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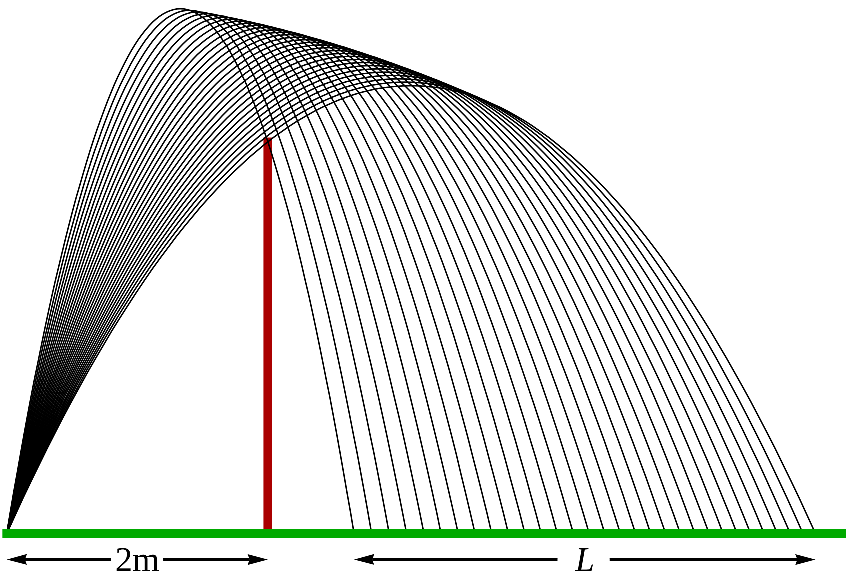

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M500 Revision Weekend 2021 In the light of the continuing coronavirus pandemic, the M500 committee has reluctantly decided that at this point we should not be taking bookings for the 2021 Revision Weekend, planned for the 7th – 9th May. We will review the decision in early spring 2021, and keep you informed. We have thoroughly researched the possibility of running an electronic weekend equivalent but have decided it would not be possible to do so.

Who discovered Snell's law?

Alan Davies

1 Introduction

As a tutor at one of the 1980 M101 Summer Schools, I presented the Friday morning lecture *The mathematics of rainbows.* It was written by John Mason and he based it on the book by Carl Boyer (1959). When I got home I bought a copy of the book and wrote a lecture/workshop on rainbows, which I have developed continuously and presented many times to a wide variety of audiences over the past forty years. While reading Boyer's book I was a little surprised to find out that there were other contributions to the understanding of the law of refraction that I had learned at school. However, to me it was still Snell's law. Sometime later my wife and I presented a series of workshops in schools in south west France. When I showed the slide giving Snell's law in the usual form

$$\sin i/\sin r \;=\; k \tag{1}$$

I was very surprised at the reaction "non! ce n'est pas vrai, c'est la loi de Descartes". So, to avoid an international incident, I used the French version for the remaining workshops. That sparked an interest to find out more and on seeing the article in *Physics World* (Kwan *et al.* 2002) I started to look further.

2 Early thoughts on refraction

The use of highly polished curved mirrors, to focus sunlight to form fire, has been known for at least three thousand years. By the fifth century BC the Greeks had a knowledge of burning glasses as rudimentary lenses, again to focus sunlight to form fire. However, the phenomenon of refraction remained unconsidered for some centuries.

A good place to start is with Euclid (*c.*325BC–*c.*265BC). He is, of course, best known for his work on geometry, which he presented in his treatise *Elements.* He wrote a text, *Optics*, in which he described light as moving in a straight line but made no mention of refraction. Ptolemy (*c.*85–*c.*165) built on Euclid's ideas and produced a text on optics in which he describes the processes of reflection and refraction. He seems to understand that both reflection and refraction take place in the plane containing the incident ray and the normal, the dashed line in Figure 1. He demonstrates the law of reflection using a *goniometer*, a copper disc with graduations, to measure the incident and reflected angles simultaneously. For refraction he describes an experiment in which a coin is placed at the bottom of a *baptistir*, a type of bowl. He then explains that the observer's eye is placed in such a position that the coin can't be seen. Water is poured gently into the vessel until the coin just becomes visible. This experiment is easy to do at home with a large mug and often causes surprise for those who haven't seen it before. He then goes on to describe a quantitative method using the goniometer of his reflection experiment to measure the incident and refracted angles. Ptolemy obtained the values shown in Table 1. He also published values of air/glass and water/glass. Very good descriptions of Ptolemy's experiments can be found in Pederson (1993) and in Callie Lane's essay (2018), where she provides some very helpful diagrams. The numerical values in Table 1

Table 1: Ptolemy's incident and refracted angles for air/water

n	1	2	3	4	5	6	7	8
i_n (°)	10	20	30	40	50	60	70	80
r_n (°)	8	15.5	22.5	29	35	40.5	45.5	50

were used subsequently up until a mathematical law was developed in the seventeenth century.

It is often said that the Greeks didn't do experiments but, according to Feynman *et al.* (2011), Ptolemy couldn't have obtained these values without performing many observations. The values are not the result of careful measurement for each angle but are interpolations from a few measurements. It is not difficult to see from the second difference values (-0.5) in the table that Ptolemy's experimental results imply a quadratic relation and Godet (2013) writes it in the form

$$r_n \;=\; nr_1 - 0.5\,\frac{n\,(n-1)}{2} \tag{2}$$

from which we can write r as a quadratic function of i as

$$r \;=\; ai - bi^2$$

with $a \;=\; 0.1r_1 + 0.025$ and $b \;=\; 0.0025$.

Ptolemy would not have written the rule in this form because such algebraic representation was still some time away. However, he did believe that there was a relationship between i and r and that his table showed this relation numerically. Also, since the result is dependent on the value of r_1, errors here will be felt in all values. Using the correct sine law yields a value $r_1 \approx 7.5$ and Ptolemy's value is out by about seven percent but the others are remarkably accurate with an average error of less than two percent. The earliest trigonometric ideas were developed some two hundred years before Ptolemy and he would have known them but he didn't relate them to refraction. Ptolemy was aware that refraction occurred when light passed from a rare medium into a dense medium and *vice versa* but he had no concept of a physical parameter such as the refractive index.

The Islamic scholar Abu Said Ibn Sahl (940–1000) published *On the burning instruments* in which he developed what we would recognise as the sine law. He followed Ptolemy's optical ideas, together with the theory of conics, to show that parallel light beams are focused on a burning point by a hyperboloidal lens. He obtained a constant geometrical ratio, which we would now see immediately as the ratio of sines; see equation (1). Ibn Sahl didn't provide any experimental data. According to Godet (2013), he was interested only in the theory of burning glasses and probably did not realise that this constant ratio is a general law and that the constant is a physical parameter which we know as the *refractive index*. During this Golden Age of Islamic science Ibn al-Haytham (*c*.965–*c*.1040), also known as Alhazen, was a leading mathematician and astronomer who, some four hundred years before the Renaissance, proposed the scientific method of systematic observation, measurement and experiment together with the formulation, testing and modification of hypotheses. He translated Ibn Sahl's work but missed the important law. He also translated Ptolemy's work and thus perpetuated the wrong law for another six hundred years.

3 Into the 17$^{\text{th}}$ century

All the usual suspects: Johannes Kepler (1571–1630), René Descartes (1596–1650), Christiaan Huygens (1629–1695) and Isaac Newton (1643–1727) had something to say about refraction. Let's start with Kepler: he developed the formula

$$i - r = \mu i \sec r,$$

where μ is a constant. How he came to this complicated result is not clear, however for small angles it approximates to the sine law if $\mu = 1 - 1/k$. Worse still, the formula is not symmetric in i and r as it should be since the path of a ray of light between two points should be the same whichever of them is the source. Kepler's main interest was in the development of lenses and he refers to the burning point as a fireplace leading us, from Latin, to the word focus. Descartes, in his 1637 *Dioptrique*, described the sine law which he developed from some physical mechanical processes using a conservation of momentum approach. He imagined that light behaves like small particles which experience a force when reaching the boundary of the two media. His argument is very difficult to accept, especially as he believed that light travelled instantaneously, i.e. at infinite speed, and that it increased speed on entering a denser medium! Newton also had the erroneous view that light travelled faster in denser materials; however his main interest was in the decomposition of white light into its constituent colours and not in the description of refraction *per se*. He favoured a particle theory of light and explained light phenomena in terms of mechanical processes. By the mid 1660s he was clearly familiar with the sine law. In 1678 Huygens stated his principle of light as a wave: every point on a wavefront is itself the source of a secondary wave. He used this principle to obtain the sine law in 1673.

Now we come to two lesser-known characters. Firstly we have the Dutch astronomer and mathematician Willebrord Snellius (1580–1626) who, in 1621, developed the eponymous sine law using a geometrical argument (Lane 2018). However, he didn't publish it and it remained unpublished until after his death. Secondly we have the English mathematician and astronomer Thomas Harriot (1560–1621), whose unpublished work of 1602 shows that he too had the sine law.

At this stage there was no concept of the refractive index. Newton looked for a characteristic property of a body called the refractive power in a thin layer near the surface of the medium. He then classified materials according to the ratio of the refractive power to the density. This refractive power is related to the refractive index but is not the same thing. The refractive index, as shown in equation (1), is given in terms of the speeds of light, v_i for the incident ray and v_r for the refracted ray, as

$$k = v_i/v_r. \tag{3}$$

However, that was not known at the time. The constant in the law was just that: a constant and different observers had different ways of expressing it. For example, Newton wrote it as the ratio of two integers, others wrote it as a fraction with a fixed numerator and others with a fixed denominator, usually 1. There was no indication that the light speed had anything to do with refraction. It had to wait until Huygens's wave approach for the speed of light to be associated with refraction.

Nevertheless, we can now attempt to answer the question posed in the title of this article. Contenders for discovery of the law, in chronological order:

- Ibn Sahl *c*.984
- Harriot 1602
- Snell 1621
- Descartes 1637

I leave it to the reader to judge. In France it will always be 'la loi de Descartes' and elsewhere 'Snell's law' although, perhaps, in the UK we should call it 'Harriot's law'. Whichever we choose we shall have an example of Stigler's law of eponymy (Stigler 1980), which states 'No scientific discovery is named after its original discoverer'. Snell's law is not alone here, the Wikipedia page (2020) lists some one hundred and twenty such incorrectly attributed laws, including that of Stigler.

There were two schools of thought concerning the nature of light. Newton, and others, believed that light was a beam of particles and this led to the explanation of refraction in geometrical terms. However, others such as Huygens believed in the wave nature of light. In his 1678 *Traité de la lumière* Huygens proposed that every point reached by a light disturbance becomes itself the source of a wave and this became known as *Huygens's Principle*. Using this principle he was able to show that refraction can be explained by treating light as a wave and thus relating the physical process to the property of light itself. The fact that the incident and refracted rays lie in the same plane also follows from Huygens's analysis of light as a wave.

A mathematical development of the sine law was first given by Pierre de Fermat (1607–1665). His *Principle of least time*, proposed in 1662, states that light travels between two points in such a way that it takes the minimum time. Strictly speaking it says that the time taken is stationary with respect to variations in the path but minimum time is sufficient for us here. Suppose that light travels from the point A, with speed v_1 in a rare medium, to the point B, with speed v_2 in a dense medium, see Figure 1. Also, that it crosses the interface at the point P with coordinates (x, b).

The time, T, to travel from A to B is given by

$$T = \frac{1}{v_1}\sqrt{a^2 + x^2} + \frac{1}{v_2}\sqrt{b^2 + (d - x)^2}.$$

So that

$$\frac{dT}{dx} = \frac{1}{v_1}\frac{x}{\sqrt{a^2 + x^2}} - \frac{1}{v_2}\frac{(d - x)}{\sqrt{b^2 + (d - x)^2}}.$$

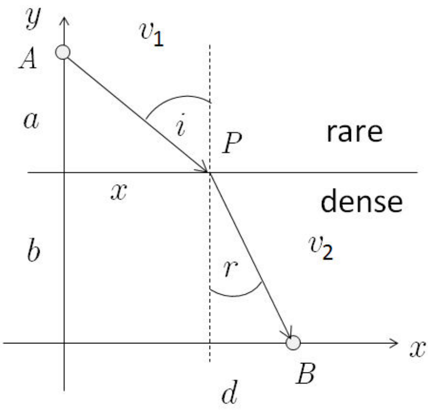

Figure 1: Light traveling from point A to point B with speeds v_1 and v_2 in the rare and dense medium respectively

For minimum time we require $dT/dx = 0$ from which it follows that

$$\frac{1}{v_1}\frac{x}{\sqrt{a^2+x^2}} = \frac{1}{v_2}\frac{(d-x)}{\sqrt{b^2+(d-x)^2}}$$

and hence that

$$\frac{\sin i}{\sin r} = \frac{v_1}{v_2} \tag{4}$$

and so establishing the sine law.

For the specific establishment of a physical parameter associated with the refraction properties of materials we have to wait until the nineteenth century.

4 Beyond the 17th century

The main impetus in optics was the development of better and better lenses. A major problem was chromatic aberration in which different colours are focused at different points. John Dolland (1706–1761) was one of the earliest to make an achromatic lens in 1757 which he made by combining crown and flint glass lenses. This combination of different materials made it very important to be able to understand the optical properties of different materials as opposed to simply using the ratio of two sines.

Thomas Young (1773–1829), in a series of lectures in 1807, introduced the concept of the refractive index and the parameter would be a particular characteristic property of each transparent material. He also stated that at the interface between two materials the combined index of refraction would be the quotient of the two respective indices. Independently, Joseph von Fraunhofer (1787–1826), in 1814, developed the same idea and introduced the symbol n. If two different materials have indices n_1 and n_2 then the parameter k in equation (1), the index of refraction, is given by

$$k = \frac{n_1}{n_2}. \tag{5}$$

This idea of Young and Fraunhofer was important because it gave each medium its own refractive property rather than treating pairs of connected media as single entities.

We define the refractive index of a medium as

$$n = \frac{c}{v}, \tag{6}$$

where c is the speed of light in vacuum and v is the speed of light in the medium. Equation (5) then yields the refractive index at the interface of two materials in terms of the ratio, v_1/v_2, of the velocities.

The refractive index given by equation (5) allows us to write Snell's law in a symmetric form. Consider the set-up in Figure 1. The law should be the same for light travel from B to A as it is from A to B. If, in Figure 1, we write θ_1 for i and θ_2 for r then equations (1) and (5) yield

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2) \tag{7}$$

and the law shows the correct symmetry between incident and refracted ray. Indeed, in equation (7) it doesn't matter which is the incident angle and which is the refracted angle.

With the refractive index given by equation (6) we can write it in terms of the electromagnetic properties of the material. In 1861, James Clerk Maxwell (1831–1879) published his eponymous equations from which he deduced that the speed of light in a vacuum was related to the electrical permittivity, ϵ_0, and the magnetic permeability, μ_0, by the equation

$$c = (\epsilon_0 \mu_0)^{-1/2}. \tag{8}$$

Now we can use equations (6) and (8) to relate the refractive index to the properties of a material, in which light speed is v and whose permittivity and permeability are ϵ and μ respectively, as

$$n = \frac{(\epsilon\mu)^{-1/2}}{(\epsilon_0\mu_0)^{-1/2}}$$

and this is usually written in terms of the relative permittivity and permeability as

$$n = (\epsilon_r\mu_r)^{-1/2}, \tag{9}$$

where $\epsilon_r = \epsilon/\epsilon_0$ and $\mu_r = \mu/\mu_0$.

That's as far as we shall go here. However, equation (9) is the starting point for the modern development of refractive optics beginning in the twentieth century. The equation allows for the possibility of a negative refractive index. Also, we mentioned earlier that colours are refracted at different angles. This is, of course, true for the electromagnetic spectrum in general and from the wave nature we can relate the refractive index to frequency and for some materials it is convenient to write the wave nature in complex form and hence have a complex refractive index. The study of such materials is currently an important area of research.

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Solution 294.9 – Mod 240

> Show that $(P^2 - Q^2)\sqrt{PQ} \equiv 0 \pmod{240}$ if $P - Q$ is even and PQ is a square.

Ted Gore

We have $240n = (P^2 - Q^2)\sqrt{PQ} = (P - Q)(P + Q)\sqrt{PQ}$, where $P - Q$, $P + Q$ are both even and PQ is a square.

Let $P - Q = 2a$, let $P + Q = 2b$ and let $\sqrt{PQ} = c$. Then $P = a + b$ and $Q = b - a$, so that $PQ = c^2 = b^2 - a^2$.

Thus c, a and b are the sides of a right-angled triangle with b as the hypotenuse. Since every Pythagorean triple has at least one component divisible by 3, one by 4 and one by 5 it is always the case that $2a2bc$ is divisible by 240. The table gives solutions for c from 3 to 10.

c	a	b	n
3	4	5	1
4	3	5	1
5	12	13	1
6	8	10	8
7	24	25	70
8	6	10	2
9	12	15	27
10	24	26	104

Problem 297.1 – Matrix square root

Reinhardt Messerschmidt

Suppose $n \in \{2, 3, 4, \ldots\}$ and α, β are real numbers such that $\alpha - \beta \geq 0$ and $\alpha + (n-1)\beta \geq 0$. Let M be the $n \times n$ matrix whose diagonal entries are all α and whose other entries are all β. Find a square root for M, i.e. a matrix S such that $S^2 = M$.

Problem 297.2 – Two trigonometric integrals

Tony Forbes

Evaluate

$$\int \frac{\sin^7 x}{\cos^5 x}\,dx \quad \text{and} \quad \int \frac{\sin^5 x}{\cos^7 x}\,dx.$$

In the opinion of me exactly one of these is not easy.

Solution 292.6 – Tracks

> This is like Problem 257.4 – Tracks (which was answered by Reinhardt Messerschmidt in M500 **259**) but the final question is different. Your MP3 player has tracks, T_0, T_1, ..., T_n of lengths t_0, t_1, ..., t_n respectively. The device selects tracks at random and plays them in full. The probability of track T_i getting selected is proportional to t_i. What is the expected minimum playing time to hear each track at least once?

Reinhardt Messerschmidt

General case

Assume that the device stops at the moment when each track has been played in full at least once. Let

(i) E be the expected total playing time;

(ii) $p_i = t_i/(t_0 + t_1 + \cdots + t_n)$, i.e. p_i is the probability that T_i is selected;

(iii) $A_i = \{0, 1, \ldots, n\} - \{i\}$;

(iv) $A_{ij} = \{k \in A_i : k < j\}$.

Consider the following event:

(i) T_i is the last track that is played;

(ii) m tracks are played in total before T_i, i.e. $m \geq n$;

(iii) for each $j \in A_i$, track T_j is played m_j times before T_i, i.e. $m_j \geq 1$ and $\sum_{j \in A_i} m_j = m$.

If this event occurs, then the total playing time is

$$t_i + \sum_{j \in A_i} m_j t_j.$$

The probability of the event is

$$\left[\prod_{j \in A_i} \binom{m - \sum_{k \in A_{ij}} m_k}{m_j} {p_j}^{m_j} \right] p_i.$$

It follows that

$$E = \sum_{i=0}^{n} E_i;$$

where

$$E_i = \sum_{m=n}^{\infty} \sum_{\substack{(m_j)_{j\in A_i} \\ m_j \geq 1 \\ \sum_{j\in A_i} m_j = m}} \Big[t_i + \sum_{j\in A_i} m_j t_j\Big]\Big[\prod_{j\in A_i} \binom{m - \sum_{k\in A_{ij}} m_k}{m_j} {p_j}^{m_j}\Big] p_i.$$

It is not clear if this expression can be simplified. We will look at some special cases.

Generalized geometric series

We will use the following formula repeatedly: if r is a positive integer and a, b, p are real numbers with $|p| < 1$, then

$$\begin{aligned}
\sum_{m=r}^{\infty} (a + bm)p^m &= a\sum_{m=r}^{\infty} p^m + b\sum_{m=r}^{\infty}\sum_{k=1}^{m} p^m \\
&= a\sum_{m=r}^{\infty} p^m + b\Big(\sum_{k=1}^{r-1}\sum_{m=r}^{\infty} p^m + \sum_{k=r}^{\infty}\sum_{m=k}^{\infty} p^m\Big) \\
&= \frac{ap^r}{1-p} + b\Big(\sum_{k=1}^{r-1} \frac{p^r}{1-p} + \sum_{k=r}^{\infty} \frac{p^k}{1-p}\Big) \\
&= \frac{ap^r}{1-p} + \frac{b(r-1)p^r}{1-p} + \frac{bp^r}{(1-p)^2}. \qquad (*)
\end{aligned}$$

Two tracks

Suppose that the device has two tracks. Without loss of generality, assume that $t_1 = 1$; and let $t = t_0$. In this case,

$$E_0 = \sum_{m=1}^{\infty} (t + m)\Big(\frac{1}{t+1}\Big)^m \Big(\frac{t}{t+1}\Big)$$

and

$$E_1 = \sum_{m=1}^{\infty} (1 + mt)\Big(\frac{t}{t+1}\Big)^m \Big(\frac{1}{t+1}\Big).$$

By $(*)$,

$$\sum_{m=1}^{\infty} t\left(\frac{1}{t+1}\right)^m\left(\frac{t}{t+1}\right) = \frac{t^2}{t+1}\sum_{m=1}^{\infty}\left(\frac{1}{t+1}\right)^m$$
$$= \frac{t^2}{t+1}\left(\frac{1}{t+1}\right)\left(\frac{t+1}{t}\right) = \frac{t}{t+1};$$

and

$$\sum_{m=1}^{\infty} m\left(\frac{1}{t+1}\right)^m\left(\frac{t}{t+1}\right) = \frac{t}{t+1}\left(\frac{1}{t+1}\right)\left(\frac{t+1}{t}\right)^2 = \frac{1}{t};$$

therefore

$$E_0 = \frac{t}{t+1} + \frac{1}{t}.$$

Similarly,

$$E_1 = \frac{t}{t+1} + t^2;$$

therefore

$$E = \frac{t^2 + (t+1) + t^2 + t^3(t+1)}{t(t+1)} = \frac{t^4 + t^3 + 2t^2 + t + 1}{t(t+1)}.$$

For tracks of equal length, i.e. $t = 1$, we have $E = 3$. The function E achieves a minimum of 2.7419 at $t = 0.6928$ (with rounding to four decimal places). The asymptotic behaviour of E is

$$E \sim t^2 \text{ as } t \longrightarrow \infty, \qquad E \sim t^{-1} \text{ as } t \longrightarrow 0;$$

by which we mean

$$\lim_{t\to\infty}(E/t^2) = 1, \qquad \lim_{t\to 0}(E/t^{-1}) = 1.$$

Three tracks, two of the same length

Suppose that the device has three tracks, with two of the same length. Without loss of generality, assume that $t_1 = t_2 = 1$; and let $t = t_0$. In this case,

$$E_0 =$$
$$\sum_{m=2}^{\infty}\sum_{m_1=1}^{m-1}(t + m_1 + (m - m_1))\binom{m}{m_1}\left(\frac{1}{t+2}\right)^{m_1}\left(\frac{1}{t+2}\right)^{m-m_1}\left(\frac{t}{t+2}\right)$$

and

$$E_1 \;=\; \sum_{m=2}^{\infty}\sum_{m_0=1}^{m-1}(1+m_0t+(m-m_0))\binom{m}{m_0}\left(\frac{t}{t+2}\right)^{m_0}\left(\frac{1}{t+2}\right)^{m-m_0}\left(\frac{1}{t+2}\right).$$

By symmetry, $E_2 = E_1$. The inner sums in the expressions for E_0, E_1, E_2 can be simplified using the binomial theorem. The outer series can then be simplified using $(*)$. The algebra involved in adding up the results is very messy; using a symbolic computer algebra system is recommended. The final answer is

$$E \;=\; \frac{3t^5+9t^4+12t^3+22t^2+12t+8}{2t(t+1)(t+2)}.$$

For tracks of equal length, we have $E = 5\frac{1}{2}$. The function E achieves a minimum of 5.2627 at $t = 0.7632$. The asymptotic behaviour of E is

$$E \sim \tfrac{3}{2}t^2 \text{ as } t \longrightarrow \infty, \qquad E \sim 2t^{-1} \text{ as } t \longrightarrow 0.$$

Ted Gore

It is useful to think of the process as a series of $n+1$ stages. Each stage consists of several tracks (possibly 0) which have been played previously and ends with a track being played for the first time.

We can think of the selection of previously played tracks as a series of failures (the selected track has already been played) followed by a success. Such a process follows a geometric distribution: $(1-p)^{k-1}p$, where p is the probability of a success. For each stage, the expected value of the number of failures before a success is $1/p-1$.

Let R be the run time of the whole album. For each stage i let P_i be the run time of the tracks that have been played at least once and Q_i the time of the tracks yet to be played. The probability of a success is Q_i/R and the expected number of repeat tracks played is $R/Q_i - 1 = P_i/Q_i$.

We use Reinhardt Messerschmidt's answer to problem 257.4 to calculate, at each stage, the playing time of each replay of a previously played track.

For each track j that has already been played at least once we have that the proportion of playing time for that track is T_j^2/P^*, where

$$P^* \;=\; \sum_j T_j^2.$$

And the estimated playing time for each repeated track is $\sum_j T_j^3/P^*$. Then for stage i the total time taken for repeated tracks is

$$U_i \;=\; \frac{P_i}{Q_i} \cdot \sum_j \frac{T_j^3}{P^*}.$$

Let the total time of stages 0 to N be X_s, where we include the time for each track being played for the first time. Then

$$X_s \;=\; R + \sum_{i=1}^{N} U_i.$$

Now X_s is the calculated running time of the album, with repeats, for a particular permutation s of the track times and there will be $(N+1)!$ permutations.

The average playing time for a given combination of track times is therefore $\sum_{X_s} /(N+1)!$.

In order to check the validity of the calculation, I wrote a program that simulates the process laid down in the question many times while randomly changing the permutation of the given track times in order to arrive at an average. A second part of the program performs the calculation for comparison. This second part includes the assumption that the order in which each track is played for the first time is the order of the relevant permutation.

Using randomly generated track times that average 3.5 (the average length of pop songs according to the internet) with $N = 4$, I ran the program 10 times with the following results.

Average time by simulation = 42.37.

Average time by calculation = 41.51.

We can get some idea of the spread of times depending on the particular permutation

Average calculated time for a permutation with times arranged in strictly ascending order = 37.94.

Average calculated time for a permutation with times arranged in strictly descending order= 48.19.

These spreads suggest a refinement to the calculation. They indicate that the more probable the permutation is to arise, the greater is the time it takes. We apply Reinhardt's result to the X_s values of all permutations giving a final average calculated time of 42.54. The average difference between the simulated time and calculated time without this refinement was 0.86 minutes; with the refinement 0.17.

Taking the permutation providing the shortest time for each run, the average is 33.11 and the longest time is 54.22. The permutation giving the least time tended to correlate more highly with the ascending order and the highest with the descending.

Re: Problem 276.6 – Alphabetic sum

$$\text{Compute } \sum_{n=1}^{24}\sum_{i=1}^{n}\left((X-A)^{1-i}(X-B)^{2-i}\ldots(X-Z)^{26-i}\right)^{24-n}.$$

It looks as if this problem should have some kind of hypothesis concerning the nature of the variables A, B, …, Z. Well, in fact there originally were conditions but I (TF) deliberately omitted them in order to invite readers to study the problem generally. Such analysis would be very interesting. Ideally—but admittedly it would be somewhat demanding—one should plot the expression as a function of A, B, …, Z and look at the graph.

With no assumptions MATHEMATICA gives up and returns the answer `Indeterminate` because it does not assign a value to 0^0. However, if you force $0^0 = 1$, a sensible choice if you consider the problem to be set in the domain of discrete mathematics, then MATHEMATICA returns `24`, presumably by treating the variables as general and assuming that A, B, …, W, Y, $Z \neq X$.

On the other hand, if you are working with matrices, then MATHEMATICA is quite happy with $[0]^0$: `MatrixPower[{{0}},0] = {{1}}`.

While we are on the subject: …

Re: Problem 288.3 – Digit powers

> Show that $n = 1$ and $n = 3435$ are the only instances of $\sum_{d,\ d \text{ runs through the digits of } n} d^d = n$. Or find another.

… One or two readers discovered $n = 438579088$ somewhere in the popular mathematical literature. Unfortunately it doesn't work because the d^d sum is 438579089. Recall that in arithmetic an empty sum is 0, but an empty product such as $0^0 = \prod_{j=1}^{0} 0$ or $0! = \prod_{j=1}^{0} j$ is equal to 1.

Solution 294.1 – Quartic

Show that $\tan^2(\pi t/16)$, $t = 1, 3, 5, 7$, are the roots of

$$x^4 - 28x^3 + 70x^2 - 28x + 1 \;=\; 0.$$

Stuart Walmsley

To simplify notation let $t_t = \tan(\pi t/16)$, $c_t = \cos(\pi t/16)$, $s_t = \sin(\pi t/16)$. Then it is required to show that

$$t_k^8 - 28t_k^6 + 70t_k^4 - 28t_k^2 + 1 \;=\; 0, \qquad k = 1, 3, 5, 7.$$

Using $t_k = s_k/c_k$ (and reversing the order) this is replaced by

$$c_k^8 - 28c_k^6 s_k^2 + 70c_k^4 s_k^4 - 28c_k^2 s_k^6 + s_k^8 \;=\; 0.$$

Using

$$(c_k^2 + s_k^2)^4 \;=\; c_k^8 + 4c_k^6 s_k^2 + 6c_k^4 s_k^4 + 4c_k^2 s_k^6 + s_k^8 \;=\; 1$$

this becomes

$$1 - 32c_k^6 s_k^2 + 64c_k^4 s_k^4 - 32c_k^2 s_k^6 \;=\; 0$$

and hence

$$1 - 32c_k^2 s_k^2 (c_k^4 - 2c_k^2 s_k^2 + s_k^4) \;=\; 0,$$

which can be factored

$$1 - 32c_k^2 s_k^2 (c_k^2 - s_k^2)^2 \;=\; 0.$$

Then using $c_{2k} = c_k^2 - s_k^2$ and $s_{2k} = 2c_k s_k$ this becomes $1 - 8s_{2k}^2 c_{2k}^2 = 0$. And since $s_{4k} = 2c_{2k}s_{2k}$, we have $1 - 2s_{4k}^2 = 0$; so that $s_{4k}^2 = 1/2$.

To summarize, if $\tan 2\theta$ is a root of

$$x^4 - 28x^3 + 70x^2 - 28x + 1 \;=\; 0,$$

then $\sin^2 4\theta = 1/2$. This equation is satisfied when

$$4\theta \;=\; \pi/4, \;\; 3\pi/4, \;\; 5\pi/4, \;\; 7\pi/4$$

and

$$\theta \;=\; \pi/16, \;\; 3\pi/16, \;\; 5\pi/16, \;\; 7\pi/16$$

which proves the required result.

Peter Fletcher

If we substitute $x = \sec^2(\theta) - 1$ in the given quartic and tidy up the resulting expression, we obtain

$$\frac{128\cos^8(\theta) - 256\cos^6(\theta) + 160\cos^4(\theta) - 32\cos^2(\theta) + 1}{\cos^8(\theta)}.$$

We could now use de Moivre's identity to write this equation in terms of cosines of multiple angles, but it's much easier to simply put each of the four powers of $\cos(\theta)$ into Wolfram Alpha and click the '='. Plugging the four sums so-obtained into the numerator of the above equation turns it into

$$\frac{\cos(8\theta)}{\cos^8(\theta)}.$$

Now putting $\theta = \pi t/16$ in the numerator makes it $\cos(\pi t/2)$: so now if $t = 1, 3, 5, 7$, in each case we get the cosine of an odd multiple of $\pi/2$, which is zero. It is obvious that the denominator in these four cases is not zero.

Therefore, what we have shown is that $\tan^2(\pi t/16)$, $t = 1, 3, 5, 7$, are indeed the four roots of the given quartic.

Solution 187.2 – 29

Find all solutions in integers n, a_0, a_1, $\ldots$, a_n and b of

$$29\sum_{k=0}^{n} a_k 10^k \;=\; 10\sum_{k=0}^{n} a_k 10^k + b(10^{n+2} + 1),$$

where $n \geq 1$, $1 \leq a_n, b \leq 9$ and $0 \leq a_0, a_1, \ldots, a_{n-1} \leq 9$.

Peter Fletcher

It is not clear why the title of this problem is '29' and not '19' since the given equation is exactly the same as

$$19\sum_{k=0}^{n} a_k 10^k = b(10^{n+2} + 1).$$

If we divide both sides of this equation by 19, it becomes a search for expressions of the form

$$\frac{10^{n+2} + 1}{19}$$

that evaluate to integers.

By trying $n = 1, 2, \ldots 43$, it quickly becomes apparent that when $n+2$ is an odd multiple of 9, $10^{n+2}+1$ is then divisible by 19.

If we put $n = 7$ so $n+2 = 9$ in the top equation and solve for a_0, we find that

$$a_0 = 52\,631\,579b - a_1 10^1 - a_2 10^2 - a_3 10^3 - a_4 10^4 - a_5 10^5 - a_6 10^6 - a_7 10^7,$$

which is true with the given conditions only if $b = 1$, and $a_0, a_1 \ldots, a_7$ are the digits of 52 631 579 in reverse order.

With $n = 25$ so $n+2 = 27$, we again have $b = 1$ and $a_0, a_1, \ldots, a_{25}$ *etc.* are the digits of

$$52\,631\,578\,947\,368\,421\,052\,631\,579$$

in reverse order.

Some subsequent integer values of $(10^{n+2}+1)/19$ are:

52 631 578 947 368 421 052 631 578 947 368 421 052 631 579 with $n+2 = 45$;

52 631 578 947 368 421 052 631 578 947 368 421 052 631 578 947 368 421 052 631 579 with $n+2 = 63$;

52 631 578 947 368 421 052 631 578 947 368 421 052 631 578 947 368 421 052 631 578 947 368 421 052 631 579 with $n+2 = 81$;

52 631 578 947 368 421 052 631 578 947 368 421 052 631 578 947 368 421 052 631 578 947 368 421 052 631 578 947 368 421 052 631 579 with $n+2 = 99$.

We conclude that b is always 1, $a_0 = 9$, and $n+2 = 18m+9$ or $n = 18m+7$ for integer m; and, from close study of the above numbers, that once n is big enough such that the following are no longer zero,

$$\begin{array}{lll}
a_1 = a_{18q+1} = 7 & a_7 = a_{18q+7} = 5 & a_{13} = a_{18q+13} = 6 \\
a_2 = a_{18q+2} = 5 & a_8 = a_{18q+8} = 0 & a_{14} = a_{18q+14} = 3 \\
a_3 = a_{18q+3} = 1 & a_9 = a_{18q+9} = 1 & a_{15} = a_{18q+15} = 7 \\
a_4 = a_{18q+4} = 3 & a_{10} = a_{18q+10} = 2 & a_{16} = a_{18q+16} = 4 \\
a_5 = a_{18q+5} = 6 & a_{11} = a_{18q+11} = 4 & a_{17} = a_{18q+17} = 9 \\
a_6 = a_{18q+6} = 2 & a_{12} = a_{18q+12} = 8 & a_{18} = a_{18q+18} = 8
\end{array}$$

where $q = 0, 1, 2, \ldots$.

Solution 288.6 – Icosahedron in a cylinder

> What is the smallest radius of a cylinder into which you can insert a regular icosahedron with edge length 1?

Chris Pile

The regular icosahedron has 20 equilateral triangular faces and 12 vertices. Each of the vertices can be viewed as the summit of a pentagonal pyramid. The diagram (1) on page 21 is looking down on vertex P with the base of the pyramid $ABCDE$. The point O is directly below P in the place $ABCDE$.

The internal angles of the pentagon, each $3\pi/5$, are trisected by lines between alternate vertices such as AC, BD, BE, AD, EC (forming a pentagram). Isosceles triangles ADC (etc.) and DTC (etc.) are similar, where T is the intersection of BD and AC. Therefore $AC/DC = DT/TC$, and

$$AT = DT = DC = AB = 1, \qquad \frac{1 + TC}{1} = \frac{1}{TC}, \qquad TC^2 + TC - 1 = 0.$$

Therefore $TC = \tau - 1$, where

$$\tau \;=\; (\sqrt{5} + 1)/2.$$

Therefore AC (etc.) $= \tau$. If M is the mid-point of BC, then

$$\frac{CM}{OC} \;=\; \sin\frac{\pi}{5} \quad \text{and} \quad \frac{CM}{CT} \;=\; \cos\frac{\pi}{5}.$$

Let $OC = r$, the radius of the circle through the vertices in the plane $ABCDE$. Then

$$\frac{CM}{OC} \;=\; \frac{1}{2r} \quad \text{and} \quad \frac{CM}{CT} \;=\; \frac{\tau}{2}.$$

Therefore

$$\frac{1}{4r^2} \;=\; 1 - \frac{\tau^2}{4} \;=\; \frac{4 - \tau^2}{4}, \quad r^2 \;=\; \frac{1}{4 - \tau^2},$$

using $\sin^2(\pi/5) = 1 - \cos^2(\pi/5)$. Hence

$$r \;=\; \frac{1}{\sqrt{4 - \tau^2}} \;=\; 0.8506508084.$$

The inverted pentagonal pyramid with vertex Q, diametrically opposite P, is rotated by $\pi/5$ from $ABCDE$ so that the vertices appear as shown, $FGHJK$. Thus PQ forms the axis of a cylinder, radius r, into which the icosahedron can be inserted. All vertices except P and Q touch the inside of the cylinder; so it appears to be the smallest radius as the other orthogonal views have a greater diameter.

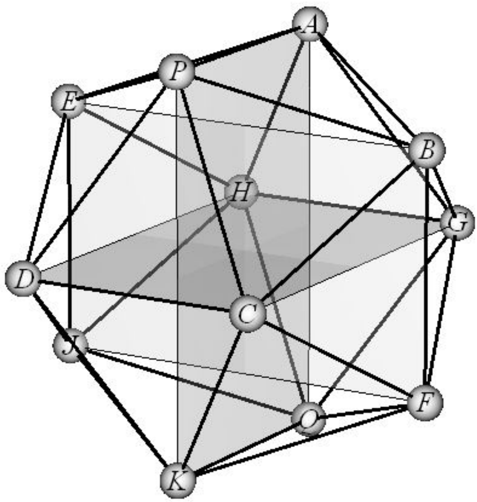
The 12 vertices can be chosen to form three mutually orthogonal golden rectangles with sides τ and 1, $APKQ$, $EBFJ$, $HGCD$. They intersect at the centre of the icosahedron, S, which is the centre of the sphere that passes through the vertices—the circumsphere. Its radius is given by $R_{\mathrm{V}}^2 = R_{\mathrm{E}}^2 + 1/4$, where R_{E} is the radius to a mid-point of an edge;

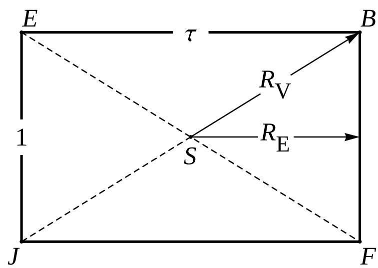
$$R_{\mathrm{E}} \;=\; \frac{\tau}{2} \;=\; 0.8090169944, \qquad R_{\mathrm{V}} \;=\; \frac{\sqrt{\tau^2+1}}{2} \;=\; 0.9510565163.$$

The radius of the sphere which touches the faces is R_{F}; see diagram 3.

$$R_{\mathrm{F}}^2 \;=\; R_{\mathrm{V}}^2 - \frac{1}{3}, \qquad R_{\mathrm{F}} \;=\; 0.7557613141.$$

The distance between the opposite faces is $2R_{\mathrm{F}} = 1.511522628$.





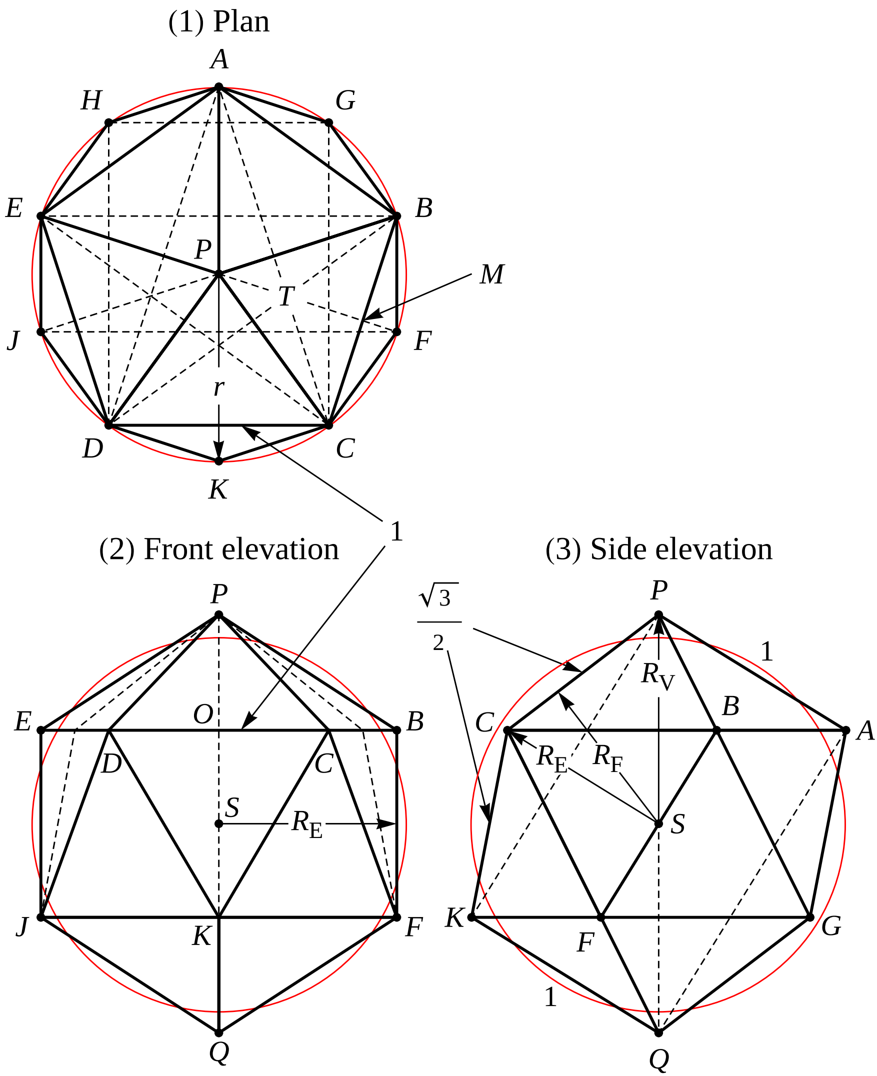
Conclusion The radius of the smallest cylinder is

$$r \;=\; 1/\sqrt{4-\tau^2} \;=\; 0.8506508084.$$

A hoop of radius $R_{\mathrm{E}} = 0.8090169944$ can be fitted around the waist of the icosahedron touching the edges of the ring of ten equilateral triangles. The icosahedron can be posted through a letter box of height 1.511522628. The sphere which encloses the icosahedron has a radius of $R_{\mathrm{V}} = 0.9510565163$.

The icosahedron can be inserted into a cylinder of radius r =

0.8506508084 centred on the axis PQ. The cylinder, shown by the circle in the plan view, touches five vertices in the plane $ABCDE$ and the parallel plane $FGHJK$. The circle is superimposed on the other two orthogonal views. In diagram (2) CD is but $\triangle CDK$ is not parallel to the paper.

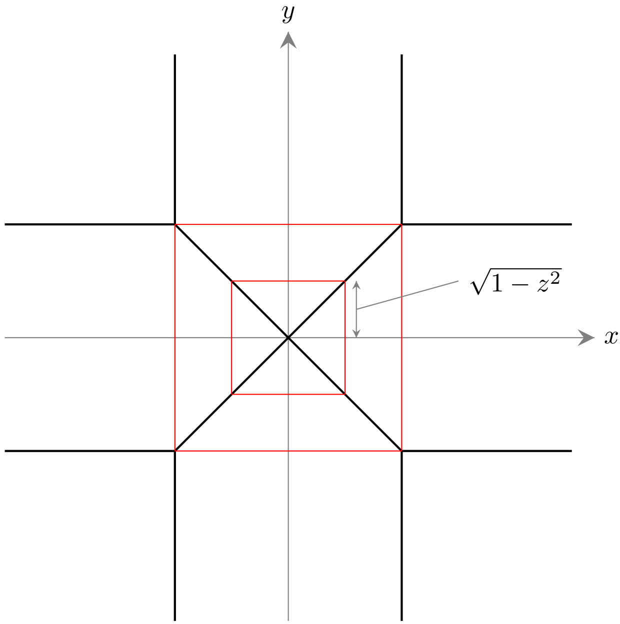


Solution 13.1

> Two circular cylinders of unit radius intersect at right angles. What is the volume common to both cylinders? If that's too easy what volume is common to three cylinders of unit radius with axes mutually perpendicular?

Peter Fletcher

Let the two cylinders have the x and y axes as centre lines and their cross-sections have equations $y^2 + z^2 = 1$ and $x^2 + z^2 = 1$ respectively. From above, we have



where the visible parts are in bold and we have marked in red, the non-visible edges of two slices parallel to the (x, y)-plane through the volume in common between the two cylinders. As can be seen, these slices are squares with sides varying between 2 in the (x, y)-plane and 0 at $z = \pm 1$. For intermediate z, the side length of the square cross-section is $2\sqrt{1-z^2}$.

This can be seen by considering the cross-section through the cylinders by the plane given by the line $y = x$ and the z-axis, which must be an ellipse. If w is the line $y = x$, then this ellipse has equation

$$\frac{w^2}{a^2} + \frac{z^2}{b^2} = 1 \quad \text{or} \quad \frac{w^2}{2} + z^2 = 1$$

since the semi-major axis is $\sqrt{2}$ and the semi-minor axis is 1. Solving for w gives $w = \pm\sqrt{2(1-z^2)}$, which is half the diagonal of a square cross-section at height $\pm z$. This gives half the side length as $\sqrt{1-z^2}$.

Therefore the volume common to two cylinders of radius 1 intersecting at right angles is

$$\int_{-1}^{1} \left(2\sqrt{1-z^2}\right)^2 \mathrm{d}z = 4\int_{-1}^{1} (1-z^2)\mathrm{d}z = 4\left[z - \frac{z^3}{3}\right]_{-1}^{1} = \frac{16}{3}.$$

With three cylinders intersecting at right-angles, we have a central cube that needs to fit inside a cylinder of radius 1. Considering a face of this cube, the distance corner to corner is 2, so its side length is clearly $\sqrt{2}$ and its volume is $2\sqrt{2}$.

Each face of this cube will have a 'cap', the intersection of two cylinders inside the third, but outside the central cube. Thus we want the same integrand as before, but the half the side length of the cube as lower limit:

$$\int_{\sqrt{2}/2}^{1} \left(2\sqrt{1-z^2}\right)^2 \mathrm{d}z = 4\left[z - \frac{z^3}{3}\right]_{\sqrt{2}/2}^{1} = \frac{8-5\sqrt{2}}{3}.$$

Therefore the volume common to three cylinders of radius 1 intersecting at right angles is

$$2\sqrt{2} + 6\left(\frac{8-5\sqrt{2}}{3}\right) = 8(2-\sqrt{2}).$$

For inspiration with the above, Wolfram's page on the Steinmetz solid (the volumes of two of which we found above) was very helpful:
`https://mathworld.wolfram.com/SteinmetzSolid.html`

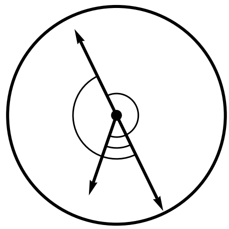
See also: Problem 189.3 – Amazing object, Andrew Pettit; Solution 189.3, Dick Boardman, Barbara Lee, David Kerr, M500 **192**; Problem 242.6 – Three cylinders; Solution 242.6, Tamsin Forbes, Tony Forbes, M500 **245**, Steve Moon, M500 **248**; Problem 245.2 – Intersecting cylinders; Solution 245.2, Richard Gould, M500 **248**. — TF

Problem 297.3 – Telling the time

Tony Forbes

Show that it is possible to tell the time on a standard 12-hour analogue clock with no markings just by measuring the three angles between the pairs of hands. For example, if the angles between the hands are (hour : minute, hour : second, minute : second) = $(225°, 45°, 180°)$, then it must be half past one. Note, however, that this type of clock cannot distinguish between a.m. and p.m.

It is probably a good idea to be consistent about measuring angles. If you want the hour : minute angle, for instance, orientate the clock so that the minute hand is pointing north and then measure the bearing of the hour hand clockwise from north. So the correct angles for 03:00:00 and 09:00:00 are $(90°, 90°, 0°)$ and $(270°, 270°, 0°)$ respectively.



The above assumes that you can tell which hand is which. If you can't, show that it is usually still possible to tell the time and that exceptions occur at half past the hours.

00:30:00	$(195°, 15°, 180°)$	06:30:00	$(15°, 195°, 180°)$
01:30:00	$(225°, 45°, 180°)$	07:30:00	$(45°, 225°, 180°)$
02:30:00	$(255°, 75°, 180°)$	08:30:00	$(75°, 255°, 180°)$
03:30:00	$(285°, 105°, 180°)$	09:30:00	$(105°, 285°, 180°)$
04:30:00	$(315°, 135°, 180°)$	10:30:00	$(135°, 315°, 180°)$
05:30:00	$(345°, 165°, 180°)$	11:30:00	$(165°, 345°, 180°)$

Problem 297.4 – Bridge

Tony Forbes

A problem for Bridge fans. Devise an arrangement of the cards and a bidding sequence where you could realistically and sensibly end up as declarer playing in 8$\diamondsuit$. Assume such a bid is legitimate. Assume you and the other three players always bid and play intelligently.

Whether bids at the eighth level should be allowed is a debatable point. The practice is currently banned by World Bridge Federation rules, but that has not always been the case.

Problem 297.5 – A gun and a wall

A wall is separated from a gun by 2 m. The height of the wall is $h > 2$ m. A bullet is fired with velocity v m/s orthogonally towards it. See the front cover for a picture. With the usual assumptions show that if the bullet clears the wall, then it will land somewhere on a line of length

$$L = \frac{4h\sqrt{v^4 - 4g^2 - 2hv^2g}}{g(h^2+4)} \text{ m.}$$

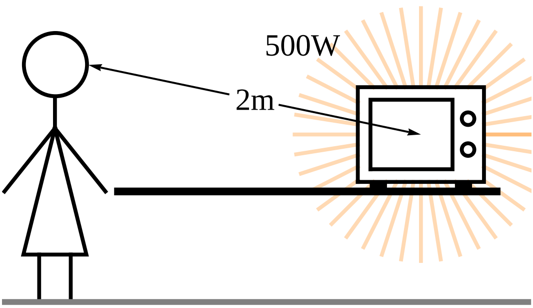
Problem 297.6 – Polygons

Prove that if polygon A fits inside polygon B, then the fitting can be done with at least one side or diagonal of A parallel to at least one side or diagonal of B. Or find a counter-example.

Mathematics in the kitchen – XII

Tony Forbes

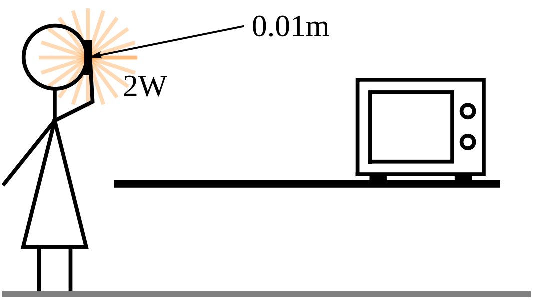
The inverse square law in pictures.



Absorbed radiation

$$\frac{500C}{2^2} = 125C \text{ watts}$$

Very dangerous
[Standards for leaky ovens]



Absorbed radiation

$$\frac{2C}{0.01^2} = 20000C \text{ watts}$$

Extremely dangerous
[Compare above]

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04.00

Problem 297.7 – Two queens

Two queens are placed on an 8 metre square chessboard at random given that they must be separated from each other by at least 2 metres. What's the probability that they do not attack each other.

Front cover Firing bullets over a wall. See Problem 297.5, page 25.