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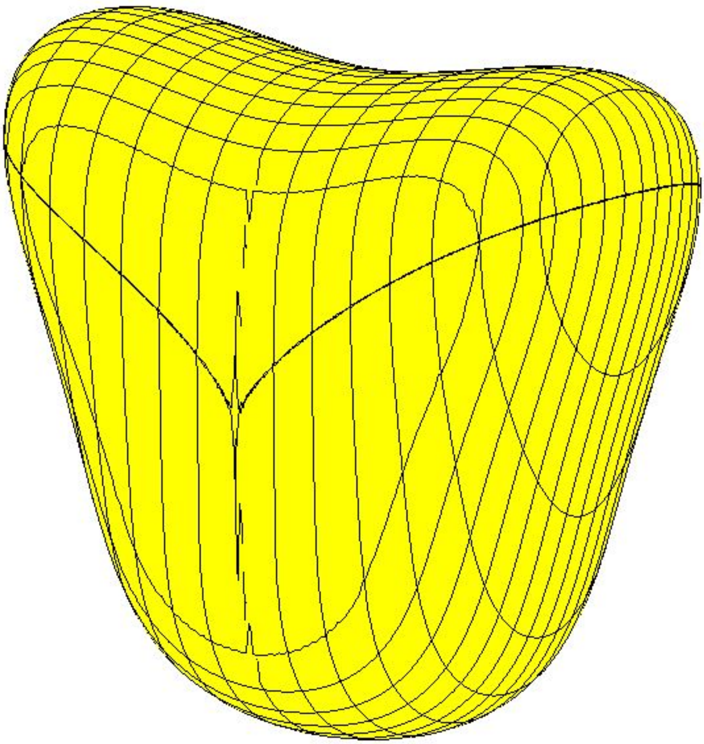
M500 294



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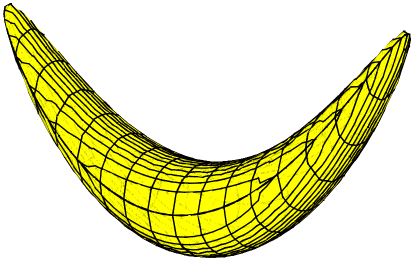
Solution 290.2 – Horn

This thing appeared on the front cover of M500 **21** ...



... accompanied by its defining formula,

$$64(z^2+y^2)+x^4-16zx^2+4x^2-256 \;=\; 0.$$

What's its volume?

Ted Gore

We have

$$\begin{aligned} 64(z^2+y^2)+x^4-16zx^2+4x^2-256 &= 0 \\ \Rightarrow\; 64\left(z-\frac{x^2}{8}\right)^2+64y^2 &= 256-4x^2 \\ \Rightarrow\; \left(z-\frac{x^2}{8}\right)^2+y^2 &= \frac{64-x^2}{16}. \end{aligned}$$

This is the equation of an ellipse which is a section through the horn at x.

The semi-major and semi-minor axes are both $\dfrac{\sqrt{64-x^2}}{4}$ so that the area of the ellipse is $\dfrac{\pi(64-x^2)}{16}$. The area becomes zero when $x=8$ so that the volume of half the horn is

$$\frac{\pi}{16}\int_0^8 (64-x^2)\,dx \;=\; \frac{64\pi}{3}$$

and the volume of the horn is $\dfrac{128\pi}{3} \approx 134.04$.

A rational approach to teaching irrationality

Ben Mestel

Back in those halcyon days when I was still a young and enthusiastic mathematics lecturer, I taught level-1 classes in the foundations of mathematics. I don't mean Zermelo–Fraenkel set theory, the axiom of choice and all that jazz, but rather the standard introduction to pure mathematics: integers, rational, real and complex numbers, sets and relations, types of proof etc. And one of the first things we covered was the irrationality of $\sqrt{2}$, the positive square root of 2.

This proof (by contradiction) is quite well known. Suppose, for a contradiction, that $\sqrt{2}$ is rational. Then there are integers $r, s > 0$ *with no common factors greater than* 1 *(because we divide them all out)*, such that $r/s = \sqrt{2}$. Then $r^2 = 2s^2$ and r^2 is even. But that means that r must be even because the square of an odd number is odd. So we can write $r = 2t$, where t is a positive integer. Then $4t^2 = r^2 = 2s^2$ so, dividing by 2, we get $s^2 = 2t^2$ and s^2 is even and so again s must be even and so has a factor 2. Hence we conclude that r and s have a common factor 2, which is a contradiction because r *and* s *have no common factors greater than* 1 *by assumption*.

For a budding mathematician, fresh at university, this isn't the easiest argument to understand, as became obvious when we received the solutions to the first exercise sheet, in which students were asked to prove the irrationality of $\sqrt{3}$. A few students started their proofs as follows: Suppose, for a contradiction, that $\sqrt{3}$ is rational. Then there are integers $r, s > 0$ with r and s, with no common factors greater than 1, such that $r/s = \sqrt{3}$. Then $r^2 = 3s^2$ and so r^2 is odd ... Ouch !

Of course, if all we want to do is find an example of an irrational number, then a much better choice is the positive number $\log_2 3$. Because if this is rational, then $\log_2 3 = r/s$ for positive integers r and s, then, by definition, $3 = 2^{\log_2 3} = 2^{r/s}$ so that $3^s = 2^r$, which is impossible for several reasons, the most obvious being that 3 divides the left-hand side but doesn't divide the right-hand side. More generally, we can consider $\log_m n$. Indeed, rationality of $\log_m n$ is readily seen to be equivalent to the condition $m^r = n^s$ for positive integers r and s, so a necessary (but, of course, not sufficient) condition for rationality is that m and n have a common set of prime divisors.

Returning to $\sqrt{2}$, I was convinced then, and am still now, that the solution to the student misconception was not to prove that $\sqrt{2}$ is irrational, but instead to prove the more general result that, as we shall explain below, *any rational algebraic integer is necessarily an integer.* From this result we can show, in one fell swoop, that $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, ... are all irrational (but happily not $\sqrt{4}$), and so demonstrate the power of generalization. Regrettably, I was unable to persuade my colleagues, even though several of them were researchers in algebraic number theory.

So, how does it go?

An *algebraic number* is a number x, real or complex, that is a root of a polynomial with integer coefficients, that is,

$$a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0 \;=\; 0, \tag{1}$$

where the degree $k \geq 1$ and a_k, a_{k-1}, ..., a_0 are all integers. Examples of algebraic numbers are integers, rational numbers, surds such as $\sqrt{2}$, i, roots of unity, but not transcendental numbers such as π and e. Now an algebraic number x is an *algebraic integer* if it is the root of a *monic* polynomial, that is one for which $a_k = 1$, so that

$$x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0 \;=\; 0. \tag{2}$$

It is important to note that $a_{k-1}, a_{k-2}, \ldots, a_0$ must be *integers*, not rational numbers.

Examples of algebraic integers are any ordinary integer, $i = \sqrt{-1}$ and, more generally the Gaussian integers $\{m + in : m, n \in \mathbb{Z}\}$. To see this, consider the Gaussian integer $z = m + in$. Then z satisfies the quadratic equation

$$z^2 - 2mz + (m^2 + n^2) \;=\; 0, \tag{3}$$

as some straightforward algebra will show. This is equation (2) with $k = 2$, $a_1 = -2m$ and $a_0 = m^2 + n^2$, both of which are integers, because m and n are. Other examples of algebraic integers are $\sqrt{2}$ and, surprisingly, the golden ratio $(\sqrt{5} + 1)/2$ (it is a root of the monic polynomial $x^2 - x - 1$). However, neither $1/2$ nor $\sqrt{5}/2$ are algebraic integers, as can be proved with a bit of work.[1]

So, what is the result that I wanted my colleagues to present to our level-1 students?

[1] To show that a number is not an algebraic integer we have to show that it is not the zero of any monic polynomial with integer coefficients. Since there are (countably) infinitely many of them, we can't check them all, but nevertheless it can be shown (using Theorem 1 and its corollaries) that neither $1/2$ nor $\sqrt{5}/2$ is an algebraic integer although both are algebraic numbers.

Theorem 1 *Every rational algebraic integer is an integer.*

This theorem is simple to state but is quite profound. It says that any algebraic integer that is also a rational number must also be an integer, so that, for example, $1/2$ is not an algebraic *integer* although it is a rational number and hence an algebraic *number*. The theorem is not *a priori* obvious, because, as we have remarked, numbers such as $(\sqrt{5}+1)/2$ are algebraic integers even though $\sqrt{5}/2$ and $1/2$ are not.

The proof of the theorem is conceptually no harder than the proof that $\sqrt{2}$ is irrational, although it is more involved. It goes like this.

Proof Let x be an algebraic integer which is also a rational number. Then we may find integers r and s with no common factors such that $s \geq 1$ and $x = r/s$. To show that x is an integer, we need to show that $s = 1$. So, to obtain a contradiction, we suppose that $s > 1$. Now, being an algebraic integer, x satisfies a monic polynomial equation

$$x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0 \;=\; 0, \tag{4}$$

where $k \geq 1$ and each coefficient $a_{k-1}, \ldots, a_0$ is an integer. Note that the coefficient of x^k is 1 since x is an algebraic *integer*. Substituting $x = r/s$, it follows that

$$\left(\frac{r}{s}\right)^k + a_{k-1}\left(\frac{r}{s}\right)^{k-1} + \cdots + a_1\left(\frac{r}{s}\right) + a_0 \;=\; 0, \tag{5}$$

so, clearing fractions by multiplying by s^k, we have the integer equation

$$r^k + a_{k-1}r^{k-1}s + \cdots + a_1 r s^{k-1} + a_0 s^k \;=\; 0, \tag{6}$$

which, on rearranging, gives

$$r^k \;=\; -\,a_{k-1}r^{k-1}s - \cdots - a_1 r s^{k-1} - a_0 s^k. \tag{7}$$

Now, we are assuming the integer s is greater than 1, so s has at least one prime factor $p > 1$. Then p divides s and hence every positive power of s. Then from equation (7), we see that p divides the right-hand side of the equation because p divides every term on the right-hand side. Hence p also divides the left-hand side of the equation and so p divides r^k. Because p is a prime number, we conclude that p divides r. But that means the prime $p > 1$ divides both r and s, which is impossible because r and s have no common factors greater than 1. From this contradiction, we deduce that $s = 1$ and so $x = r/s = r$, an integer. $\square$

As an immediate corollary we can prove that all square roots of integers that are not integers themselves are irrational.

Corollary 1 *Let m be a positive integer that is not a perfect square and let x be a square root of m, i.e., $x^2 = m$. Then x is irrational.*

Proof Suppose x is a rational number. Then, since x satisfies the equation $x^2 - m = 0$, x is an algebraic integer. From Theorem 1, we deduce that x is an integer. But since $m = x^2$ we then have m a perfect square, which is a contradiction. We deduce that x is not a rational number. □

Corollary 2 *Let $n > 1$ be a positive integer and let m be a positive integer that is not a perfect nth power, and let x be a real nth root of m, i.e., $x^n = m$. Then x is irrational.*

We leave the proof as an exercise for the reader.

The beauty of Theorem 1 is that it reduces the irrationality question to the solution of Diophantine (i.e. integer) equations in number theory. And, as we know, there are many number-theoretic problems that are simple to state but are hard to solve.

One example is Brocard's problem, which is to find positive integers m and n which satisfy the equation $m^2 = n! + 1$. At the time of writing, there are only three known solutions, namely $(m, n) = (5, 4)$, $(11, 5)$ and $(71, 7)$, and it is believed there are no others. So the problem of finding all positive integers n for which $\sqrt{n! + 1}$ is rational is still open.

On the other hand, this third corollary is established:

Corollary 3 *Let $n > 2$ be a positive integer, let y and z be positive integers, and let x be a real nth root of $y^n + z^n$. Then x is irrational.*

Wiles-ing away my time happily when I should have been doing other work, I have discovered a truly marvellous proof of this, which the margin on this page is too narrow to contain.

Problem 294.1 – Quartic

Show that $\tan^2(\pi t/16)$, $t = 1, 3, 5, 7$, are the roots of

$$x^4 - 28x^3 + 70x^2 - 28x + 1 \;=\; 0.$$

Solution 256.4 – Two septics

Two equations to solve. Like the octics in M500 **254** and the bisextics in M500 **256** they have only a few terms. As usual, exact expressions for the seven roots are required in each case:

$$x^7 + 7x^4 + 16 \;=\; 0, \qquad\qquad x^7 + x^6 + 3x^5 + 3 \;=\; 0.$$

Finding these things can be quite fun, too. At M500 we would be interested if you discover a 3- or 4-term equation of high degree that admits an exact solution but does not split into small polynomials with integer coefficients.

Peter Fletcher

The first septic

In Maple, we can use the `factor` function to find that

$$x^7 + 7x^4 + 16 \;=\; (x+2)(x^6 - 2x^5 + 4x^4 - x^3 + 2x^2 - 4x + 8)$$

so the first root is obviously -2.

If we now use the `factor` function with the `complex` option on the sextic, we find its roots as

$$\begin{aligned} r_2 &= -0.8061\ldots - (0.8583\ldots)\mathrm{i}, & r_5 &= 0.8459\ldots + (1.6905\ldots)\mathrm{i},\\ r_3 &= -0.8061\ldots + (0.8583\ldots)\mathrm{i}, & r_6 &= 0.9601\ldots - (0.8322\ldots)\mathrm{i},\\ r_4 &= 0.8459\ldots - (1.6905\ldots)\mathrm{i}, & r_7 &= 0.9601\ldots + (0.8322\ldots)\mathrm{i}, \end{aligned}$$

but we want the exact roots. By trial and error, we find that

$$\begin{aligned} z_1 &= (x-r_2)(x-r_5)(x-r_6)\\ &= x^3 - x^2 + 1.5x - (1.3228\ldots)\mathrm{i}x + 1 + (2.6457\ldots)\mathrm{i},\\ z_2 &= (x-r_3)(x-r_4)(x-r_7)\\ &= x^3 - x^2 + 1.5x + (1.3228\ldots)\mathrm{i}x + 1 - (2.6457\ldots)\mathrm{i}. \end{aligned}$$

Using 50 digits, we found that it was obvious that, ignoring signs, the imaginary parts of the constant terms are each exactly twice the imaginary parts of the coefficients of x.

What this means is that we can write

$$z_1 \;=\; x^3 - x^2 + \left(\frac{3}{2} - a\mathrm{i}\right)x + (1 + 2a\mathrm{i})$$

$$z_2 \;=\; x^3 - x^2 + \left(\frac{3}{2} + a\mathrm{i}\right) x + (1 - 2a\mathrm{i}).$$

Then

$$z_1 z_2 \;=\; x^6 - 2x^5 + 4x^4 - x^3 + \left(a^2 + \frac{1}{4}\right) x^2 + (3 - 4a^2)x + (1 + 4a^2).$$

Equating coefficients,

$$a^2 + \frac{1}{4} \;=\; 2, \qquad 3 - 4a^2 \;=\; -4, \qquad 1 + 4a^2 \;=\; 8.$$

Solving the first of these gives $a = \pm\sqrt{7}/2$, which does satisfy the other two equations.

Then taking the positive root (it does not matter which we take),

$$z_1 \;=\; x^3 - x^2 + \left(\frac{3}{2} - \mathrm{i}\frac{\sqrt{7}}{2}\right) x + \left(1 + \mathrm{i}\sqrt{7}\right),$$

$$z_2 \;=\; x^3 - x^2 + \left(\frac{3}{2} + \mathrm{i}\frac{\sqrt{7}}{2}\right) x + \left(1 - \mathrm{i}\sqrt{7}\right).$$

Now solving these two cubics in Maple gives, after $x_1 = -2$,

$$x_2 \;=\; \frac{w_+}{6} - \frac{\left(7 + 3\mathrm{i}\sqrt{7}\right)}{3w_+} + \frac{1}{3},$$

$$x_3 \;=\; \frac{w_-}{6} - \frac{\left(7 - 3\mathrm{i}\sqrt{7}\right)}{3w_-} + \frac{1}{3},$$

$$x_4 \;=\; -\frac{\left(1 - \mathrm{i}\sqrt{3}\right) w_+}{12} + \frac{7 - 3\sqrt{21} + \mathrm{i}\left(7\sqrt{3} + 3\sqrt{7}\right)}{6w_+} + \frac{1}{3},$$

$$x_5 \;=\; -\frac{\left(1 + \mathrm{i}\sqrt{3}\right) w_-}{12} + \frac{7 - 3\sqrt{21} - \mathrm{i}\left(7\sqrt{3} + 3\sqrt{7}\right)}{6w_-} + \frac{1}{3},$$

$$x_6 \;=\; -\frac{\left(1 + \mathrm{i}\sqrt{3}\right) w_+}{12} + \frac{7 + 3\sqrt{21} - \mathrm{i}\left(7\sqrt{3} - 3\sqrt{7}\right)}{6w_+} + \frac{1}{3},$$

$$x_7 \;=\; -\frac{\left(1 - \mathrm{i}\sqrt{3}\right) w_-}{12} + \frac{7 + 3\sqrt{21} + \mathrm{i}\left(7\sqrt{3} - 3\sqrt{7}\right)}{6w_-} + \frac{1}{3},$$

where

$$w_+ \;=\; \left(-154+90\mathrm{i}\sqrt{7}+6\sqrt{-1134-714\mathrm{i}\sqrt{7}}\right)^{1/3},$$

$$w_- \;=\; \left(-154-90\mathrm{i}\sqrt{7}+6\sqrt{-1134+714\mathrm{i}\sqrt{7}}\right)^{1/3}.$$

We have changed the labelling of the roots above as we cannot guarantee that they are in the same order as the decimal representations. We do the same with the second septic's roots.

The second septic

In Maple, we can use the `factor` function to find that

$$x^7+x^6+3x^5+3 \;=\; (x+1)(x^6+3x^4-3x^3+3x^2-3x+3)$$

so the first root is obviously -1. If we now use the `factor` function with the `complex` option on the sextic, we find its roots as

$$\begin{aligned} r_2 &= -0.4958\ldots-(1.5852\ldots)\mathrm{i}, & r_5 &= -0.2869\ldots+(1.0853\ldots)\mathrm{i},\\ r_3 &= -0.4958\ldots+(1.5852\ldots)\mathrm{i}, & r_6 &= 0.7828\ldots-(0.4999\ldots)\mathrm{i},\\ r_4 &= -0.2869\ldots-(1.0853\ldots)\mathrm{i}, & r_7 &= 0.7828\ldots+(0.4999\ldots)\mathrm{i}, \end{aligned}$$

but we want the exact roots. By trial and error, we find that

$$\begin{aligned} z_1 &= (x-r_2)(x-r_5)(x-r_7)\\ &= x^3+1.5x-(0.8660\ldots)\mathrm{i}x-1.5-(0.8660\ldots)\mathrm{i},\\ z_2 &= (x-r_3)(x-r_4)(x-r_6)\\ &= x^3+1.5x+(0.8660\ldots)\mathrm{i}x-1.5+(0.8660\ldots)\mathrm{i}, \end{aligned}$$

which we can also write as

$$z_1 \;=\; x^3+\left(\frac{3}{2}-a\mathrm{i}\right)x-\left(\frac{3}{2}+a\mathrm{i}\right),$$

$$z_2 \;=\; x^3+\left(\frac{3}{2}+a\mathrm{i}\right)x-\left(\frac{3}{2}-a\mathrm{i}\right).$$

Then

$$z_1z_2 \;=\; x^6+3x^4-3x^3+\left(\frac{9}{4}+a^2\right)x^2-\left(\frac{9}{2}-2a^2\right)x+\left(\frac{9}{4}+a^2\right).$$

Equating coefficients,

$$\frac{9}{4} + a^2 \;=\; 3, \qquad \frac{9}{2} - 2a^2 \;=\; -3, \qquad \frac{9}{4} + a^2 \;=\; 3.$$

Solving the first of these gives $a = \pm\sqrt{3}/2$, which does satisfy the other two equations.

Then taking the positive square root (it does not matter which we take),

$$\begin{aligned} z_1 &= x^3 + \left(\frac{3}{2} - \mathrm{i}\frac{\sqrt{3}}{2}\right) x - \left(\frac{3}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right), \\ z_2 &= x^3 + \left(\frac{3}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right) x - \left(\frac{3}{2} - \mathrm{i}\frac{\sqrt{3}}{2}\right). \end{aligned}$$

Now solving these two cubics in Maple gives, after $x_1 = -1$,

$$\begin{aligned} x_2 &= \frac{w_+}{6} - \frac{\big(3 - \mathrm{i}\sqrt{3}\big)}{w_+}, \\ x_3 &= \frac{w_-}{6} - \frac{\big(3 + \mathrm{i}\sqrt{3}\big)}{w_-}, \\ x_4 &= -\frac{\big(1 - \mathrm{i}\sqrt{3}\big)\, w_+}{12} + \frac{\big(3 + \mathrm{i}\sqrt{3}\big)}{w_+}, \\ x_5 &= -\frac{\big(1 + \mathrm{i}\sqrt{3}\big)\, w_-}{12} + \frac{\big(3 - \mathrm{i}\sqrt{3}\big)}{w_-}, \\ x_6 &= -\frac{\big(1 + \mathrm{i}\sqrt{3}\big)\, w_+}{12} - \frac{2\mathrm{i}\sqrt{3}}{w_+}, \\ x_7 &= -\frac{\big(1 - \mathrm{i}\sqrt{3}\big)\, w_-}{12} + \frac{2\mathrm{i}\sqrt{3}}{w_-}, \end{aligned}$$

where

$$\begin{aligned} w_+ &= \left(162 + 54\mathrm{i}\sqrt{3} + 18\sqrt{54 + 38\mathrm{i}\sqrt{3}}\right)^{1/3}, \\ w_- &= \left(162 - 54\mathrm{i}\sqrt{3} + 18\sqrt{54 - 38\mathrm{i}\sqrt{3}}\right)^{1/3}. \end{aligned}$$

Cycles

Ted Gore

In Cycling sequences, [M500 **292**, 11], Tommy Moorhouse defines a sequence of integers as follows. Choose two relatively prime integers $m > 1$ and $n > m$ and a 'seed' integer a_0. Set

$$a_k = \begin{cases} a_{k-1} + m & \text{if } n \text{ does not divide } a_{k-1}, \\ a_{k-1}/n & \text{otherwise.} \end{cases}$$

Here we examine the question at the end of the article. Do all such sequences end in a cycle?

Definition A *block* is a sequence of consecutive results conforming to the function that starts with a given value and ends with a multiple of n.

Let x be the first value in the block.

Let k be the number of steps required to get to the final value which is the first multiple of n in the sequence. Since m and n are co-prime there will always be such a value.

Let yn be the final value in the block. Then $x + km = yn$.

The y of one block becomes the starting value of the next block. If $y = x$ then the next block is a repeat of this one and therefore cyclic.

If $y \neq x$ then each subsequent block will have a different starting value to its predecessor. Eventually, however, there is an upper limit to the value of y, which means that eventually there will be a block with the same starting value as a previous block. In this case, a group of blocks form a cyclic process. A proof follows.

The greatest number of steps required to get from x to yn is $(n-1)$. So that $yn \leqslant x + (n-1)m$, which can be rearranged as $y \leqslant m + \dfrac{(x-m)}{n}$. When $x > m$ this results in y being closer to m than x was.

If $x = m$, then $y = m$. If $x < m$, then $y < m$ and therefore, for all subsequent blocks, $x < m$.

Examples

If $x = m$, then $y = m$ and there is only one repeated block. The triplet $(m, n, a_0) = (3, 11, 3)$ gives block [3 6 9 12 15 18 21 24 27 30 33].

If $x < m$, there are two or more blocks that taken together form a cycle. Triplet $(3, 11, 1)$ gives [1 4 7 10 13 16 19 22] [2 5 8 11].

If $x > m$, there are one or more 'introductory' blocks followed by a repeating group of blocks. Since an 'introductory' block may be a fragment of a repeating block, it can also be the start of a cycle.

Triplet (3,11,5) gives [5 8 11] [1 4 7 10 13 16 19 22] [2 5 8 11].

Triplet (5,7,1) gives four blocks [1 6 11 16 21] [3 8 13 18 23 28] [4 9 14] [2 7], which form a repeating group.

Triplet (3,11,500) shows how large starting values are reduced. It has three 'introductory' blocks of which the third is a fragment of a repeating block: [500 503 506] [46 49 52 55] [5 8 11] [1 4 7 10 13 16 19 22] [2 5 8 11].

Gravitational light

Colin Aldridge

I refer to the gravitational energy device, one of the exhibits of the museum at Woolsthorpe Manor House, Isaac Newton's birthplace, and which I described in M500 **292**, page 10, especially with regard to the generation, by a falling 10 kg mass, of electricity sufficient to light two LEDs thereby providing just about adequate illumination to read by.

We want to find out how much energy is released when the weight drops through 1 metre. The relevant formula is $J = mgh$, where J is the energy, m is the mass of the weight, and h is the height. So in this case $J = (10\,\mathrm{kg}) \times (9.8\,\mathrm{m/s^2}) \times (1\,\mathrm{m})$, which without recourse to a calculator gives 98 joules.

The weight takes half an hour, or 1800 seconds, to drop; so at 100 per cent efficiency that is a power output of $98/1800 = 0.054$ joules per second. Happily we have chosen the right units since a watt is a joule per second.

I doubt that the mechanism is 80 per cent efficient; so we cannot get 0.04 watts out of the system and therefore the lights are either 2×0.01 watt LEDs or one 0.01 and one 0.02 watt LEDs.

Solution 278.4 – Polynomial integration

(i) Find a polynomial $Q(x, y)$ of degree 2 in x and y such that for any quadratic $P(x)$,

$$\int_{-1}^{1} P(x)Q(x,y)\,dx \;=\; P(y). \qquad (1)$$

(ii) Find a polynomial $Q(x, y)$ of degree 1 in x and y such that (1) holds for any linear function $P(x)$. Hint: Do (ii) first.

Peter Fletcher

We take the hint and do Part (ii) first.

Let $P_1(x) = ax + b$ and $Q_1(x, y) = cx + ey + f$ with the subscripts indicating the degree of each polynomial. Then

$$\begin{aligned}\int_{-1}^{1} P_1(x)Q_1(x,y)\mathrm{d}x &= \int_{-1}^{1} (ax+b)(cx+ey+f)\mathrm{d}x \\ &= \int_{-1}^{1} \big(acx^2 + aexy + (af+bc)x + bey + bf\big)\,\mathrm{d}x \\ &= \left[\frac{acx^3}{3} + bexy + bfx\right]_{-1}^{1} \\ &= \frac{2ac}{3} + 2bey + 2bf.\end{aligned}$$

We want

$$a \;=\; 2be \quad \text{and} \quad b \;=\; \frac{2ac}{3} + 2bf;$$

so

$$e \;=\; \frac{a}{2b} \quad \text{and} \quad f \;=\; \frac{b - 2ac/3}{2b};$$

but we can choose $c = 0$ so that

$$f \;=\; \frac{1}{2} \quad \text{and} \quad Q_1(x,y) \;=\; \frac{ay+b}{2b},$$

i.e. $Q_1(x, y)$ is a function of y only.

This suggests that $Q_2(x, y)$ in Part (i) may also be a function of y only.

Let $P_2(x) = ax^2 + bx + c$ and $Q_2(x,y) = ey^2 + fy + g$ so that

$$\begin{aligned}
\int_{-1}^{1} P_2(x)Q_2(x,y)\mathrm{d}x &= \int_{-1}^{1} (ax^2+bx+c)(ey^2+fy+g)\mathrm{d}x \\
&= \int_{-1}^{1} \big((ey^2+fy+g)ax^2 \\
&\qquad + (ey^2+fy+g)bx + (ey^2+fy+g)c\big)\mathrm{d}x \\
&= \left[\frac{(ey^2+fy+g)ax^3}{3} + (ey^2+fy+g)cx\right]_{-1}^{1} \\
&= \frac{2(ey^2+fy+g)a}{3} + 2(ey^2+fy+g)c \\
&= 2e\left(\frac{a}{3}+c\right)y^2 + 2f\left(\frac{a}{3}+c\right)y + 2g\left(\frac{a}{3}+c\right).
\end{aligned}$$

We want

$$a = 2e\left(\frac{a}{3}+c\right) \quad \text{so that} \quad e = \frac{a}{2\left(\frac{a}{3}+c\right)};$$

$$b = 2f\left(\frac{a}{3}+c\right) \quad \text{so that} \quad f = \frac{b}{2\left(\frac{a}{3}+c\right)};$$

and

$$c = 2g\left(\frac{a}{3}+c\right) \quad \text{so that} \quad g = \frac{c}{2\left(\frac{a}{3}+c\right)}.$$

Therefore

$$Q_2(x,y) = \frac{ay^2+by+c}{2\left(\frac{a}{3}+c\right)}.$$

$230 - 220 \times 0.5 = ?$ Believe it or not, the answer is 5!

—Sent by Jeremy Humphries

Problem 294.2 – Columns

Roger Thompson

Initially, columns 1, 2, 3, ... are all empty. For the integers $X = 1, 2, 3, \ldots$, add X to the appropriate column according to the following rules.

1. Starting at $C = 1$, add X to column C if this has less than two entries, or no two different entries in C add up to X.
2. If X could not be added, move on to the next column.

For example, after 1 to 9 have been added, the columns look like this.

Column	1	2	3
	1	3	8
	2	5	9
	4	6	
	7		

Which column is 314159261234568 in?

Richard Kenneth Guy

Tony Forbes

I was sad to learn that Richard Guy died on 9th March 2020 at the age of 103. He graduated at Cambridge in 1938, served in the Royal Air Force during the Second World War, and in 1965 emigrated with his wife Nancy to Canada. He took up a permanent position at the University of Calgary, where he remained for the rest of his life. There is more at `https://calgaryherald.remembering.ca/obituary/richard-guy-1078841321`.

He was well known throughout the number-theoretic community for his highly influential book *Unsolved Problems in Number Theory*. I bought the second edition soon after it was published in 1994, and about a decade later I was at a mathematics conference, where I noticed that the third edition was on offer at quite a generous discount. I agonized about buying it, weighing the advantages (new results) against the disadvantages (duplication, personal wealth reduction). The deciding factor was a conversation with Richard, who was also at the conference. We both agreed that I would buy the book iff he autographed it.

I also remember him making a \$20.00 bet with John Conway (with whom he co-authored *The Book of Numbers*) that a new completely factorized Fermat number would appear before his 100th birthday. Alas! It didn't happen, the bet was lost, and it is still the case that the only numbers $2^{2^n} + 1$ for which all the prime factors are known have $n \leq 11$.

Erdős

Mike Grannell

Most mathematicians do not fit the popular stereotype of being highly eccentric and absent minded, but a few do, and one of these was Paul Erdős. This is a story told to me by a colleague from an American University in the Deep South, who had better be nameless—let's call him Fred. Erdős was accustomed to roam the world with a few personal items in a suitcase, and arrive at a Mathematics Department at a random University, unannounced.

Such was his fame and ability that he was invariably welcomed and invited to stay and join in some collaborative research. So it happened at the University of X, where Fred was a Maths Professor, and Fred offered to host Erdős at his house. One night, Fred in bed with his wife, woke up and became dimly aware that someone was moving around in his bedroom. He switched on the bedside light to see Erdős advancing towards the bed. Erdős reached the bed, bent over Fred who was still half asleep, and said, "Take a positive integer n ...."

Actually Erdős was quite fortunate, since this was Deep South USA, and Fred had an arsenal of guns for dealing with intruders. At the end of two weeks, Erdős moved on elsewhere and Fred and his wife were quite relieved, but the story had a sequel. A few months later at a conference attended by Fred and by Erdős, a young mathematician (let's call him Joe) asked Fred to introduce him to Erdős. So Fred went up to Erdős with Joe in tow and said, "Hello Paul, can I introduce my colleague, Joe." Erdős replied politely to Joe and then turned to Fred and asked, "And who are you?"

Problem 294.3 – Even and odd

Suppose $28 \leq a < b$ and $a \equiv b \equiv 0 \pmod{28}$. Let S be the set of integers in the range $[a, b]$ that are congruent to a modulo 3 but not divisible by 4 or 7. For example, if $a = 28$ and $b = 56$, then $S = \{31, 34, 37, 43, 46, 55\}$. Show that S contains twice as many odd integers as even integers, or find a counter-example.

Problem 294.4 – Mod 9

Two three-digit numbers, a and b, are chosen at random. What's the probability of them being anagrams of each other given that $a \equiv b \pmod 9$? Does the answer explain why I (TF) get the impression that this sort of thing happens quite often?

Solution 173.1 – Binomial coefficients squared

Show that

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2}\binom{n}{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Henry Ricardo

We have

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}^2 = \sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{n}{n-r},$$

which is the coefficient of x^n in

$$(1-x^2)^n = (1-x)^n(1+x)^n = \sum_{r=0}^{n}(-1)^r\binom{n}{r}x^r \cdot \sum_{r=0}^{n}\binom{n}{r}x^r.$$

If n is even, let $n = 2m$ and $y = x^2$. Then the coefficient of y^m in $(1-y)^{2m}$ is $(-1)^m\binom{n}{m} = (-1)^{n/2}\binom{n}{n/2}$. However, $(1-x^2)^n$ has only terms of even degree, so if n is *odd*, then clearly the coefficient of x^n is 0.

TF — There is (at least) one other binomial power where MATHEMATICA gives a simple sum, i.e. one that does not involve a non-simplifiable hypergeometric function. We formally state it as M500 Problem 294.5, below.

Problem 294.5 – Binomial coefficients cubed

Tony Forbes

Show that

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}^3 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2}\dfrac{(3n/2)!}{((n/2)!)^3} & \text{if } n \text{ is even.} \end{cases}$$

Notice that if you replace the three threes by twos, you recover the formula in Problem 173.1, above. So one might be tempted to conjecture that for n divisible by 4, we have

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}^m = \frac{(mn/2)!}{((n/2)!)^m}.$$

But this appears to be not true unless $m = 0$, 2 or 3 (or $n = 0$).

Problem 294.6 – Graph smash

Tommy Moorhouse

This puzzle involves graphs. We will use fairly standard terminology as set out in the reference for example. A graph is a set of vertices, some of which are connected by edges. Multiple edges and edges starting and finishing at the same vertex are not allowed, and an edge starts and finishes at distinct vertices but has no other vertices attached to it. A graph is said to be connected if it cannot be drawn as two or more separate components with no edges connecting the components. If a graph is not connected it is said to be disconnected. The complement $\overline{G}$ of a graph G is the graph having the same vertices as G, with edges connecting two vertices of $\overline{G}$ only where G has no edge connecting those same vertices.

We will define the 'smash product' $G \vee H$ between two graphs G and H as follows. Draw the graphs side by side and connect every vertex of G to every vertex of H, making no other changes. This is $G \vee H$. We also define $G + H$ to be the disconnected graph consisting of G and H (drawn separately).

Show that

$$(G \vee H) \vee W \;\simeq\; G \vee (H \vee W),$$

where $A \simeq B$ means A and B are isomorphic.

Show that

$$\overline{G \vee H} \;\simeq\; \overline{G} + \overline{H}. \tag{1}$$

As an example, show that $K_n \vee K_m \simeq K_{n+m}$ where K_n is the graph with n vertices, all connected to one another. You could use the isomorphism (1) and the fact (prove this) that $\overline{K_n} \simeq N_n$ where N_n is the graph with n vertices and no edges.

Not every graph can be written as the smash product of two other nonempty graphs. For example the cyclic graph C_n cannot be so expressed for $n > 4$ (try it out). Find a criterion for a graph J to be expressible as $G \vee H$ for nonempty G and H if and only if J has some simple property (hint – think about $\overline{J}$).

Reference Richard J. Trudeau, *Introduction to Graph Theory*, Dover, 1993.

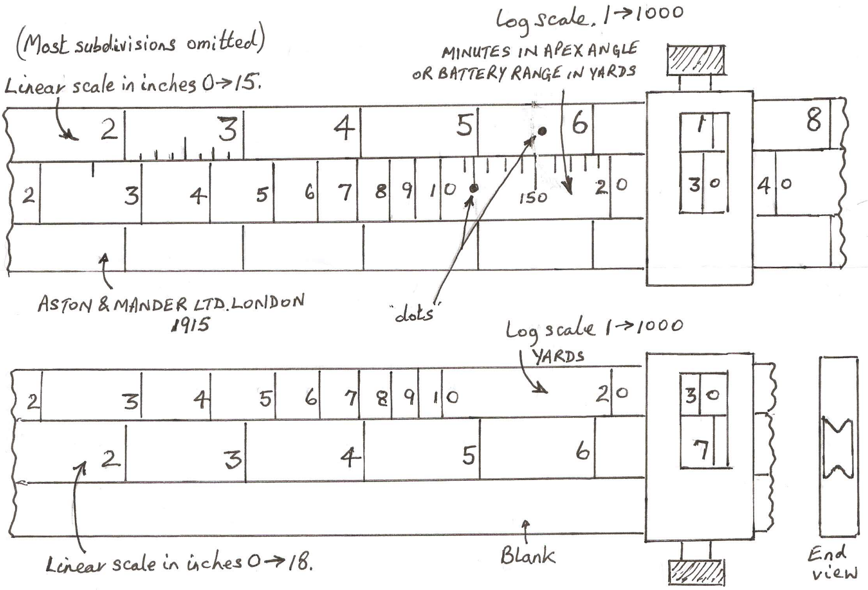
Chemist: What's that in $CO(NH_2)_2$?
Mathematician: It's like a field but with the postulates weakened a bit.

— Sent by JRH

WW1 artillery slide rule

Chris Pile

Last year my neighbour was clearing out his garage prior to moving house and he invited people to sort through, and take whatever they liked from, a box of old tools, etc. before it was consigned to the skip. I picked out a slim leather case containing, what turned out to be, a WW1 artillery slide rule, stamped 'ASTON & MANDER LTD. LONDON 1915.'



The fixed, outer, scales are marked in inches from 0 to 15, subdivided into eights of an inch. The central, sliding, scale protrudes by three inches and is marked in a 3-cycle log scale from 1 to 1000, each cycle being slightly more than $4^7/_8$ inches. This scale is marked 'MINUTES IN APEX ANGLE / BATTERY RANGE IN YARDS.' The subdivisions on each cycle are $^1/_2$ yd from 1 to 10, 1 yd from 10 to 150 and 10 yds from 100 to 1000. The last three inches are blank. On the linear scale there are two 'dots' at just over $5^1/_2$ inches and just under $10^1/_2$ inches (i.e. 1 log scale apart). Similarly, on the log scale there are two 'dots' at 11.4 yds and 114 yds. On the reverse, the log scale is on top, fixed, marked 'YARDS' and the sliding scale is n inches from 0 to 18 subdivided into eighths. The thumb screw on the cursor window locks the cursor and both scales.

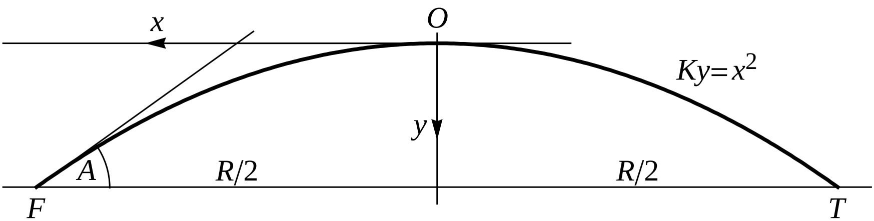
The log scale equates the apex angle in minutes to the range in yards for values from 1 to 1000. Assuming a simple parabolic trajectory with a shallow angle, such that $Ky = x^2$ (see diagram), let the range be R and the apex angle be A. Then

$$K\frac{dy}{dx} \;=\; 2x;$$

therefore at F,

$$\frac{dy}{dx} \;=\; \tan A \qquad \text{and} \qquad x \;=\; \frac{R}{2}.$$

Hence $K \tan A = R$. Taking a mid value (on the log scale) of $30'$ and 30 yards, $K \tan(1/2)^\circ = 30$. Therefore $K \approx 3437.7$. This value of K defines the parabola approximately from $1'$ and 1 yd ($K = 3437.75$) to 1° and 60 yds ($K = 3437.4$). At 1000 yds and 16.67°, $K = 3340$.



Explanations please! What is the slide rule intended to calculate? What is the linear inch scale for, and the significance of the 'dots'? What is the practical use of extending the scale through very small values to 1 yard and 1 minute of angle? Does the value of K relate to any specific characteristic (e.g. muzzle velocity)?

Problem 294.7 – Triangles

Tony Forbes

You and your opponent take it in turn to select an edge of the complete graph K_4. An edge cannot be selected by both players. You go first. The winner is the first person to have selected three edges that form a triangle. A draw results either by mutual agreement or if a player dies.

For example, labelling the vertices of the graph 1, 2, 3, 4, and denoting edge $\{x, y\}$ by $10x + y$, the edges selected might go like this:

12, 13; 12, 13; 12, 13; 12, 34; 23, 34; 23, 34; 24, 34; 23, 14;

and you lose because your opponent now has a triangle, $\{13, 34, 14\}$.

Show that your opponent can always force a draw. What if K_4 is replaced by K_5? Or K_6?

Letter

Gravitational lamp

Dear Eddie,

Many thanks for M500 **292**. I was intrigued by the gravitational lamp, a splendidly medieval idea of something powered by a falling weight (I use the word ‘weight’ in its everyday sense, so don’t start moaning about it being a mass). I suppose that the weight, which looks like a bag, is filled with locally available stones so that only the relatively light mechanism has to be sent to the places where it’s needed.

For readers of the paper copy, the red loop is the one to the right of the black loop.

I wonder, though, how easy it is to wind up by pulling the chain. There must be a lot of frictional loss in this complex mechanical system. Winding up an old weight-powered clock is quite laborious. It would be interesting to see whether the generator and storage of one of Trevor Baylis’s hand-cranked radios, which are much easier to use, would power the same two LEDs for as long with less effort needed to wind the thing up.

Glad to see, on the same page 10, that there are ‘persons whose business it is to trisect angles’. I hope they have a union to look after their interests in this threatened trade.

Best wishes,

Ralph Hancock

Letters

Arrange these letters in a sensible manner.

```
a a a a a a a a A A A A A A A A b b b b b b b B B B
B B B B c c c c C C C C C C C C C C C C d d d d d D
D D e e e e e e e e e e E E E f f f F F F F F g g g
g g g G G G h h h h H H H H H H i i i i i I I I k K
K l l l l L L L L L m m m m m m M M M M M M n n n n
n n N N N N N N N N N o o o o o O O O p P P P P P P
P P P r r r r r r r r r r r R R R R R R R R s s s s
s s s S S S S S S S S S t t t T T T T T T T T T u u
u u u u U v V W X y Y Y Z Z
```

Answer in the next issue.

John Horton Conway

As this issue was going to press we learned of the sudden death on 11 April 2020 at age 82 of group theorist and inventor of LIFE, John Conway. He was a victim of the Covid-19 epidemic. You can read more about his fascinating life and mathematical work at `https://www.theguardian.com/science/2020/apr/23/john-horton-conway-obituary`.

Jeremy Humphries

I met John Conway back in the 1970s. The Open University engaged him to give a Saturday afternoon popular mathematics talk in Cambridge, which was not far for me to go when I was living in Luton. He talked about all sorts of stuff—high dimensional sphere packing, Leech lattice, and so on. Also he reminisced about his mother making tray bakes when he was a boy, and how he would investigate various ways of cutting them up. He grew up to invent a few mathematical games involving cake. And something I've always remembered—after the talk he came to explain a point to me as I sat at the desk, which involved him writing a few lines of words and maths on my notepaper. Since he was standing facing me, he wrote it all, very fluently and naturally, joined up writing, upside down and right to left.

A few of us walked back with him across Cambridge from the lecture venue to his college. We were there, all star struck, and I was puzzled to note that the general populace took no notice of us. I mean—come on, Cambridge shoppers—don't you know who this man is??

Problem 294.8 – Digits and divisors

Tony Forbes

Let $b \geq 2$ and let n be a number consisting of b distinct digits when written in base b. Suppose also that n has the property that $\lfloor n/b^{n-r} \rfloor \equiv 0 \pmod r$ for $r = 1, 2, \ldots, b$.

Show that (i) b must be even, or find a counter-example, and (ii) when $b = 14$ there is a unique solution:

$$n = 7838911147538198 = \mathrm{9C3A5476B812D0}_{14}.$$

The (mod r) condition can be rephrased by saying that when you express n in base b the number consisting of its first r digits is divisible by r. For example, take $b = 6$. Then $n = 13710 = 143250_6$ satisfies all of the conditions. Clearly, 1_6, 14_6, 143_6 and 143250_6 are divisible by 1, 2, 3 and 6 respectively. Also 4 divides $380 = 1432_6$ and 5 divides $2285 = 14325_6$.

As *Guardian* readers might remember, the case $b = 10$ featured in Alex Bellos's Monday puzzle, where the problem and its unique solution were attributed to John Conway.

Contents

04.00

Front cover $64\left(z^4+y^2\right)+x^4-16zx^2+4x^2=256$; compare page 1.

Problem 294.9 – Mod 240

Show that $(p^2-q^2)\sqrt{pq}\equiv 0 \pmod{240}$ if $p-q$ is even and pq is a square.