{k=1}^{a} P(k, b+1-k)P(a-k, k-1) \\ &=\; \sum_{k=1}^{a} \binom{b+1}{k}\binom{a-1}{k-1} \;=\; \sum_{k=1}^{a} \binom{a-1}{k-1}\binom{b+1}{k}. \end{aligned}$$

This is the expression for $S(b,a)$ derived by ADF. In summary, we have shown that

$$S(b,a) = P(a,b) = \frac{(a+b)!}{a!\,b!}.$$

Now we swap a and b in (1):

$$P(b,a) \;=\; \sum_{k=m-a}^{b} P(k, m-k)P(b-k, a-(m-k)), \tag{3}$$

provided $0 \leq m \leq a+b$. We want the lower limit of the summation to be 1, so we need $1 = m-a$; hence $m = a+1$. Substituting $a+1$ for m in (3) we get

$$\begin{aligned} P(b,a) &= \sum_{k=1}^{b} P(k,a+1-k)P(b-k,k-1) \\ &= \sum_{k=1}^{a} \binom{a+1}{k}\binom{b-1}{k-1} = \sum_{k=1}^{a} \binom{b-1}{k-1}\binom{a+1}{k}, \end{aligned}$$

the expression for $S(a,b)$ derived by ADF. This completes our task;

$$S(b,a) = P(a,b) = \frac{(a+b)!}{a!\,b!} = P(b,a) = S(a,b),$$

as required.

Now let us look at original problem in which there were eight items. Suppose we have c objects of type C, b objects of type B, and a objects of type A. Altogether there are $(a+b+c)!/(a!\,b!\,c!)$ different sequences of these objects. Let us define a function $P(a,b,c)$ as follows:

$$P(a,b,c) = \begin{cases} \dfrac{(a+b+c)!}{a!\,b!\,c!} & \text{if} \quad a,b,c \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

When $a \geq 0$, $b \geq 0$ and $c \geq 0$, the values of $P(a,b,c)$ make up 'Pascal's tetrahedron'. For example, the 1 at the top of the tetrahedron is $P(0,0,0)$. Also, eight levels further down at the base of the tetrahedron, we find $P(4,3,1) = P(4,1,3) = P(3,4,1) = P(3,1,4) = P(1,4,3) = P(1,3,4) = 280$.

We can decompose $P(a,b,c) = \dfrac{(a+b+c)!}{a!\,b!\,c!}$ into three parts:

$$\begin{aligned} P(a,b,c) &= \frac{(a+b+c-1)!}{(a-1)!\,b!\,c!} + \frac{(a+b+c-1)!}{a!\,(b-1)!\,c!} + \frac{(a+b+c-1)!}{a!\,b!\,(c-1)!} \\ &= P(a-1,b,c) + P(a,b-1,c) + P(a,b,c-1). \end{aligned}$$

Continuing this process by decomposing each term into three parts:

$$\begin{aligned}
P(a,b,c) &= P(a-1,b,c) + P(a,b-1,c) + P(a,b,c-1) \\
&= P(a-2,b,c) + 2P(a-1,b-1,c) + P(a,b-2,c) \\
&\quad + 2P(a,b-1,c-1) + P(a,b,c-2) + 2P(a-1,b,c-1) \\
&= 6P(a-1,b-1,c-1) + P(a-3,b,c) + 3P(a-2,b-1,c) \\
&\quad + 3P(a-1,b-2,c) + P(a,b-3,c) + 3P(a,b-2,c-1) \\
&\quad + 3P(a,b-1,c-2) + P(a,b,c-3) + 3P(a-1,b,c-2) \\
&\quad + 3P(a-2,b,c-1)
\end{aligned}$$

and so on for further decompositions, until we get an expression which contains $P(0,0,0)$. This is the 1 at the summit of Pascal's pyramid, which cannot be decomposed further.

Again the coefficient of each term can easily be calculated from the negative numbers appearing inside the brackets of the function P. For example, the coefficient of $P(a-1,b-1,c-1)$ is $6 = P(1,1,1)$. The coefficient of $P(a,b-1,c-2)$ is $3 = P(0,1,2)$. Hence the equations above can be written more succinctly as

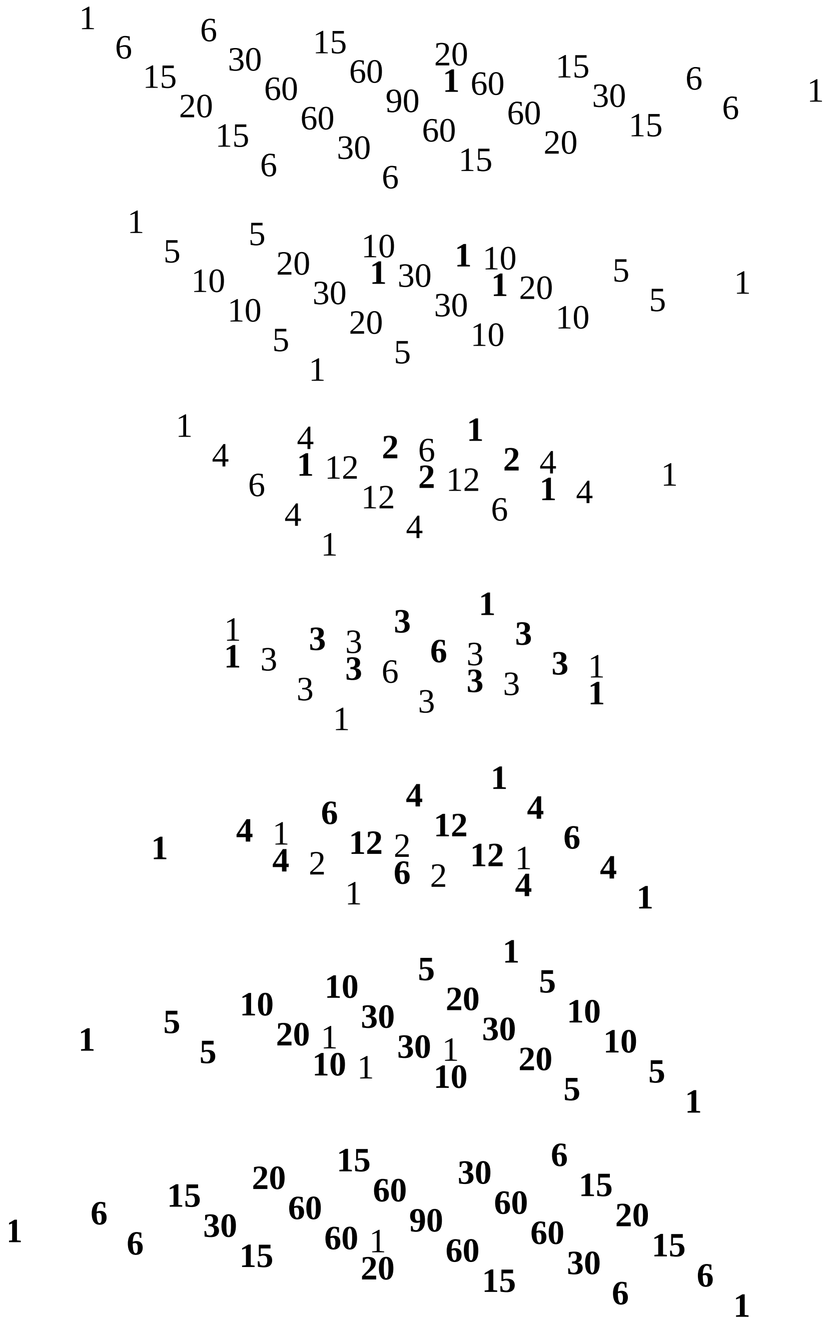
$$\begin{aligned}
P(a,b,c) &= \sum_{k=0}^{1}\sum_{j=0}^{1} P(j,k,1-j-k)P(a-j,b-k,c-(1-k)) \\
&= \sum_{k=0}^{2}\sum_{j=0}^{2} P(j,k,2-j-k)P(a-j,b-k,c-(2-k))
\end{aligned}$$

and so on. This can be written even more concisely as

$$P(a,b,c) = \sum_{k=0}^{m}\sum_{j=0}^{m} P(j,k,m-j-k)P(a-j,b-k,c-(m-j-k)) \quad (4)$$

for $0 \le m \le a+b+c$.

If we alter the summation limits of (4) a little we can get an attractive pattern. For example when $a=3$, $b=2$ and $c=1$ we get the $3+2+1+1=7$ horizontal groups of numbers shown on the next page; the top group is generated when $m=0$, the bottom when $m=a+b+c$.



Pairs of adjacent numbers (bold on the left and light on the right) should be multiplied, then summed in each group. If there is only a single number rather than a pair, the unseen number is a zero. The bold numbers form Pascal's tetrahedron. The light numbers form an inverted Pascal's tetrahedron. From the top group of numbers downwards we have equal sums 60, $10+20+30$, $12+2\cdot 6+2\cdot 4+2\cdot 12+4$, $3+3\cdot 3+3\cdot 6+6\cdot 3+3\cdot 3+3$.

To interpret this look, for example, at the numbers in the fourth group, $3+3\cdot 3+3\cdot 6+6\cdot 3+3\cdot 3+3$. The first pair of numbers gives the number of ways of arranging AAA, $P(3,0,0)$, times the number of ways of arranging BBC, $P(0,2,1)$. The second pair of numbers is the number of ways of arranging AAC, $P(2,0,1)$, times the number of ways of arranging ABB, $P(1,2,0)$. The third pair of numbers is the number of ways of arranging AAB, $P(2,1,0)$, times the number of ways of arranging ABC, $P(1,1,1)$, and so on. If we add these up we get the total number of sequences of any three items together with the other three items; that is, we get the total number of sequences of all six items.

Let us alter the summation limits in (4) so that the arguments of $P(j,k,m-j-k)$ are always non-negative. We need $m-j-k \geq 0$; hence $m-k \geq j$. This gives

$$P(a,b,c) = \sum_{k=0}^{m} \sum_{j=0}^{m-k} P(j,k,m-j-k) P(a-j, b-k, c-(m-j-k)), \quad (5)$$

$0 \leq m \leq a+b+c$. To check this, we choose $a=4$, $b=3$, $c=1$, as in ADF's example, and $m=2$ for example. This gives

$$\begin{aligned} P(4,3,1) &= \sum_{k=0}^{2} \sum_{j=0}^{2-k} P(j,k,2-j-k) P(4-j, 3-k, 1-(2-j-k)) \\ &= P(0,0,2)P(4,3,-1) + P(1,0,1)P(3,3,0) \\ &\quad + P(2,0,0)P(2,3,1) + P(0,1,1)P(4,2,0) \\ &\quad + P(1,1,0)P(3,2,1) + P(0,2,0)P(4,1,1) \\ &= 280. \end{aligned}$$

This is the result found by ADF.

If we had a further group of items, we would need to define a function $P(a,b,c,d) = (a+b+c+d)!/(a!\,b!\,c!\,d!)$ for $a,b,c,d \geq 0$, $P(a,b,c,d) = 0$ otherwise. Then $P(a,b,c,d)$ for different values of a,b,c,d can be arranged in a 4-dimensional 'Pascal's simplex'.

Differentiation over non-commutative algebras

Dennis Morris

Let us differentiate $y = x^2$ from first principles without assuming multiplicative commutativity. This expression is meaningful because, even in non-commutative algebra, $x \cdot x = x \cdot x$;

$$\begin{aligned} y + \delta y &= (x + \delta x)^2 \\ &= (x + \delta x) \cdot (x + \delta x) \\ &= x \cdot x + x \cdot \delta x + \delta x \cdot x + \delta x \cdot \delta x \\ &= x^2 + x \cdot \delta x + \delta x \cdot x + \delta x \cdot \delta x. \end{aligned}$$

Ignoring the higher-order term, this leads to

$$\delta y = x \cdot \delta x + \delta x \cdot x.$$

We now need to divide by δx. However, division is not defined over any algebra. What is defined, and we call division, is multiplication by the inverse. In our case, multiplication is not commutative, and we can either post-multiply by the inverse of δx or we can pre-multiply by it. Let us do both.

Post multiplication:

$$\delta y \cdot \frac{1}{\delta x} = x \cdot \delta x \cdot \frac{1}{\delta x} + \delta x \cdot x \cdot \frac{1}{\delta x}.$$

By the definition of inverse,

$$\delta x \cdot \frac{1}{\delta x} = 1 = \frac{1}{\delta x} \cdot \delta x,$$

where we use 1 to denote the multiplicative identity. So

$$\delta y \cdot \frac{1}{\delta x} = x + \delta x \cdot x \cdot \frac{1}{\delta x}.$$

Now, what $\delta x \cdot x \cdot 1/\delta x$ actually means depends upon the particular rules of multiplication within our non-commutative algebra. For demonstration purposes, we pick the frequently found rule: $a \cdot b = -b \cdot a$. In this case we have

$$\delta x \cdot x \cdot \frac{1}{\delta x} = -x \cdot \delta x \cdot \frac{1}{\delta x} = -x,$$

leading to

$$\delta y \cdot \frac{1}{\delta x} = x - x = 0.$$

Pre-multiplication:

$$\frac{1}{\delta x}\cdot\delta y \;=\; \frac{1}{\delta x}\cdot x\cdot\delta x+\frac{1}{\delta x}\cdot\delta x\cdot x.$$

So

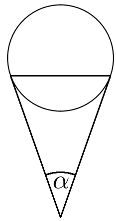
$$\frac{1}{\delta x}\cdot\delta y \;=\; \frac{1}{\delta x}\cdot x\cdot\delta x+x.$$

For demonstration purposes, we continue with the rule $a\cdot b=-b\cdot a$, giving

$$\frac{1}{\delta x}\cdot\delta y \;=\; x-x \;=\; 0.$$

In this demonstration case, $\dfrac{1}{\delta x}\cdot\delta y=\delta y\cdot\dfrac{1}{\delta x}$, but it need not be that way. The multiplication rules might be more complicated (matrices for example) or the functions might be more complicated. In the case of the quaternions, which form a non-commutative algebra, in general $\dfrac{1}{\delta x}\cdot\delta y\neq\delta y\cdot\dfrac{1}{\delta x}$.

Problem 205.1 – Sphere in a cone



Given a finite cone of fixed height 1 m and apex angle α, a sphere is inserted as far as possible into its open end. What is the maximum volume of that part of the sphere which is inside the cone?

The maximum must exist. Clearly, the volume tends to zero as the sphere's volume tends to either zero or infinity. The answer may be of interest to people who eat ice-cream.

Problem 205.2 – Ants

Let k and n, $1\leq k<n$, be a fixed integers. There are n ants, $A_0,A_1,\ldots,A_{n-1}$, situated at the vertices of a regular n-gon of side 1 m, arranged in anticlockwise order. At time 0 the ants start walking at speed 1 m/s, ant A_i always heading in the direction of ant $A_{i+k \bmod n}$, $i=0,1,\ldots,n-1$. When do they meet?

Problem 205.3 – Reciprocals

Show that if $1\leq m<n$, the following expression cannot be an integer:

$$\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{n}.$$

Scepticism in mathematics

Sebastian Hayes

Dick Boardman (Letters, M500 **197**, p. 27) expresses surprise that I am 'unhappy about being a sceptical non-believing mathematician' since this is 'an excellent thing to be'. I have found it a very inconvenient thing to be. I have already recounted the anecdote of the Oxford mathematician who declined to meet me socially after discovering my views on infinity and, quite frankly, I don't think I would have been welcome at the M500 Winter Workshop on the transfinite if I'd raised questions of actuality.

Obviously I have no problems, social, physical or otherwise, with textbook solutions to problems in statics which assume pulleys to be frictionless and cords to be made of silk. Why not? Because in such cases the discrepancy between the simplified mathematical model and reality is openly avowed and is moreover quantifiable: we can work out within what sort of limits the mathematical solution will be valid. Even here, however, there is danger—there are plenty of people, perhaps, God forbid, even some engineering students, who believe that the pull of gravity on a body *only* operates at the (strictly non-existent) point we baptize the centre of mass. (A molecule situated at this point is not subject to any greater pull from the Earth than any other molecule composing the body.)

The situation is more serious when we come to calculus. As I have repeatedly pointed out in this magazine calculus does not, and cannot, faithfully represent the true state of affairs in the real world since no known physical processes are actually continuous. But such is the prestige of calculus that the implication is that it's somehow the real world that is in the wrong—and this of course is precisely what mathematical Platonism is frank enough to state. I do not know how many times I have come across in books and newspapers the 'argument' that 'modern mathematics shows that it is possible for a body to pass through an "infinite" number of positions in a finite lapse of time.' This is not even *mathematically* true: all that analysis shows is that certain indefinitely extendable series have a finite *limit*.

Of course, in the bulk of modern mathematics there is no problem about a possible discrepancy between the simplifying mathematical treatment and the underlying reality because there simply is no underlying reality, not even in theory. This would be quite acceptable if what we had before us really was just a mass of 'meaningless marks on paper manipulated according to fixed rules' (Hilbert): higher mathematics would then be a superior sort of embroidery and we would at least know where we were. But in practice not even the most outlandish branches of modern mathematics manage to get along without numerical and geometrical notions which they subsequently negate, trample underfoot and contemptuously cast aside. The Banach-

Tarski two sphere theorem, for example, asserts that it is possible to dissect a sphere in such a way that the fragments can be reassembled to form two spheres each the size of the original one. This theorem has been proved.

The great classical figures Newton, Leibnitz, Descartes *et al.* were all realists and were at once mathematicians, observational scientists and philosophers, since the approaches of all these three are necessary if we want to determine what's real and what isn't. What we need today is a mathematics which is humble enough to recognize its origins in the outside world and, conversely, a physics which dares to attack head on 'philosophic' issues beyond its specific remit. The unspoken message in science courses—and OU ones are no exception—is: 'If the student can get out the right answers, who cares about the underlying rationale?' There is, it is true, a fair bit of semi-philosophical speculation in areas such as cosmology but while the mathematical level here is incredibly high (too high) the philosophical level is low—I found the best-selling *A Short History of Time* vague, trivial and dull. And when you do get scientists, such as David Bohm and Rupert Sheldrake, tackling competently issues of importance they are simply ignored because they are outside the mainstream and to boot not sufficiently mathematical.

I believe some sort of 'return to basics' is in order. As an example of how a feature of 'elementary' mathematics reaches into physics, metaphysics and maybe even technology, take unique prime factorization. To judge from most books on number theory this is a purely technical issue of no great import outside mathematics (except perhaps for devising codes). This is quite wrong, however—unique prime factorization is about the most basic and important physical law we have. I have seriously wondered whether there could be a universe where multiple prime factorization was the norm, or at least permissible. It would be a very strange place indeed. It would be a world where, having a certain amount of tennis balls, say $\oslash\oslash\oslash\oslash\oslash\oslash\oslash\oslash\oslash\oslash\oslash\oslash\oslash\oslash\oslash$, I put them into bins $\bigcup\bigcup\bigcup$ with equal quantities in each bin. I now tip the whole lot out onto the floor. In both this world and the multiple prime factorization world I could bin up the tennis balls differently by placing $\oslash\oslash\oslash$ tennis balls into each of $\bigcup\bigcup\bigcup\bigcup\bigcup$ bins. But in the multiple prime factorization world I would be able to bin up the same original mass of tennis balls using a *different* amount of bins i.e. neither $\bigcup\bigcup\bigcup$ nor $\bigcup\bigcup\bigcup\bigcup\bigcup$. Is this conceivable? I'm not sure. In such a world aggregates would not have specific numerical properties such as 'baggableness' fixed in advance, or not all of them would. They would acquire these numerical properties when placed in certain numerical situations, but lose them in others, just as massive bodies behave differently on the Earth and on the Moon.

Alternatively, if we reject this, we have reached a pretty amazing conclusion, namely that features such as unique prime factorization are invariant throughout universes which in other respects may be quite different physically from ours, with different values for g and so on. This doesn't make unique prime factorization transcendent—it remains a *physical* law—but it does make it far more basic than all the other physical laws we're familiar with, gravity, Boyle's Law, &c., &c. No wonder Pythagoras worshipped numbers.

Still, maybe one *should* take scepticism as far as this. The world around us exhibits an astonishingly high degree of what one might call 'numerical rationality', I don't think anyone would quarrel with that. There is the intriguing question of where this rationality came from in the first place since by all accounts there is no intelligent Being running the show—but let's leave that issue aside for the moment. What has occurred to me is that this numerical rationality might be to some degree malleable much in the way that the number (*sic*) of protons in an atom is malleable (isotopes). After all, lots of people today view the universe as in some sense a giant computer and it *is* possible to interfere with the workings of a computer. This line of thought is worrying though: it is perhaps not wholly inconceivable that an 'anti-rationality virus' could be invented, inflicting chosen physical targets with dysfunctional incoherence. It would be a very deadly sort of virus, the numerico/rational equivalent of AIDS. At this very moment a group of clean-shaven curly-haired Harvard number theorists are perhaps sitting round a table discussing the development of a 'rationality bomb' with Donald Rumsfeld and Dick Cheney. I can't say I like the sound of that too much.

Erratum. In my article 'The Fibonacci series and the golden section' [M500 **201**, p. 12], the passage at the bottom of the page just before the diagram should have read 'During month n, the ratio of the productive pairs (marked in black) to the unproductive pairs is very nearly equal to the ratio of the total number of pairs to the productive pairs', and not '... is equal to ...'. — SH

Numerical coincidences

ADF

Is there is anything other than coincidence at work in the following assortment of near-integer expressions?

$\exp\left((5/2)^{-2/5}\right) = 1.9999953\ldots$	$163/\log 163 = 31.9999987\ldots$
$\log\log(5^5)! = 10.000042\ldots$	$\log 41958! = 404666.99999918\ldots$
$\log(21\sqrt{10} + 82) = 4.999964\ldots$	$\pi\exp(-\pi^2/214) = 2.9999938\ldots$

Solution 201.1 – Continued fraction

Show that
$$\tan\theta = \cfrac{2}{\cot\dfrac{\theta}{2} - \cfrac{2}{\cot\dfrac{\theta}{4} - \cfrac{2}{\cot\dfrac{\theta}{8} - \dots}}}.$$

Basil Thompson

Use the tangent double-angle formula,
$$\tan\theta = \frac{2}{\cot\theta/2 - \tan\theta/2},$$
recursively expanding the expression $\tan(\theta/2^n)$ in the denominator.

This is an interesting continued fraction. It can be terminated at any n by replacing the nth denominator with $\cot(\theta/2^n) - \tan(\theta/2^n)$, or it can continue to infinity. There are an infinite number of finite continued fractions and one infinite continued fraction. Unique? Well, no. We also have
$$\sin\theta = \frac{2\tan\theta/2}{1+\tan^2\theta/2},\ \tanh\theta = \frac{2\tanh\theta/2}{1+\tanh^2\theta/2} \quad\text{and}\quad \sinh\theta = \frac{2\tanh\theta/2}{1-\tanh^2\theta/2},$$
which can be expressed as continued fractions in the same way as $\tan\theta$. But $\cos\theta$ and $\cosh\theta$ are not so easy. Thus
$$\cos\theta = \frac{\cot\theta/2}{\cot\theta/2+\tan\theta/2} - \frac{\tan\theta/2}{\cot\theta/2+\tan\theta/2},$$
$$\cosh\theta = \frac{\coth\theta/2}{\coth\theta/2-\tanh\theta/2} + \frac{\tanh\theta/2}{\coth\theta/2-\tanh\theta/2}$$
and the denominators are expanded as previously.

ADF — We also had contributions from Jim James, Sebastian Hayes, Peter Fletcher, David Porter, John Spencer and John Bull. However, Sebastian thinks that we have all brushed aside the difficult part. What happens at odd multiples of $\frac{1}{2}\pi$? Bearing in mind that $\tan\theta \to \pm\infty$ as θ tends to an odd multiple of $\frac{1}{2}\pi$ from below or above, is it obvious that the continued fraction behaves in the same way? Observe, for example, that if you put $\theta = \frac{25}{2}\pi$ and calculate the first few convergents, you get approximately
$$-2, -0.34314, 1.71106, -0.15411, -1.67990, -0.32063, 0.92187, 11.547,$$
$$427.87, 67757, 43958814, 114711068940, 1198977341053386, \dots.$$

Solution 201.2 – Sine series

Prove that

$$\theta \;=\; (\sin\theta)(\cos\theta) + \sum_{n=1}^{\infty} 2^n \sin\frac{\theta}{2^{n-1}}\sin^2\frac{\theta}{2^n}.$$

Jim James

For all $\alpha, \beta \in \mathbb{R}$,

$$\sin\alpha\sin^2\beta = \frac{\sin\alpha(1-\cos 2\beta)}{2} = \frac{2\sin\alpha - \sin(\alpha-2\beta) - \sin(\alpha+2\beta)}{4}.$$

So when $\alpha = \theta/2^{n-1}$, $\beta = \theta/2^n$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} 2^n \sin\frac{\theta}{2^{n-1}}\,\sin^2\frac{\theta}{2^n} \;&=\; \sum_{n=1}^{\infty}\frac{2^n}{4}\left(2\sin\frac{\theta}{2^{n-1}} - \sin\frac{\theta}{2^{n-2}}\right)\\ &=\; \sum_{n=1}^{\infty}\left(2^{n-1}\sin\frac{\theta}{2^{n-1}} - 2^{n-2}\sin\frac{\theta}{2^{n-2}}\right).\end{aligned}$$

The first few terms of this summation are as follows:

$$\begin{array}{llcll} n=1: & \sin\theta & - & (\sin 2\theta)/2 & \\ n=2: & 2\sin(\theta/2) & - & \sin\theta & \\ n=3: & 4\sin(\theta/4) & - & 2\sin(\theta/2) & \\ n=4: & 8\sin(\theta/8) & - & 4\sin(\theta/4) & \quad \text{etc.} \end{array}$$

The right-hand side of the given expression then reduces to

$$\begin{aligned}\sin\theta\cos\theta + \lim_{n\to\infty}\left(2^{n-1}\sin\frac{\theta}{2^{n-1}}\right) - \frac{\sin 2\theta}{2} \;&=\; \lim_{n\to\infty} 2^{n-1}\sin\frac{\theta}{2^{n-1}}\\ &=\; \theta\lim_{n\to\infty}\frac{2^{n-1}}{\theta}\sin\frac{\theta}{2^{n-1}}\\ &=\; \theta,\end{aligned}$$

as required.

Solved in a similar manner by Basil Thompson, Steve Moon, David Porter and John Bull.

The incompleteness of the hyperbolic Euler relations and the completion thereof

Dennis Morris

The Euler relations for the hyperbolic trigonometric functions are usually given as

$$\cosh\chi = \frac{e^{\chi}+e^{-\chi}}{2}, \qquad \sinh\chi = \frac{e^{\chi}-e^{-\chi}}{2}.$$

We consider

$$e^{\hat{r}\chi} = 1+\hat{r}\chi+\frac{\hat{r}^2\chi^2}{2!}+\frac{\hat{r}^3\chi^3}{3!}+\ldots,$$

where $\hat{r}=\sqrt{+1}\neq\pm1$. This leads to $e^{\hat{r}\chi}=\cosh\chi+\hat{r}\sinh\chi$. Similarly,

$$e^{-\hat{r}\chi} = 1-\hat{r}\chi+\frac{\hat{r}^2\chi^2}{2!}-\frac{\hat{r}^3\chi^3}{3!}+\ldots$$

leads to $e^{-\hat{r}\chi}=\cosh\chi-\hat{r}\sinh\chi$. Combining these results gives

$$\cosh\chi = \frac{e^{\hat{r}\chi}+e^{-\hat{r}\chi}}{2}, \qquad \sinh\chi = \frac{e^{\hat{r}\chi}-e^{-\hat{r}\chi}}{2}.$$

In the case that we allow $\hat{r}=\sqrt{+1}=\pm1$, these reduce to the usually given versions.

Marks

ADF

Browsing through old M500s, as I often do, I notice that one can easily extend the marking scheme described by **Ron Potkin** in M500 **194**, p. 28. This where a maths teacher gives her students an exam with 100 true-or-false questions; Arthur gets 100 correct, Ford gets 50 correct, Marvin gets 100 wrong. But the teacher awards full marks to Marvin as well as Arthur, and 0 to Ford (who obviously guessed).

The generalization is clear. Award +1 for each correct answer and −1 for each incorrect answer. Then take the absolute value of their sum.

Assuming the examination consists of questions with true/false answers, candidates who rely on pure guesswork and candidates who know most of the answers get treated appropriately. But the scheme also automatically rewards those persons who, as I did once, ignore the clear instructions to indicate the correct options, and mark what they perceive to be the wrong answers instead. Isn't this fair?

Solution 202.5 – Interesting equality

Find interesting equalities like

$$\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{19}\right) = \frac{4}{3}\sqrt{\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\left(1-\frac{1}{19^2}\right)}.$$

Ian Adamson

The right-hand side of the equality is some rational fraction (greater than unity) multiplied by a rational square root; so the problem is necessarily and sufficiently that of finding products of the form $(x^2-1)(y^2-1)\ldots$ which are integer squares.

Clearly and trivially $x = y$ is a solution; so let us seek $x < y < z < \ldots$.

One way of finding a few solutions is to list $m^2 - 1$ for $m > 1$ and note its prime divisors, writing $[q_1, q_2, \ldots]$, where $q_i \equiv a_i \pmod 2$, $m^2 - 1$ is some product of the form $p_1^{a_1} p_2^{a_2} \ldots$ and p_n is the nth prime. It is thus straightforward to find pairs, triads, etc. by inspection: seeing that every column totals a multiple of 2.

For example, $6^2 - 1 = [0,0,1,1]$, $8^2 - 1 = [0,0,0,1]$, $9^2 - 1 = [0,0,1]$; so (6, 8, 9) is a solution. Once we have two solutions we find a larger from their union and intersection.

Some solutions are $(2,7)$, $(3,17)$, $(2,3,5)$, $(2,4,9)$, $(6,8,9)$, $(2,3,4,9,17)$ and $(2,3,5,6,8,9)$. I have made no attempt to find them all.

ADF

Also we received this from **Mick Bromilow**:

> Try 3 and 17 with the factor 3/2; or 5, 7 and 17 with a factor, again, 3/2. I suspect there are many.

In fact, a simple computer search reveals similar equalities in great abundance—far too many to maintain a high level of interest. Thus it appears that Mick is suspiciously correct; so we shall now make the problem more difficult.

As **Ken Greatrix** suggests (page 22), rearrange the original equality to get

$$\left(\frac{7+1}{7-1}\right)\left(\frac{11+1}{11-1}\right)\left(\frac{19+1}{19-1}\right) = \left(\frac{4}{3}\right)^2.$$

Now generalize to

$$\left(\frac{p_1+1}{p_1-1}\right)\left(\frac{p_2+1}{p_2-1}\right)\cdots\left(\frac{p_n+1}{p_n-1}\right) \;=\; \alpha^k, \tag{1}$$

where $p_1, p_2, \ldots, p_n$ are distinct primes, α is a rational number and k is an integer greater than 1. Defining

$$\mathcal{P}(S) \;=\; \prod_{p \in S}^{n} \left(\frac{p+1}{p-1}\right),$$

we can write (1) more succinctly as

$$\mathcal{P}(\{p_1, p_2, \ldots, p_n\}) \;=\; \alpha^k.$$

For instance, with ten primes and $k = 9$ we have this expression:

$$\mathcal{P}(\{3, 7, 13, 19, 31, 41, 97, 199, 769, 4801\}) \;=\; \left(\frac{7}{6}\right)^9.$$

We ask: Is it possible to find a more interesting equality either by increasing the exponent or by reducing the number of primes?

Big Bang by Simon Singh (Fourth Estate)

Eddie Kent

You will know that Simon Singh wrote *Fermat's Last Theorem* and *The Code Book*, both best sellers. On each occasion he came to Aston University to talk to us about the book, and incidentally to sell copies of it.

He has now written *Big Bang*, and very generously agreed to come again. *Big Bang* is on sale at the moment at £20, but a paperback published by Harper Perennial at around £8 is due out. In any case Simon will be bringing some copies to Aston for those who can wait till September.

I shall be writing a review of *Big Bang* soon, but at the moment I am only a few of the 532 pages in (with glossary, bibliography and index). I can tell you that he starts with a few creation myths, then quickly gets on to the scientific method, in the course of which he tells you how to measure the distance to the moon from your back garden.

Every concept is explained simply and clearly, as is to be expected from Simon. He is a most excellent communicator. I sincerely urge that however you go about it, you must not go to your grave without owning a copy of *Big Bang; The Most Important Discovery of All Time and Why You Need to Know About it.* Because you do.

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196 revisited

Tony Forbes

Recall that problem where you start with a positive integer p, reverse its digits (in base 10) to get q, replace p with $p+q$, and repeat the process, stopping as soon as you reach a palindrome. For example, $155 \to 155+551 = 706 \to 1313 \to 4444$. Does the process always terminate? A possible counter-example is $p = 196$ [M500 **184**, p. 9].

Anyway, I thought it would be interesting (even if others might disagree [M500 **184**, p. 29]) to find large palindromes generated from small numbers. So here is a short list. Since the probability of a randomly chosen $2n$- or $(2n+1)$-digit number being a palindrome is 10^{-n}, I am puzzled by the great abundance of these things when $n \geq 29$. Perhaps the mechanics of the algorithm explains why.

palindrome	start p	steps
1225556542892363298245655521	1007393	53
1779924975996323699579429971	1903598	50
1784545987840363048789545487 1	10103975	48
3867966324268444862423669768 3	10049507	48
7146686399817060718993686641 7	1005351	53
8889320236747505747632023988 8	10801962	49
36779978965656556565698799776 3	10059868	52
49847736768988448898676377489 4	10089190	64
56872549858548778458589452786 5	10405948	53
135315537958736663785973551353 1	1048955	63
155320352459912421995425302355 1	10079918	54
664412218949755255794981221446 6	10019261	55
883445332484167476148423354438 8	147996	58
1475872457859888888958754278574 1	1003569	65
1552156138757988889757831651255 1	1998999	75
1896633685246796697642586336698 1	1007953	60
8918800674499127721994476008819 8	1010968	58
11672256985511894981155896522761 1	1006086	60
68204956946555012105556496594028 6	150296	64
465630567978445478745448797650365 64	1008595	69
688756568797089796979807978656578 86	10305983	66
1467444396014326533335623410693444764 1	1007601	80
3569548797677843358853348776797845965 3	10905963	71
6858637865565696499994696565568736858 6	7008899	82
79658988432496694564654966942348898569 7	1000689	78
5554587740837266745808622680854766273804778545 55	9008299	96

Higher and higher mathematics

Bob Newman

I remember reading in an undergraduate mathematical magazine called *Eureka* an article, purportedly reprinted from the *Proceedings of the Laotian Philosophical Society*, which proved many remarkable things, taking as its starting point the 'well-known result' that 'any quasitropic hedroid is diatonically sub-similar to itself.' Many earnest students read it all the way through without giving the slightest sign of either incomprehension or amusement. The article was, of course, complete gobbledygook from beginning to end, as is this poem.

This result's a real beauty, far too picturesque for prose.
I can put it in plain English—leave your text books on the shelf.
It's a theorem of importance greater than you might suppose:
Any quasitropic hedroid is sub-similar to itself.

There's an Element of Euclid's that was ranked quite unimportant
Till the great Jacobi thought a new condition to impose.
If you find this unexciting, I assure you that you oughtn't—
This result's a real beauty, far too picturesque for prose.

It was Gauss who christened hedroids, and defined their permutations,
Proved them stunted but well-ordered—as I later did myself.
Tarted up Jacobi's theory by diffracting both equations.
I can put it in plain English—leave your text books on the shelf.

Then MacLaurin squared the matrices and saw what they would yield.
How he thought of splining transcendental subgroups, heaven knows!
But he proved that every couple has its moment in a field.
It's a theorem of importance greater than you might suppose.

You see, Euclid's epimorphism's a set of measure zero,
Both imaginary and real, though this may harm your mental health!
Whence this wonderful result of Lobachevski—he's my hero—
Any quasitropic hedroid is sub-similar to itself.

It's an epoch-making breakthrough; it's the new Pons Asinorum.
Of ingenious applications you will find there is a wealth.
Here's a sure-fire opening line for any scientific forum:
'Any quasitropic hedroid is sub-similar to itself!'

Under the skin

Colin Davies

My article 'Under the skin' in M500 **197** drew a number of comments from M500 readers, but since then an article has appeared in the magazine *Nature* [1]. The questions I originally posed were as follows.

(1) If you take two British resident people at random (but exclude obvious relatives and recent immigrants), how far back in time would you have to go to be 99 percent certain that they have a common ancestor?

(2) If you are of apparent British ancestry, what is the probability of having any particular historical person as an ancestor?

(3) How closely is any particular person in Britain likely to be related to anybody else? (But again excluding close relatives and recent immigrants.)

The principal question posed by the *Nature* article is a variation of (1) and (2) above. To paraphrase it, they ask: 'In a population size n, how many generations back do you find the Most Recent Common Ancestor (MRCA) of everybody alive today?'

The authors give no mathematical argument in answering this, but simply state that: 'For a population size n, assuming random mating, probabilistic analysis [2] has proved that the number of generations back to the MRCA, T_n, has a distribution that is sharply concentrated around $\log_2 n$.' They then go on to show that if n is the population size and T_n is the number of generations to MRCA, $T_n/\log_2 n$ approaches 1 as n approaches infinity. From which it follows that in a population of 1 million, the MRCA would have lived about 20 generations ago (presumably because $\log_2$ of 1 million is about 20.)

As the number of one's ancestors doubles every generation back, this result does look highly plausible. Using my population table from M500 **197**, the population of Britain around 20 generations ago was around 3 million, which suggests that our MRCA lived 21 or 22 generations ago, around 1350. One would expect the most recent CA to live later than a randomly selected historical person chosen as a CA. This is what I did in my Q2 result, which shows that one needs to go back about 24 generations to find near certainty for such a person to be a CA.

This is supported by the *Nature* article, which then goes on to point out that 'as genealogical ancestry is traced back beyond the MRCA, a growing percentage of people in earlier generations are revealed to be CAs of the present-day population.' 'Tracing back further, there was a threshold U_n generations ago before which ancestry of the present-day population was an all or nothing affair. That is, each individual living at least U_n generations ago was either a CA of all today's humans, or an ancestor of no human alive today.' They call that time the IA point.

The authors then infer that $U_n/(1.77 \log_2 n)$ approaches 1 as n approaches infinity. They do not explain how the figure 1.77 has been derived, but give reference [2] again.

The bulk of the paper goes on to explain how, by using graph theory and computer simulations to explore probable human migration over the last few thousand years, they have calculated the date of the MRCA and the IA point for everyone alive today. 'With 5 percent of individuals migrating out of their home towns, 0.05 percent migrating out of their home country, the simulations produce a mean MRCA date of 1,415 BC, and an IA date of 5,353 BC.' But the authors think that the above assumptions are too conservative, and have tried the effect of a simulation based on what they regard as a more plausible set of assumptions. These give results suggesting a world MRCA date of 55 AD, and an IA date of 2158 BC. They end their paper: 'Our findings suggest a remarkable proposition: no matter the languages we speak or the colour of our skin, we share ancestors who planted rice on the banks of the Yangtze, who first domesticated horses on the steppes of the Ukraine, who hunted giant sloths in the forests of North and South America, and who laboured to build the Great Pyramid of Khufu.'

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[1] Douglas L. T. Rohde, Steve Olsen and Joseph T. Chang, Modelling the recent common ancestry of all living humans, *Nature* **431** (30 September 2004),

[2] J. T. Chang, Recent common ancestors of all present-day individuals, *Adv. Appl. Probab.* **31** (1999), 1002-1026, 1027-1038.

Answers to *What's next?* (i) 1, 2, 3, 4, 5, ...; (ii) 1, 3, 5, 7, 9, ...; (iii) 192, 384, 768, 1536, 3072, ...; (iv) 25, 28, 31, 34, 37, ...; (v) 107624, 109573, 132485, 138624, 159406, ...; (vi) 60, 61, 67, 71, 73, ...; (vii) 88, 142, 228, 367, 590, ... Source: Neil Sloane's *On-Line Encyclopedia of Integer Sequences* at www.research.att.com/~njas/sequences.

Letters to the Editor

M500 202

Tony,

Here are some comments concerning issue **202**

202.2 – Five spheres. Take the five Platonic solids and place a sphere at each vertex of such radius that these spheres are touching (the radius of these spheres would be half the length of an edge). The radius of the circumscribed sphere is available as an 'off-the-shelf' calculation, so subtract half the length of an edge from this radius to give the radius of the enclosed sphere. Then compare this with unity.

If I might correct Colin Davies (page 17). It's not exactly a West Indian song—it was performed by Lance Percival, who also had a hit with *Gossip Calypso* sometime in the early 60s. The story is about a young man who asks his father's advice about marriage and is told, 'That girl is your sister but your mama don't know' (on several occasions!). In the end he asks his mother ....

An Elliptic gardening problem (Donald Preece, page 24). It would seem that the ellipse could be extended symmetrically along each of its axes, and then the new ellipse completed by guesswork. If you then avoid inviting mathematicians for tea on the terrace, I'm sure no one would notice.

202.4 – Commas and brackets. (i) If $n > 0$, then the number of commas is $2^{n-1} - 1$; (ii) For $n \geq 0$, the number of brackets is 2^{n+1}. I worked this out long-hand, and offer these results without an actual proof—which I am sure would follow easily.

202.5 – Interesting equality. Take three integers (possibly primes) p, q, r. By noting that the RHS contains the 'difference of two squares', The given relationship can be transposed and then reduced to

$$\frac{(p+1)(q+1)(r+1)}{(p-1)(q-1)(r-1)} = \frac{A}{B},$$

where A, B are squares of integers. I have found that $(p, q, r) = (3, 5, 7)$ gives $A = 4$, $B = 1$.

Regards,

Ken Greatrix

Dates

28 August 888 was the last date expressible entirely with even numbers until 2 February 2000. Or, to put it another way, all the dates in between used an odd number somewhere.

John Reade

ADF writes — After thinking about it for a length of time I am too embarrassed to own up to, I concluded that they use digits which read the same upside down on a calculator using the traditional seven-bar display (1 doesn't count because it flips to the other side). Therefore a third and more precise answer is that they consist of only calculator-invertible even digits.

Health and safety

Dear Eddie and Tony,

I admire the new health warnings about the dangers of matches, knives and polluted doughnuts. However, I think there should also have been a warning about little bits of string (Problem 203.6), citing that well known study, Belloc, H., *Cautionary Tales*, London 1896, pp 17–20. Not to mention injuries caused by bouncing snooker balls (Problems 200.5, 203.7) — my uncle lost all his front teeth playing a game called billiard fives. As for cutting sheet metal into a heptagon (Problem 203.5), where one is exposed to the hideous dangers of sharp-pointed dividers, finger-crushing vices, lethally fanged hacksaws and horribly abrasive files, words fail me. Please also note that before inverting a cup, one should call a safety officer to certify that it is not full of burning petrol; and octonions should never, never be used in cooking.

Best wishes,

Ralph Hancock

Problem 205.4 – abc

For integers n, a and b, define

$$q(n) = \prod_{\substack{p|n \\ p \text{ prime}}} p \quad \text{and} \quad L(a,b) = \frac{\log(a+b)}{\log q(ab(a+b))}.$$

For example, $q(96) = 6$ and $L(96, 29) = \log(125)/\log(2 \cdot 3 \cdot 5 \cdot 29) \approx 0.713$.

Find triples of positive integers (a, b, c) for which $c = a+b$, $\gcd(a, b) = 1$, and $L(a, b)$ is as large as possible. Note that without the gcd condition the task is trivial; for instance, $L(2^k, 2^k) = k + 1$.

What is the next number?

Eddie Kent

This is a problem that keeps surfacing and ought to be laid to rest. In M500 **193** (August 2003) Sheldon Attridge mentioned that Wittgenstein poured scorn on the concept of there being a right answer to the question in the title or, indeed, to any question. And now in M500 **203** Chris Jones asks, What's next?

While not being prepared to hunt for sources, I do recollect the example used in this field by Polya, possibly following Wittgenstein. Find the next number in the sequence 1, 2, 4, 8, 16, ... The answer Polya gave is 31; something to do with joining dots placed on the circumference of a circle. But it can be shown that a satisfactory answer is 19.

How would you go about solving a problem of this kind? Suppose you are given 1, 2, 3, 4, ... and asked to find the next number. One way might be to plot them on a graph, letting n be the point (n, n) in the (x, y)-plane. Clearly the points fall on a straight line so it is logical to continue this line. Then an easy extrapolation takes you to number 5, which you can call 'the next number'. Try some more: 2, 4, 6, 8, ... leads to 10; -1, -3, -5, -7, ... well, you can do that. The method seems to work!

However, let the sequence be 1, 2, 4, 8, 16, ... Plotting these on the graph does not produce a straight line, frustratingly; but you might realize that taking the base-2 logarithms of the numbers, that is, $\log_2 1 = 0$, $\log_2 2 = 1$, $\log_2 4 = 2$, $\log_2 8 = 3$ and $\log_2 16 = 4$, does give numbers in a straight line. Another easy extrapolation furnishes the number 5 and the simple calculation 2^5 produces 32.

But now consider the polynomial

$$1 + \frac{7}{12}\,x + \frac{11}{24}\,x^2 - \frac{1}{12}\,x^3 + \frac{1}{24}\,x^4.$$

Evaluate this at $x = 0$ and $1 + 0 + 0 + 0 + 0 = 1$. For $x = 1$ we get $1 + 7/12 + 11/24 - 1/12 + 1/24 = 2$. I leave the rest of the calculations for the interested reader who will find that the sequence produced is thus: 1, 2, 4, 8, 16, 31.

Now 31 is not 32, and since we are looking for the 'correct' answer we have to decide which of the two to accept. Both are good, and follow from correct mathematics. They are both entirely logical. But for the first, one has to rely on the unlikely chance of stumbling on a method using logarithms. Whereas the second is simple and easy to use, and is obtained by a much more general method.

However, whatever answer suits our own personal preference, the fact remains that we have illustrated an important point. There is no 'correct answer', and so Wittgenstein (if it was he) was correct. Since unique correctness does not apply we might as well pick any number, so long as it can be justified.

Briefly, here is how it works. You are given p, q, r, ... and have to find the next number. Pick your favourite, say 19. Then your sequence is p, q, r, 19 and you need to validate this. Write down the polynomial

$$f(x) = ax^3 + bx^2 + cx + d$$

and evaluate it successively at the integers. Then

$$\begin{aligned} f(0) &= d = p, \\ f(1) &= a + b + c + d = q, \\ f(2) &= 8a + 4b + 2c + d = r, \\ f(3) &= 27a + 9b + 3c + d = 19. \end{aligned}$$

A neat little set of simultaneous equations. All you need to do is solve them to find values for a, b, c and d and your selection of 19 is completely vindicated.

Anyone who has wasted time on linear algebra should be able to do the same thing with very little effort, using successive differences. You merely need to know that the nth set of differences for an $(n-1)$th-degree polynomial are all zero. From here it is all downhill; the essence of mathematics.

ADF — Inspired by the Polya sequence mentioned above, I offer the following. *What's next?* (Answers elsewhere in the magazine.)

(i) $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, \ldots$

(ii) $2, 4, 6, 8, 10, 12, 14, 16, 18, \ldots$

(iii) $1, 2, 4, 8, 16, 32, 64, 128, \ldots$

(iv) $3, 6, 9, 12, 15, 18, 21, \ldots$

(v) $1, 10, 100, 1000, 10000, 100000, \ldots$

(vi) $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, \ldots$

(vii) $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

Hints: (i) the smallest number of terms to represent n as a sum of fourth powers; (ii) the sum of the (decimal) digits of n plus the product of the digits of n; (iii) the number of subsets of $\{1, 2, \ldots, n\}$ containing exactly one square; (iv) $\lfloor n\pi \rfloor$; (v) n and n^3 have the same digits; (vi) simple groups; (vii) breeding pairs of mortal rabbits; $s(n) = s(n-1) + s(n-2) - s(n-10)$.