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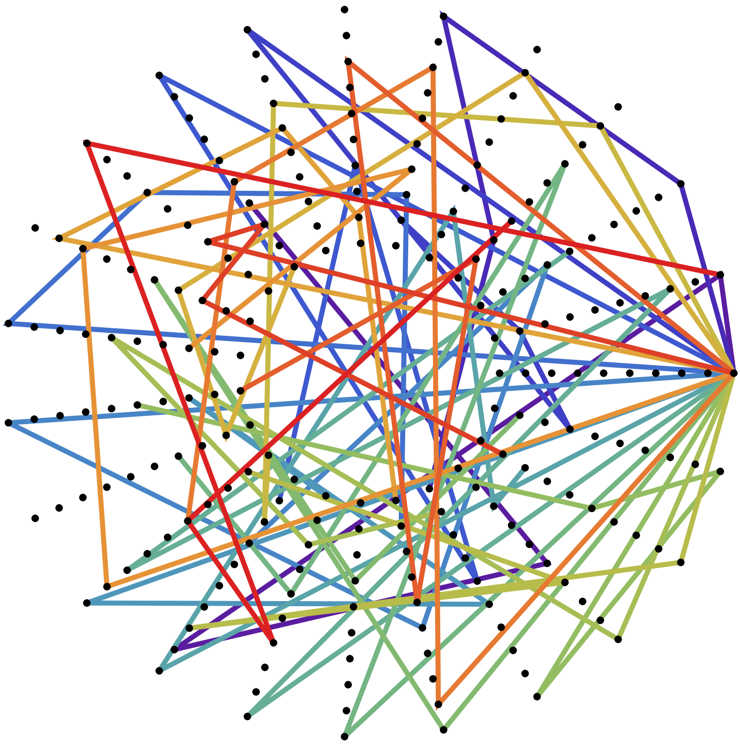
M500 296



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: `m500.org.uk`.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The Revision Weekend is a residential Friday to Sunday event providing revision and examination preparation for both undergraduate and postgraduate students. For details, please go to the Society's web site.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, please go to the Society's web site.

Advice for authors We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to the Editor, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. For more information, go to `m500.org.uk/magazine/` from where a LaTeX template may be downloaded.

M500 Mathematics Revision Weekend 2021

The M500 Revision Weekend 2021 will be held at

Kents Hill Conference Centre, Milton Keynes

from Friday to Sunday 7th - 9th May 2021.

We expect to offer tutorials for most undergraduate and postgraduate mathematics Open University modules, subject to the availability of tutors and sufficient applications. Application forms will be sent via email to all members who supplied an email address. Contact the Revision Weekend Organizer if you have any queries about this event. All arrangements are subject to the development of the coronavirus pandemic.

Chicks, eggs and advertising

J. M. Selig

Some time ago I taught Mathematics in a department of Business. I was quite surprised that most students were so afraid of Maths that they weren't even interested whether or not their enterprise was profitable. Even some of my colleagues, especially those who taught Marketing, were very shy of anything to do with Maths. I began to wonder if there were any good applications of Mathematics to Advertising. A classic problem in probability is usually phrased in term of chicks and eggs.

> My hen lays eggs according to a Poisson distribution with mean λ. The probability that she lays n eggs tomorrow is $p(n) = \lambda^n e^{-\lambda}/n!$. The eggs hatch, or not, according to a binomial distribution with p as the probability of an egg hatching into a chick. What will be the distribution of chicks?

Instead of eggs and chicks suppose we consider enquiries and orders. Suppose my company make a product that is large and expensive so that customers will order either one or none. The adverts my company buys generate enquiries; these could be phone calls, letters or most probably e-mails. Assume the enquiries can be modelled by a Poisson distribution. The mean number of enquiries, λ, is now some measure of the effectiveness of the advertising. In the enquiries potential customers (also known as 'punters') want to know the cost, delivery arrangement and so forth. On the basis of this information they may place an order. Whether or not they place an order can be modelled by a binomial distribution. Clearly the company would like to know about the distribution of orders.

Another story we could tell about the same underlying problem concerns internet shopping. Companies spend money advertising their websites, but many people who visit websites may go through most of the process of ordering items but then leave the site without completing the order. I have seen figures quoted that something like 70 per cent of visitors to some websites bail before completing their order.

One more story with the same model might be to do with the spread of a virus in a population. An infected individual might meet n susceptible people in a day. The probability that a susceptible person then becomes infected will obey a binomial distribution. If we only consider the early stages of an epidemic then we could assume that n obeys a Poisson distribution and that susceptible people will likely meet no or only one infected person in a day. This type of model might be useful to understand the reproduction number, R, for the disease.

The problem, however we state it, concerns the compounding of a Poisson distribution with a binomial distribution; see [1] for example. To get k orders we might have k enquiries and they all turn into orders, or we could have $k+1$ enquiries but only k turn into orders, and so on. So if we write $\rho(k)$ as the probability of getting k orders, then

$$\rho(k) \;=\; \sum_{i=k}^{\infty} \frac{\lambda^i}{i!} e^{-\lambda} \binom{i}{k} p^k q^{i-k}$$

where $q = (1-p)$, the probability the enquiry fails to turn into an order. To sum the series it is convenient to make a change of index variable so that the sum runs from 0 to ∞. Let $j = i - k$, so that $i = k + j$. Now the probability can be written

$$\rho(k) \;=\; \sum_{j=0}^{\infty} \frac{\lambda^{k+j}}{(k+j)!} e^{-\lambda} \binom{k+j}{k} p^k q^j.$$

Taking out the constants and expanding the binomial coefficient gives

$$\rho(k) \;=\; \frac{(\lambda p)^k}{k!} e^{-\lambda} \sum_{j=0}^{\infty} \frac{(\lambda q)^j}{j!} \;=\; \frac{(\lambda p)^k}{k!} e^{-\lambda} e^{\lambda q}.$$

Remembering that $q = 1 - p$ gives

$$\rho(k) \;=\; \frac{(\lambda p)^k}{k!} e^{-(\lambda p)}.$$

This is a Poisson distribution again, but now with mean λp.

With different advertising we may be able to increase λ, but this isn't necessarily a good thing. More enquiries from people who are not really interested in the product and are not going to place an order are not useful to the company; they will be referred to as 'time wasters'. Really, we would like to increase p; this can be done by more targeted advertising, but this would probably decrease λ. That is, we would expect targeting of our advertising to generate fewer enquiries but with a higher chance that each will turn into an order. Clearly, it is the product λp that we should attempt to increase for more orders. This could be a risky strategy. The variance of the Poisson distribution is the same as its mean, so increasing λp will also increase the variance.

These ideas can be pushed a little further by relaxing the assumption that customers only buy one or no items. To keep things simple, though, we assume that customers buy a discrete number of the same item or spend integer multiples of some basic amount. Let's use the variable k now to count the number of items ordered. We can assume that the probability that $k = 0$ is about 70 per cent, but what distribution should we use for k? I don't have a good answer for this; really I should go and find some data to see what the distribution looks like. For the purposes of this article I will just assume the spending for an individual punter is distributed according to a geometric distribution. So the probability of ordering k items is given by the probability mass function,

$$g(k) \;=\; qp^k, \qquad \text{where} \quad q = 1 - p,$$

and from the data $g(0) = q \approx 0.7$. Next, suppose that we have persuaded, by advertising, n punters to visit our website. What is the distribution giving the probability that they will collectively buy κ items? This convolution can be found by multiplying the generating functions for the individual geometric distributions. The generating function for the geometric distribution is

$$G(z) \;=\; \sum_{k=0}^{\infty} qp^k z^k \;=\; q(1 - pz)^{-1}$$

since this is the sum of a geometric series. For n punters, the generating function of the distribution is the nth power of the above,

$$G^n(z) \;=\; q^n(1 - pz)^{-n}.$$

The probability that the n punters collectively order κ items is then the coefficient of z^κ in the series expansion of $G^n(z)$. Using the binomial theorem to expand the bracket gives

$$G^n(z) \;=\; q^n(1 - pz)^{-n} \;=\; q^n \sum_{\kappa=0}^{\infty} \binom{\kappa + n - 1}{\kappa} (pz)^\kappa.$$

So we can see that the probability that n punters order κ items is distributed according to a negative binomial distribution, with probability mass function,

$$nb(n, \kappa) \;=\; \binom{\kappa + n - 1}{\kappa} q^n p^\kappa.$$

As before, the probability that the advertising lays n punters is assumed to follow a Poisson distribution. The probability that we get orders for κ items is then the compound,

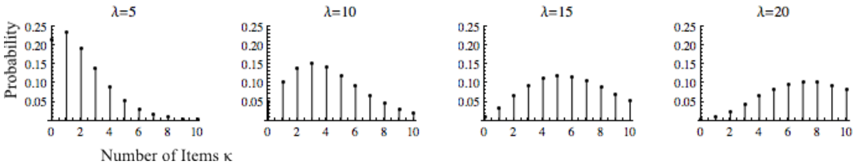


Figure 1: Probability of getting orders for 0 to 10 items. In each case $q = 0.7$, the value of λ varies from $\lambda = 5$ to $\lambda = 20$.

$$
\begin{aligned}
f(\kappa) &= \sum_{n=0}^{\infty} n b(n,\,\kappa) p(n) \\
&= \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \binom{\kappa+n-1}{\kappa} q^n p^\kappa \;=\; e^{-\lambda} p^\kappa \sum_{n=0}^{\infty} \binom{\kappa+n-1}{\kappa} \frac{(\lambda q)^n}{n!}.
\end{aligned}
$$

I have no idea how to evaluate this infinite sum, but luckily *Mathematica* does; the result is

$$
\sum_{n=0}^{\infty} \binom{\kappa+n-1}{\kappa} \frac{(\lambda q)^n}{n!} \;=\; (\lambda q)\, {}_1F_1(\kappa+1; 2; (\lambda q)).
$$

The function ${}_1F_1(a; b; x)$ is the confluent hypergeometric function, also called Kummer's hypergeometric function; see the Wikipedia article for more details, [3]. The probability that we get orders for κ items is thus

$$
f(\kappa) \;=\; e^{-\lambda} \lambda q p^\kappa\, {}_1F_1(\kappa+1; 2; (\lambda q)).
$$

Most probability distributions have a name so that we can refer to them simply. I haven't been able to find this distribution in the literature but I can't imagine it is 'unknown'.

When κ is an integer the confluent hypergeometric function is, in fact, an elementary function. The first few are

$$
\begin{aligned}
{}_1F_1(1; 2; x) &= \frac{1}{x}(e^x - 1), \\
{}_1F_1(2; 2; x) &= e^x, \\
{}_1F_1(3; 2; x) &= \frac{1}{2}(x+2)e^x, \\
{}_1F_1(4; 2; x) &= \frac{1}{6}(x^2+6x+6)e^x, \\
{}_1F_1(5; 2; x) &= \frac{1}{24}(x^3+12x^2+36x+24)e^x,
\end{aligned}
$$

and so forth. Figure 1 shows the probabilities of getting orders for 0 to 10 items when $q = 0.7$ and $\lambda = 5$, 10, 15 and 20.

To compute the mean and variance of the distribution we can use some generating function tricks. It is useful to find the generating function of the distribution $f(\kappa)$. That is, we seek

$$F(z) \;=\; \sum_{\kappa=0}^{\infty} f(\kappa) z^{\kappa}.$$

Returning to the definition of $f(\kappa)$ as an infinite sum gives

$$F(z) \;=\; \sum_{\kappa=0}^{\infty}\sum_{n=0}^{\infty} p(n) nb(n,\kappa) z^{\kappa}.$$

Substituting for the Poisson and negative binomial distributions gives

$$F(z) \;=\; e^{-\lambda} \sum_{\kappa=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q\lambda)^n}{n!} \binom{\kappa+n-1}{\kappa} (pz)^{\kappa}.$$

Assuming we can swap the order of summation and perform the sum over κ first we get

$$F(z) \;=\; e^{-\lambda} \sum_{n=0}^{\infty} \frac{(q\lambda)^n}{n!} (1-pz)^{-n}.$$

Finally the generating function for $f(\kappa)$ is

$$F(z) \;=\; e^{-\lambda} e^{\left(\frac{q\lambda}{1-pz}\right)}.$$

The mean and variance of the distribution are now easy to compute:

$$\text{mean} \;=\; \frac{dF(1)}{dz} \;=\; \frac{\lambda p}{q}$$

and

$$\text{variance} \;=\; \frac{d^2F(1)}{dz^2} + \frac{dF(1)}{dz} - \left(\frac{dF(1)}{dz}\right)^2 \;=\; \frac{\lambda p(p+1)}{q^2}.$$

(Feller [1, Chap. XII] gives some relations for compound distribution which would simplify these computation a little.)

In conclusion, there are several ways these ideas could be extended, although one should first check that the notions explored above have some validity! One could, for example, look at several different advertising channels or media: radio, TV, newspapers, internet advertising etc. Each will have its own characteristics, different probability distributions perhaps. Each channel has an associated cost. So, we could maximize the effectiveness of the advertising subject to a fixed budget. There might be non-linear effects from advertising on several media. If we advertise our product on TV and newspapers the effect may be better than the sum of advertising in each separately.

People have been advocating the use of Maths in Marketing since the 1960s, but there doesn't seem to have been a lot of progress. Most of the mathematics referred to is just calculating a ratio of two numbers to produce a performance measure for an advertising campaign. The deepest application I have come across was a model of Marketing in a duopoly based Lanchester's equations, see [2]. These models were originally developed to model warfare so the idea here is two companies fighting an advertising war for market share.

This all assumes that the purpose of advertising is to increase sales but I don't think this is always the case; sometimes advertising is for 'brand management'.

References

[1] Feller, W. (1968). *An Introduction To Probability Theory and Its Applications*, Volume 1, 3rd Edition, Wiley, New York.

[2] Jørgensen, S., Sigué, S. (2020), A Lanchester-Type Dynamic Game of Advertising and Pricing. In: Pineau, P.-O., Sigué, S., Taboubi, S. (eds), *Games in Management Science : essays in honor of Georges Zaccour*, International Series in Operations Research & Management Science, Vol. 280, Springer, Cham.

[3] Wikipedia, 'Confluent hypergeometric function', `https://en.wikipedia.org/wiki/Confluent_hypergeometric_function`

Problem 296.1 – Divisibility

Tony Forbes

Let m and n be positive integers with $m \geq 4$. Show that

$$(mn)! - m!\,(n!)^m \;\equiv\; 0 \pmod{n^{m+3}},$$

or find a counter-example.

Solution 237.4 – Continued fraction

Show that

$$\cfrac{1}{1-\cfrac{1^4}{5-\cfrac{2^4}{13-\cfrac{3^4}{25-\dots}}}} \;=\; \frac{\pi^2}{6}.$$

The number underneath n^4 is $2n^2+2n+1$.

Peter Fletcher

The first few approximations are:

$$\cfrac{1}{1-\cfrac{1}{5}} \;=\; \frac{5}{4} \;=\; 1+\frac{1}{2^2},$$

$$\cfrac{1}{1-\cfrac{1}{5-\cfrac{16}{13}}} \;=\; \frac{49}{36} \;=\; 1+\frac{1}{2^2}+\frac{1}{3^2},$$

$$\cfrac{1}{1-\cfrac{1}{5-\cfrac{16}{13-\cfrac{81}{25}}}} \;=\; \frac{205}{144} \;=\; 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2},$$

from which it is apparent that for each subsequent n, we add $1/(n+1)^2$ to the previous sum and the result follows.

Problem 296.2 – Sums of n numbers

Tony Forbes

Given integers a, b, c, $0 \le a < b < c$, and $n \ge 1$, find a general formula for $S(a,b,c;n)$, the set of integers that can be expressed as a sum of n elements of $\{a,b,c\}$. For example,

$$S(2,14,32;17) \;=\; \{34,40,46,\dots,544\} \setminus \{40,52,502,520,532,538\};$$

i.e. nearly all numbers from $17\cdot 2$ to $17\cdot 32$ that are congruent to 4 (mod 6). For those who have Mathematica, $S(a,b,c;n)$ is the set of numbers `j` such that `IntegerPartitions[j, {n}, {a, b, c}]` is not empty.

Solution 291.1 – Treasure

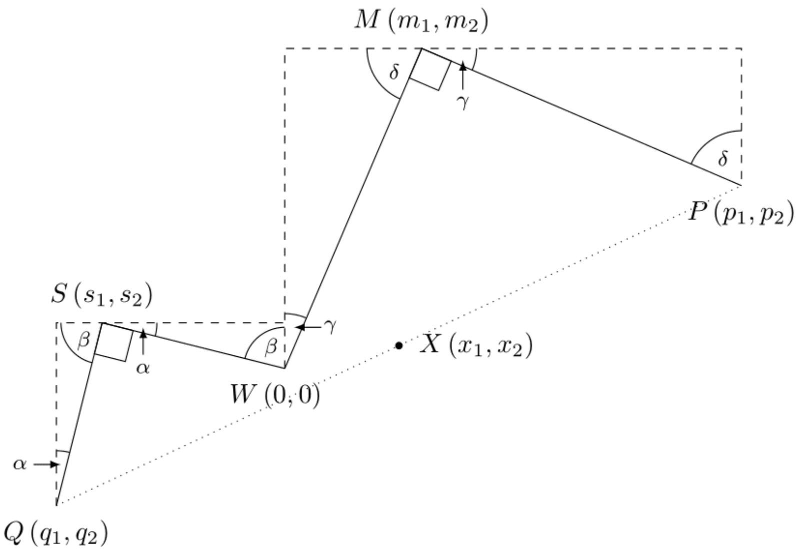
There is some valuable stuff buried on an island and to your delight you have obtained precise instructions for digging it up.

> Locate a tall wooden post. From there, not far away you will see two tall stone pillars, one made of marble and the other of sandstone. Go to the wooden post, walk to the marble pillar, turn right, walk the same distance again and mark the spot. Go to the wooden post, walk to the sandstone pillar, turn left, walk the same distance again and mark the spot. The treasure is mid-way between the two marked spots.

You arrive at the island and find the stone structures, but alas! the wooden post has disappeared without trace. However, you had the foresight to bring a replacement with you. You erect your wooden post somewhere on the island, follow the instructions and successfully acquire the treasure. How is this possible?

Peter Fletcher

Let the wooden post W be at $(0,0)$, the marble pillar M at (m_1, m_2) and the sandstone pillar S at (s_1, s_2); let also the first marked spot P be at (p_1, p_2) and the second one Q at (q_1, q_2), so that the treasure is at X given by $(x_1, x_2) = \Big((p_1+q_1)/2, (p_2+q_2)/2\Big)$. The diagram below illustrates the situation.



In this diagram, the triangles with hypotenuses WM and MP are congruent, as are those with hypotenuses WS and SQ. Therefore we can write down

$$\begin{aligned}(p_1, p_2) &= (m_1 + m_2, m_2 - m_1),\\ (q_1, q_2) &= (s_1 - s_2, s_1 + s_2)\end{aligned}$$

and

$$(x_1, x_2) = \Big((m_1 + m_2 + s_1 - s_2)/2,\ (m_2 - m_1 + s_1 + s_2)/2\Big).$$

As long as we keep S in the same position relative to M, X will then always be is the same position relative to both M and S.

Example

Let M be at $(1, 2)$ and S be at $(-4, 1)$. Then X is at $(-1, -1)$.

Note that to get from M to S, we go 5 west and 1 south and to get from M to X, we go 2 west and 3 south.

Now let W be such that M is at $(57, -35)$ and S is at $(57-5, -35-1) = (52, -36)$. Then X is at $(55, -38) = (57 - 2, -35 - 3)$.

Ted Gore

The point reached by walking to the marble pillar and turning right is A, as in the picture on the next page. The point reached by walking to the sandstone pillar and turning left is B. The distance between the two stone pillars is r. Let

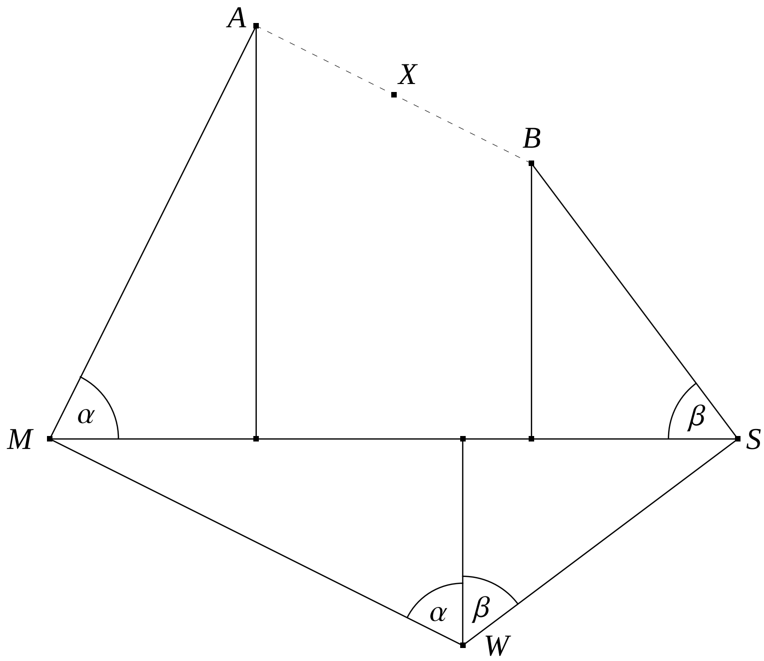
$$WM = MA = p \quad \text{and} \quad WS = SB = q.$$

Let M be the origin and let the line MS be along the x-axis. Then A is the point $(p \cos \alpha,\ p \sin \alpha)$ and B is the point $(r - q \cos \beta,\ q \sin \beta)$.

Let X be the mid point between A and B. So that X is the point

$$\left(\frac{p \cos \alpha + r - q \cos \beta}{2},\ \frac{p \sin \alpha + q \sin \beta}{2}\right).$$

From $\triangle WMS$, we have that $p \cos \alpha = q \cos \beta$ and $p \sin \alpha + q \sin \beta = r$. Then X is the point $(\frac{r}{2},\ \frac{r}{2})$, which depends only on r and not on the position of W.



Tony Forbes

This looks like a case where complex numbers might find useful employment. A right turn corresponds to multiplying by $-i$, and a left turn corresponds to multiplying by i. Referring to the picture, above, we have

$$A \;=\; M + (M - W)(-i), \qquad B \;=\; S + (S - W)\,i.$$

Therefore

$$\frac{A+B}{2} \;=\; \frac{S+M}{2} + \frac{(S-M)i}{2},$$

which is independent of W.

Chris Pile

Forget the wooden post and save yourself some walking! Go to the sandstone pillar and walk halfway towards the marble pillar. Turn right and walk the same distance. You are now at the treasure site.

Alan Davies

Suppose the stone pillars, S_1 and S_2, are distance d apart. If P is the wooden post and we walk to S_1, turn right and walk the same distance we arrive at the point A. Similarly following the instructions with regard to S_2 we arrive at the point B. Suppose that the midpoint of AB is T. Also, M and N are the perpendiculars from A and P respectively to the line S_1S_2 with $PN = a$ and $NS_1 = b$. For details, see Figure 1.

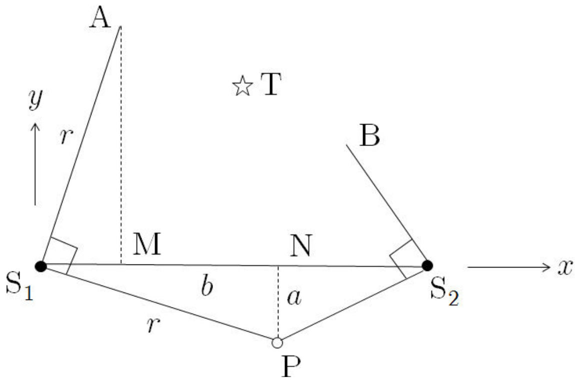


Figure 1: Two stone pillars, S_1 and S_2, and the wooden post, P.

It is clear that $\angle NS_1P = \angle MAS_1$ and hence that triangles NS_1P and MAS_1 are congruent. It follows that $S_1M = a$ and $MA = b$.

Hence, with origin at S_1 and x-axis along S_1S_2, the coordinates of A are (a, b). Similarly the coordinates of B are $(d - a, d - b)$.

If T is the mid-point of AB then the coordinates of T are $\left(\frac{1}{2}d, \frac{1}{2}d\right)$.

In Figure 1 we have placed P between S_1 and S_2.

If P is to the left of S_1 or to the right of S_2 the argument follows in a very similar manner to find, again, the coordinates of T as $\left(\frac{1}{2}d, \frac{1}{2}d\right)$.

That is, the position of T depends only on the distance between S_1 and S_2 and is independent of the position of the post, P.

So, we can place the post anywhere we like and always end up with the treasure.

Keith Brown

We regret to inform you that M500 Society member Keith Brown died on 1st September 2020. Our sympathy goes to his widow, Patricia.

Solution 231.4 – Four tans

Prove that

$$\tan 70^\circ \;=\; \tan 20^\circ + 2\tan 40^\circ + 4\tan 10^\circ.$$

Peter Fletcher

We begin by noting that

$$\tan(20^\circ) \;=\; \cot(70^\circ), \quad \tan(10^\circ) \;=\; \cot(80^\circ), \quad \cot(140^\circ) = -\cot(40^\circ),$$

$$\cot(140^\circ) \;=\; \frac{1-\tan^2(70^\circ)}{2\tan(70^\circ)} \quad \text{and} \quad \cot(80^\circ) \;=\; \frac{1-\tan^2(40^\circ)}{2\tan(40^\circ)}.$$

This means that we can write

$$\begin{aligned}
\tan(70^\circ) - \tan(20^\circ) &= \tan(70^\circ) - \cot(70^\circ) \\
&= \tan(70^\circ) - \frac{1}{\tan(70^\circ)} \\
&= \frac{\tan^2(70^\circ) - 1}{\tan(70^\circ)} \\
&= -2\left(\frac{1-\tan^2(70^\circ)}{2\tan(70^\circ)}\right) \\
&= -2\cot(140^\circ) \;=\; 2\cot(40^\circ).
\end{aligned}$$

Now subtracting $2\tan(40^\circ)$ from both sides,

$$\begin{aligned}
\tan(70^\circ) - \tan(20^\circ) - 2\tan(40^\circ) &= 2\cot(40^\circ) - 2\tan(40^\circ) \\
&= 2\left(\frac{1-\tan^2(40^\circ)}{\tan(40^\circ)}\right) \\
&= 4\cot(80^\circ) \;=\; 4\tan(10^\circ).
\end{aligned}$$

Therefore

$$\tan(70^\circ) \;=\; \tan(20^\circ) + 2\tan(40^\circ) + 4\tan(10^\circ).$$

Lastly, I should own up to finding this solution in Stack Exchange: `https://math.stackexchange.com/questions/1861288/`.

Tony Forbes

Here's an alternative if you find yourself confused and bewildered by all those complicated trigonometric identities. Let

$$Z \;=\; \frac{z^{14}-1}{z^{14}+1} - \frac{z^4-1}{z^4+1} - 2\,\frac{z^8-1}{z^8+1} - 4\,\frac{z^2-1}{z^2+1}.$$

To combine the denominators, note that $z^{14}+1$ is divisible by z^2+1, write $Z = A/B$, where

$$B \;=\; (z^{14}+1)(z^4+1)(z^8+1),$$

and use your favourite symbolic mathematics software to compute

$$A \;=\; -2(z-1)(z+1)(z^4-z^2+1)(3z^8+2z^6+4z^4+2z^2+3)(z^{12}-z^6+1).$$

Now let $z = e^{\pi i/18}$. Then $B \neq 0$ since $z^n = -1$ only if n is an odd multiple of 18. However, in the expression for A there is a factor

$$z^{12} - z^6 + 1 \;=\; e^{2\pi i/3} - e^{\pi i/3} + 1 \;=\; e^{2\pi i/3} + e^{4\pi i/3} + 1,$$

and this is zero since it is the sum of the cube roots of 1. Hence $Z = 0$. But

$$\tan 70^\circ - \tan 20^\circ - 2\tan 40^\circ - 4\tan 10^\circ \;=\; Z/i.$$

Problem 296.3 – Elliptic curve

Let a be a positive real number. Then the elliptic curve $y^2 = x(x^2-a^2)$ has two components, an unbounded curve that passes through $(a,0)$ and a closed 'bubble' that passes through $(0,0)$ and $(-a,0)$. What area does the bubble enclose?

Problem 296.4 – Cubic curve

Let a be a positive real number. Then the cubic curve $y^2 = x(x-a)^2/(3a)$ has a loop that passes through $(0,0)$ and $(a,0)$. What is its length and what area does it enclose?

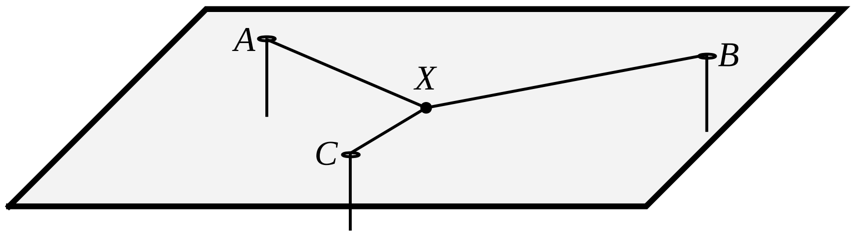
Unlike in Problem 296.3, above, the curve is not elliptic. Curiously, the denominator $3a$ in the definition of the curve has some significance. Remove it, and the loop length of the curve $y^2 = x(x-a)^2$ becomes much more difficult to compute. You are welcome to try!

Physical proof of the midpoint triangle theorem

Sebastian Hayes

The following physics based proof (?) of the mid-point triangular theorem is inspired by *The Mathematical Mechanic* by Mark Levi (see my review on page 17).

> 'Problem. Given three points, A, B, and C in the plane, find the point X for which the sum of the distances $XA+XB+XC$ is minimized.' (Levi, *The Mathematical Mechanic*, p. 6)



Start by drilling three holes at points A, B and C in a wooden board supported horizontally by trestles. Tie the three strings (or, better, silk cords) together and slip them each through a different hole. Attach equal weights W_A, W_B and W_C underneath the board. The point X where the three strings are joined together is free to move about across the board.

The work done by way of hole A is $W_A \times AX$ (since the weight W_A has been hauled up from its original position by a height equal to the length of AX).

The work done by way of hole B is $W_B \times BX$ (since the weight W_B has been hauled up from its original position by a height equal to the length of BX).

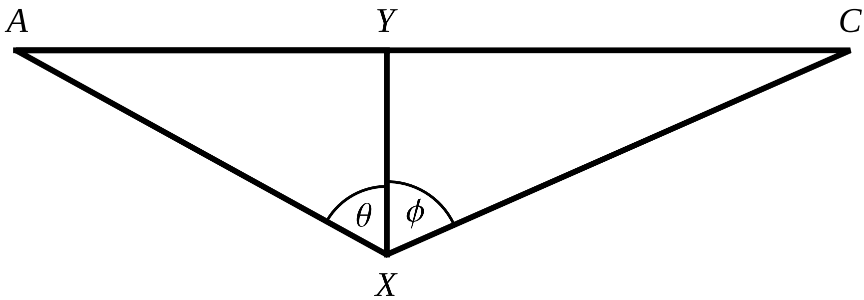
The work done by way of hole C is $W_C \times CX$ (since the weight W_C has been hauled up from its original position by a height equal to the length of CX).

The total work done on the system of three weights when taking them from their original freely hanging positions is

$$(W_A \times AX) + (W_B \times BX) + (W_C \times CX).$$

Now let $W_A = W_B = W_C = 1$ newton, so that the total work done is $AX + BX + CX$ newton-metres (or joules).

The system attains equilibrium when X, point where the three strings meet, settles into its final position. The configuration is thus the one where the potential energy of the system as a whole is at a minimum. The forces operating at X are the tensions in the strings, namely T_A, T_B, and T_C and they must cancel once the system as a whole is at rest.



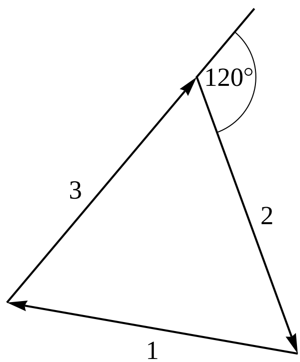
Therefore, resolving along AC, we have $T_A \sin\theta = T_C \sin\phi$.

But $T_A = T_B$ since we have equalized the weights hanging below A and C. Therefore $\theta = \phi$ (since both angles are less than $\pi/2$).

Resolving along CB, we have $T_C \sin\phi = T_B \sin\gamma$ and so $\phi = \gamma$. Resolving along AB, we have $T_A \sin\theta = T_B \sin\gamma$ and so $\theta = \phi = \gamma$. Thus, we have six equal angles meeting at X and adding to 2π or 360°. Each angle is thus 60° and the strings at X are separated by 120°.

Also $\triangle AXY = \triangle CXY$ (XY common, two angles equal counting right angle $AYX = CYX$). Therefore, $AX = CX = BX$ and X is the centre of a circle whose circumference passes through A, C and B. Moreover, the quantity $AX + BX + CX$ is the least possible since the potential energy of the system is a minimum (because it is in equilibrium).

We have thus shown, via a *mechanical* argument, that the point inside a (scalene) triangle that minimizes the distances to the three points of the triangle is the centre of a circle whose circumference passes through the points. Levi puts it more succinctly since he adds a second diagram, right, but I wanted to derive the result from first principles without even assuming the triangle of forces.

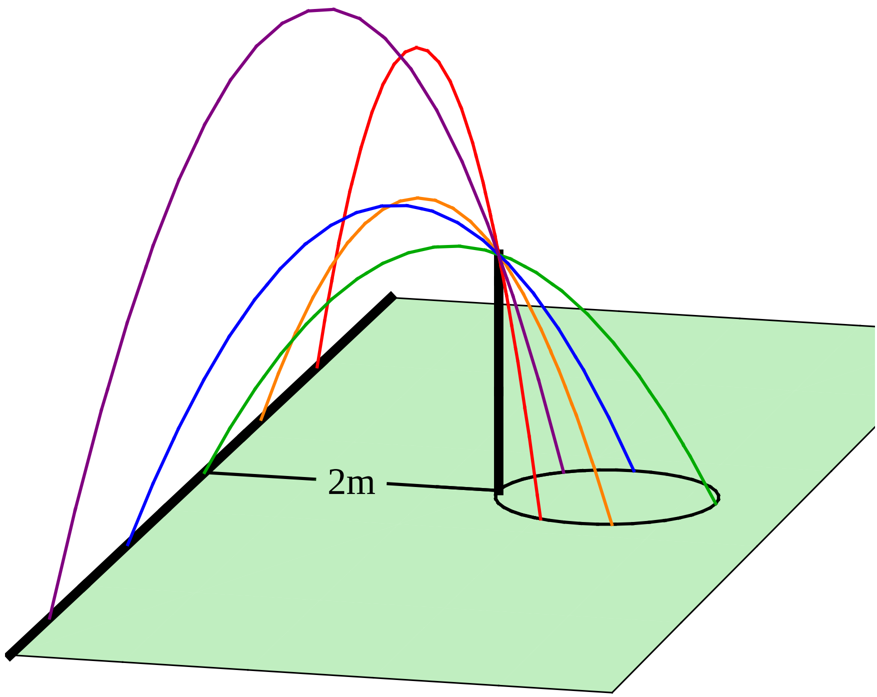


Levi summarizes: 'We endowed the sum of the distances $AX+BX+CX$ with the physical meaning of potential energy. Now, if this length/energy is minimal, then the system is in equilibrium. The three forces of tension acting on X then add up to zero and hence form a triangle (rather than an open path) if placed head-to-tail. This triangle is equilateral since the weights are equal and hence the angle between positive directions of these vectors is 120°.' *The Mathematical Mechanic*

Problem 296.5 – A line and a pole

A vertical pole of height h m is separated by 2 m from the y-axis and has its base on the (x, y)-plane. A projectile is fired with horizontal velocity component v m/s from somewhere on the y-axis and just grazes the top of the pole.

Assuming that gravity, $g = 9.80665\,\mathrm{m/s}^2$, acts vertically downwards and that the atmosphere has been removed, show that the projectile lands on the (x, y)-plane somewhere on a circle with radius $hv^2/(2g)$ m.



The Mathematical Mechanic

by

Mark Levi

Princeton University Press, 2012, ISBN 9780691154565

Sebastian Hayes

Which is more fundamental and more worthy of esteem, physics or mathematics? The ancient Greeks answered emphatically: Mathematics. Incredibly, even Archimedes, who not only founded both statics and hydrostatics but personally oversaw the defence of his native city, Syracuse, against the Romans, 'regarded as ignoble and sordid the business of mechanics and every sort of art which is directed to use and profit, [placing] his whole ambition in those speculations the beauty and subtlety of which is untainted by any admixture of the common needs of life' (Plutarch, *Life of Marcellus*).

Newton, on the other hand, sought to place 'natural philosophy' firmly on an inductive, experimental basis and was himself an indefatigable though extremely reckless experimentalist—he was lucky not to have lost the sight of his right eye by probing it with a 'bodkin' to test his ideas on vision. The 'hands-on' approach to natural science promoted by the Royal Society eventually paid huge dividends during the Industrial Revolution since it enabled Britain to overtake France, the fabled land of *savants* and *philosophes*. But in the 20th century and our own the mathematicians have so much taken their revenge that physics is in danger of being entirely swallowed up by the whale-like Mathematics Department. When a so-called 'physicist' – Tegmark – actually claims that 'all mathematical structures exist physically as well; every mathematical structure corresponds to a parallel universe', the wheel has turned full circle and we are back to Plato.

Enter Mark Levi. Not only does he argue that physical insight can, and should, be a tool of discovery in *mathematics*, he illustrates this with a panoply of 'physical proofs' of standard mathematical formulae. The range and complexity of the problems the author tackles is astounding. We don't just get the well-known 'lifeguard' 'proof' of Snell's Law but Pappus's Centroid Theorems, the Euler–Lagrange Equation, the Gauss–Bonnet Formula and a whole lot more. This is a timely and altogether laudable book that I hastened to order as soon as I heard of its existence. The approach taken is very much needed in schools today where the potential Faradays of tomorrow get put off by the dead weight of interminable formulae and finicky algebraic proofs.

'The real lesson', the author concludes, 'is that one should not focus solely on one or the other approach, but rather look at both sides of the coin. This book is a reaction to the prevalent neglect of the physical aspect of mathematics.'

Solution 293.4 – Triangular numbers

Let $T_n = n(n+1)/2$, the nth triangular number. Show that

$$(T_n - T_{n-1})^2 \;=\; T_n + T_{n-1} \;=\; n^2$$

and hence that $\sum_{n=1}^{N} n^3 = T_N^2$.

Stuart Walmsley

Given $T_n = n(n+1)/2$ and $T_{n-1} = n(n-1)/2$, we have

$$T_n + T_{n-1} \;=\; n^2, \qquad T_n - T_{n-1} \;=\; n.$$

Hence $(T_n - T_{n-1})^2 = n^2$ solving the first part of the problem:

$$(T_n - T_{n-1})^2 \;=\; T_n + T_{n-1} \;=\; n^2.$$

Further,

$$T_n^2 - T_{n-1}^2 \;=\; (T_n - T_{n-1})(T_n + T_{n-1}) \;=\; n \cdot n^2$$

giving

$$T_n^2 - T_{n-1}^2 \;=\; n^3. \tag{1}$$

To prove $\sum_{n=1}^{N} n^3 = T_N^2$, simplify the notation by letting $\sum_{n=1}^{N} n^3 = F_N$. Then using (1),

$$F_N - F_{N-1} \;=\; N^3 \;=\; T_N^2 - T_{N-1}^2$$

for all N. Also

$$F_1 \;=\; 1^3 \;=\; T_1^2;$$

that is, the result is true for $N = 1$ and hence $F_N = T_N^2$ for all N.

Writing out the first few values explicitly:

$$1 = 1, \quad 1 + 8 = 9, \quad 1 + 8 + 27 = 36, \quad 1 + 8 + 27 + 64 = 100.$$

Peter Fletcher

We have

$$T_n - T_{n-1} \;=\; \frac{n(n+1)}{2} - \frac{(n-1)n}{2} \;=\; \frac{n^2 + n - (n^2 - n)}{2} \;=\; n$$

and

$$T_n + T_{n-1} \;=\; \frac{n(n+1)}{2} + \frac{(n-1)n}{2} \;=\; \frac{n^2 + n + (n^2 - n)}{2} \;=\; n^2;$$

so the first result follows. Clearly

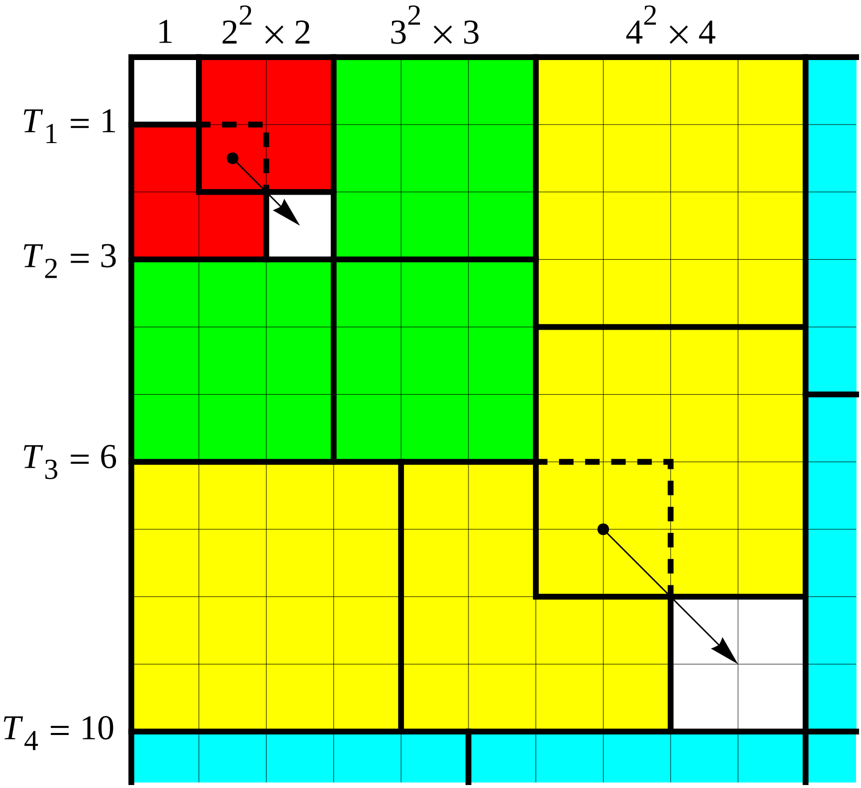
$$n^3 \;=\; n \cdot n^2 \;=\; (T_n - T_{n-1})(T_n + T_{n-1}) \;=\; T_n^2 - T_{n-1}^2;$$

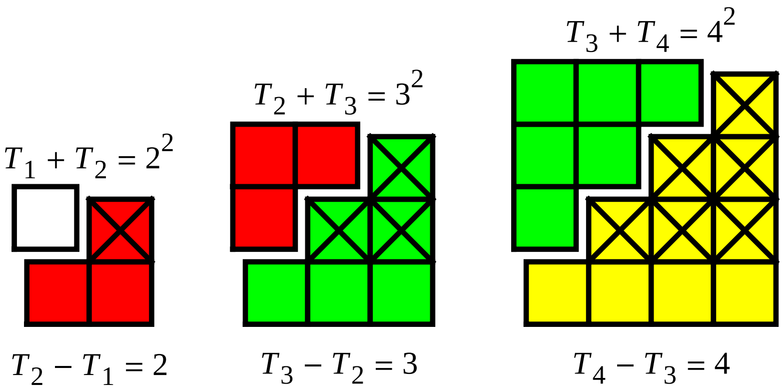
so we can write

$$\begin{aligned}\sum_{n=1}^{N} n^3 \;&=\; \sum_{n=1}^{n} \left(T_n^2 - T_{n-1}^2\right)\\ &=\; T_1^2 - T_0^2 + T_2^2 - T_1^2 + \cdots + T_N^2 - T_{N-1}^2\\ &=\; T_N^2 - T_0^2 \;=\; T_N^2.\end{aligned}$$

Chris Pile

The first few cases are neatly illustrated by pictures [see the colour version of M500 for best effect—TF].





Solution 169.4 – Functional inequality

> The function f takes a positive integer, n, as an operand, and must produce a positive integer result; that is, the function is undefined unless both n and $f(n)$ are positive integers. If, for any positive integer, n, it is always true that $f(n+1) > f(f(n))$, prove that $f(n) = n$ must follow as a consequence.

Peter Fletcher

The question does not specify that f is linear, so what if f is nonlinear and, e.g. $f(n) = n^p$ for $p > 1$?

Then $f(n+1) = (n+1)^p \not> n^{p^2} = f(f(n))$ for $n > 1$ (try $n = 2$ and $p = 2$) and we can rule out f being quadratic. We can obviously also rule out any higher powers.

There are no further nonlinear functions that have inputs and outputs both positive integers, so we are left with f being linear.

What if $f(n) = qn$ for $q > 1$? Then $f(n+1) = qn + q \not> q^2 n = f(f(n))$ for $n \geq 1$ (try $n = 1$ and $q = 2$).

What if $f(n) = n + r$ for $r \geq 1$? Then $f(n+1) = n + 1 + r \not> n + 2r = f(f(n))$ (try $n = 1$ and $r = 1$).

From the above, we can conclude that if $f(n) > n$ then the inequality cannot be satisfied. We cannot have $f(n) < n$ because $f(1) < 1$ can only be true if $f(n) \leq 0$, which is not allowed. We are therefore led to the conclusion that the only way to satisfy the inequality for all $n > 0$ and $f(n) > 0$ is if $f(n) = n$.

Solution 294.2 – Columns

Initially, columns 1, 2, 3, ... are all empty. For the integers $X = 1, 2, 3, \ldots$, add X to the appropriate column according to the following rules.

1. Starting at $C = 1$, add X to column C if this has less than two entries, or no two different entries in C add up to X.
2. If X could not be added, move on to the next column.

For example, after 1 to 9 are added, the columns look like this.

Column	1	2	3
	1	3	8
	2	5	9
	4	6	
	7		

Which column is 314159261234568 in?

Roger Thompson

Column 14.

Decimal 314159261234568 is 1112012100022222001222111221220 in ternary. An algorithm for calculating which column X goes in is as follows.

If $X = 2 \times 3^N$, then X is in column $N+1$. In all other cases, express X in ternary, and proceed as follows, noting that 'last' means 'least significant'.

If the last N digits are 1 followed by $N-1$ zeros, then X is in column N. For instance, if the last digit is 1, then X is in column 1, or if the last two digits are 10, then X is in column 2.

If the last N digits are 1 followed by M zeros followed by 2 followed by $N-M-2$ zeros, then X is in column N. For instance, if the last two digits are 12, then X is in column 2.

In the remaining cases, add a leading zero to the ternary expression for X, and set $Y = 0$. If the last N digits are 2 followed by M zeros followed by 2 followed by $N-M-2$ zeros, and this is true for no larger N, then $Y = N$. Ignoring the last Y digits, find the last 0 digit, and remove all digits before this. The column number for X is the number of digits in the remaining expression.

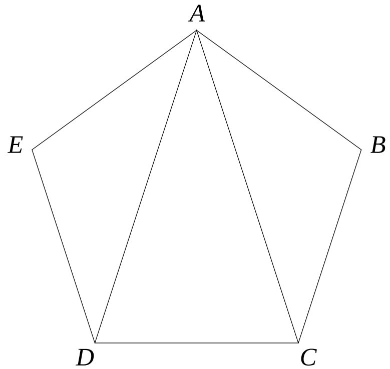
Observe that for sufficiently large X, column C contains $2 \times 3^{C-1}$, together with 2^{C-1} distinct integers modulo 3^{C-1}. It is relatively straightforward to prove the above by induction.

Solution 169.2 – Chords

If we have a regular pentagon inscribed in a circle with unit radius, show that the product of the chords from any vertex to each of the others is equal to 5. That is

$$|AB| \cdot |AC| \cdot |AD| \cdot |AE| \ = \ 5.$$

Show that a similar relation holds for any regular n-gon.



Peter Fletcher

Let $\omega_m = \exp(2\pi \mathrm{i} m/n)$, $m = 0, 1, \ldots, n-1$.

These n complex numbers lie at the vertices of a regular n-gon with the first vertex at $(1, 0)$. The number we are after is

$$\prod_{m=1}^{n-1} |1 - \omega_m|,$$

the product of the distances between $(1, 0)$ and each of the other vertices in turn. Consider

$$\prod_{m=0}^{n-1} (z - \omega_m) \ = \ z^n - 1.$$

This it true because the ω_m are the n roots of unity. If we divide this equation by $z - 1$, we get

$$\prod_{m=1}^{n-1} (z - \omega_m) \ = \ \frac{z^n - 1}{z - 1}.$$

We can recognise the RHS as being the sum of a geometric series with first term 1 and common ratio z, *i.e.*

$$\prod_{m=1}^{n-1} (z - \omega_m) \ = \ \sum_{k=0}^{n-1} z^k.$$

Now we put $z = 1$ and find that

$$\prod_{m=1}^{n-1} |1 - \omega_m| \;=\; \prod_{m=1}^{n-1} (1 - \omega_m) \;=\; \sum_{k=0}^{n-1} 1^k \;=\; n.$$

I should own up to getting some hints from
`https://math.stackexchange.com/questions/835278/`
`simple-prove-that-product-of-the-diagonals-of-a-polygon-n`.

Solution 179.1 – Two cars

> Two cars are heading towards one another from 100 miles apart on a straight road. The first is going 60 m.p.h. and the second is going 40 m.p.h. A fly starts at the front bumper of the first car and flies to the second and then back to the first, then back to the second, etc. Eventually there is a god-awful crash and our fly is squashed. If the fly can fly 50 m.p.h., how far does he fly before the smash?

Peter Fletcher

We know immediately from the information in the question that the cars will crash after 60 minutes. If we assume that the fly's velocity is 50 m.p.h. relative to the *road*, there is a simple answer. The fly travels for 50 miles and avoids the crash, which happens 10 miles away. There is no back-and-forth flying.

On the other hand, from the wording of the problem it seems reasonable that the fly should inherit the speed of whichever bumper it has launched itself from. Relative to the road its speed will therefore be 110 m.p.h. for the first car, 90 m.p.h. for the second. This is what we assume in what follows.

The speed of car 1 is 1 mile per minute (m.p.m.), the speed of car 2 is 40/60 m.p.m. or 2/3 m.p.m., the speed of the fly after launching itself from car 1 is $(60 + 50)/60$ m.p.m. or 11/6 m.p.m. and the speed of the fly after launching itself from car 2 is $(40 + 50)/60$ m.p.m. or 3/2 m.p.m.

Let the road on which the cars are travelling be labelled the x-axis with car 1 starting at $x = 0$ and car 2 starting at $x = 100$. We shall quote some fractions to four decimal places.

The time for the fly to fly from car 1 to car 2 is given by

$$\frac{11}{6}t_1 + \frac{2}{3}t_1 \;=\; 100$$

so that $t_1 = 2/5 \cdot 100 = 40$ minutes.

In the first 40 minutes, car 1 moves to $x = 40$ and car 2 moves to $x = 100 - 2/3 \cdot 40 = 73.3333$; the fly flies 73.3333 miles (which is also $11/6 \cdot 40$). The distance between car 1 and car 2 is now $73.3333 - 40 = 33.3333$ miles.

The time for the fly to fly back to car 1 is given by

$$t_2 + \frac{3}{2}t_2 \;=\; \frac{100}{3}$$

so that $t_2 = 2/5 \cdot 100/3 = 13.3333$ minutes.

In this 13.3333 minutes, car 1 moves to $x = 40 + 13.3333 = 53.3333$, car 2 moves to $73.3333 - 2/3 \cdot 13.3333 = 64.4444$; the fly flies $3/2 \cdot 13.3333 = 20$ miles. The distance between car 1 and car 2 is now $64.4444 - 53.3333 = 11.1111$ miles.

We are now effectively back at the starting position, but with 100 miles shortened to 11.1111 miles.

In subsequent flights from car 1 to car 2 and back to car 1, in each leg the distance between the cars would shorten by a factor of 3. In our equations for the time for the fly to fly from one car to the other, these distances are the RHS's; so each subsequent time would also go down by a factor of 3. We can summarize these times in the following table.

Time flying car 1 to car 2	Time flying car 2 to car 1
40.0000 min	$40.0000/3 = 13.3333$ min
$13.3333/3 = 4.4444$ min	$4.4444/3 = 1.4815$ min
$1.4815/3 = 0.4938$ min	$0.4938/3 = 0.1646$ min
$0.1646/3 = 0.0549$ min	$0.0549/3 = 0.0183$ min
$0.0183/3 = 0.0061$ min	$0.0061/3 = 0.0020$ min
$0.0020/3 = 0.0007$ min	$0.0007/3 = 0.0002$ min
Total 44.9999 min	Total 14.9999 min

It is clear that with more decimal places and more times, the two totals would get closer to 45 minutes and 15 minutes respectively. Therefore the total distance that the fly flies before being squashed is

$$\frac{11}{6} \cdot 45 + \frac{3}{2} \cdot 15 \;=\; 105 \text{ miles.}$$

Things you can't buy in shops – IV

Tony Forbes

Following on from the lists that we printed in M500 **278**, M500 **289** and M500 **293**, here is another selection of useful items you might want to ask for if your idle browsing in a shop gets interrupted by a salesperson who enthusiastically expresses a willingness to provide you with assistance.

1. An electric kettle that doesn't make a noise.
2. A pocket torch that does not switch itself on when carelessly placed in one's pocket.
3. A computer application that automatically sends 'GO AWAY' (or something similar) to the originator of any unsolicited message or advertisement that suddenly appears on your screen.
4. A pair of dice that when thrown always shows a sum of either 7 or 11 but not always with the same numbers. I have only ever seen this in films where, for example, our feisty heroine uses loaded dice to recoup her errant father's losses at the craps table by throwing in succession $\{1, 6\}$, $\{1, 6\}$, $\{5, 6\}$, $\{2, 5\}$. But how could loaded dice actually work? Obviously by some kind of quantum entanglement.
5. An electron microscope (to test yourself and your home for viruses).
6. Bananas that do not contain potassium-40. Surely here is an opportunity for an enterprising farmer to create a banana plantation where the trees are fed only with carefully prepared nutrients. A facility for enriching uranium for atomic bombs can easily be modified to work with potassium. The harmful K-40 gets removed to leave only the stable isotopes K-39 and K-41, which would then be used to manufacture suitable plant feed. The main advantage is that the final product—rather expensive bananas—would not be radioactive.
7. A cloverleaf mains connector for making a power lead for a computer.

The item that has been irritating me most of all is the last one. After searching on the high street and online I have to conclude that no such thing exists. The power lead for my computer has a cloverleaf connector at one end, a standard 3-pin plug at the other end, and 5 m of industrial-strength cable in between. It weighs 0.43 kg, it is bulky and there is no possibility of non-destructive disassembly into separate components. The people who make computer accessories evidently do not understand the basics of power transmission. The computer uses less than 120 W. Therefore the current required from the 240 V mains is less than 0.5 A. However, suppliers do not supply anything rated below 10 A. The obvious solution is to buy a suitable cable and wire it up myself to make a lightweight, compact power lead. Unfortunately, unless someone can prove otherwise, that particular option is, like the stable banana, currently unavailable.

Contents

04.00

Front cover Graphical representation of a group divisible design with block size 5 and type 10^{23} [https://arxiv.org/abs/2006.15734].