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A	F	1	8	3	6	9	G	D	C	7	2	5	4	E	B
3	6	C	D	A	F	4	5	8	1	E	B	G	9	7	2
G	9	7	2	8	1	E	B	A	F	4	5	3	6	C	D
5	4	E	B	D	C	7	2	3	6	9	G	A	F	1	8
4	A	F	5	G	3	6	9	2	D	C	7	B	1	8	E
D	7	2	C	5	A	F	4	B	8	1	E	6	G	9	3
6	G	9	3	B	8	1	E	5	A	F	4	D	7	2	C
B	1	8	E	2	D	C	7	G	3	6	9	4	A	F	5
1	8	A	F	9	G	3	6	7	2	D	C	E	B	5	4
C	D	3	6	4	5	A	F	E	B	8	1	7	2	G	9
7	2	G	9	E	B	8	1	4	5	A	F	C	D	3	6
E	B	5	4	7	2	D	C	9	G	3	6	1	8	A	F
8	5	B	A	F	E	2	3	1	4	G	D	9	C	6	7
9	C	6	7	1	4	G	D	F	E	2	3	8	5	B	A
2	3	D	G	C	9	5	8	6	7	B	A	F	E	4	1
F	E	4	1	6	7	B	A	C	9	5	8	2	3	D	G

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Quasi-magic sudoku puzzles

Tony Forbes

We begin with a sudoku puzzle. Complete the array to make a Latin square on $\{1, 2, \ldots, 9\}$ such that each of the 3×3 boxes into which the array is divided contains all of the symbols 1–9. The solution is unique.

						9		
		8	1				4	
5			7	2			8	3
1			5		2	8		
	5			3			7	
		9	8		7			2
9	1			4	8			6
	4				1	3		
		3						

Puzzle A

The presence of 3×3 squares which contain precisely the numbers 1–9 strongly suggests that one should be able to create more interesting sudoku puzzles by insisting that the solution has a magic square of order 3 in each of the nine boxes. The symbols 1–9 must be distributed in such a way that the rows, columns and diagonals of each 3×3 box sum to the same 'magic' number, namely 15. Unfortunately it's impossible; a magic square of order 3 has a 5 in its centre. This is peculiar to 3. On the other hand, there is no problem with 4—as is demonstrated by the front cover of this M500, where we have printed a 16×16 Latin square in which every 4×4 box is a magic square of order 4. So magic squares of order 3 won't work. Nevertheless, it is extremely desirable to give the boxes of a standard sudoku array some kind of magic property. Two approaches suggest themselves.

Semi-magic sudoku puzzles

We could insist that the 3×3 boxes be *semi-magic* squares, where it is only the rows and columns that must sum to 15. The diagonal sums are unrestricted. The semi-magic property is invariant under row and column permutations, thus permitting the 5 to appear in any cell of the square. This idea was taken up by John Bray of Queen Mary College, London, who has coined the name *magidoku*.

Quasi-magic sudoku puzzles

Alternatively, we retain the row-column-diagonal summation property but we relax the condition that the sum must be a precise figure. Instead *we require the row, column and diagonal sums of the* 3×3 *boxes to be any number in the range* $15 \pm \Delta$, where Δ is a fixed parameter—which must of course be made known to the solver. The sums do not have to be identical, nor is it necessary for all numbers in the permitted range to occur. We call a square with this property *quasi-magic*, and we use the same qualifier for a sudoku puzzle where all the boxes are quasi-magic squares: a *quasi-magic sudoku puzzle*. If you obtained the solution to Puzzle A at the beginning, you can verify that it is actually quasi-magic with $\Delta = 2$. Having that information at the start might have made it a lot easier to solve!

The case $\Delta = 1$

As we have seen, we cannot have $\Delta = 0$ for a quasi-magic sudoku square, and with a little more work we can show that $\Delta = 1$ is not possible either.

Suppose $\Delta = 1$. The row, column and diagonal sums are restricted to $\{14, 15, 16\}$, the box centres are restricted to $\{4, 5, 6\}$, the box centres of the whole array form a Latin square on the symbols $\{4, 5, 6\}$, and only the following patterns are permitted as valid boxes (plus variations induced by symmetry operations).

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ 1 & 4 & 9 \\ \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \cdot & \cdot \\ 1 & 6 & 9 \\ \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \cdot & \cdot \\ 1 & 5 & \cdot \\ \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccc} 1 & \cdot & \cdot \\ 9 & 5 & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

But now there is no way of arranging the 1s and 9s on the entire array.

Terminology

Before proceeding to greater things we need a few definitions.

A *Latin square* is an $n \times n$ square array of n distinct symbols where each row and each column contains precisely n distinct symbols. A *sudoku square* is a 9×9 Latin square on the symbols $\{1, 2, \ldots, 9\}$ with that extra condition—the 3×3 boxes also contain $\{1, 2, \ldots, 9\}$. You can think of a sudoku square as a solved sudoku puzzle.

For consistency, we always use the word *box* to describe the 3×3 squares. The individual squares are called *cells* or *positions* or some such word with a similar meaning. We use a coordinate system to identify cells. The rows are numbered 0–8, top to bottom, and columns 0–8, left to right. The cell at row r column c is referred to by its coordinates, (r, c). Boxes are labelled by their top left cells. Thus box $(6, 6)$ is the one at bottom right. Within each box, the cells fall naturally into three types; *centres*, *corners* and *edges*. A box has four corner cells four edge cells and one centre cell. By a *row block* we mean three adjacent rows that span three boxes. Similarly, a set of three adjacent columns that span three boxes is called a *column block*.

Finally, much use will be made of the fact that a quasi-magic sudoku square remains a quasi-magic sudoku square under any of the following *symmetry operations*: rotation by 90 degrees, reflection in the middle row or the middle column, reflection in a main diagonal, permutations of the row blocks, permutations of the column blocks, swapping the upper and lower rows of a row block, swapping the left and right columns of a column block, and permutation of the symbol set by the mapping $x \mapsto 10 - x$.

The case $\Delta = 2$

Henceforth we set $\Delta = 2$, so that the rows, columns and diagonals of the boxes must sum to any of $\{13, 14, 15, 16, 17\}$. Our task is to see if quasi-magic sudoku squares have any interesting properties which might be helpful to puzzle solvers. We start with some easy ones.

Numbers 1 and 2 cannot occur together in a box row, column or diagonal. Hence by symmetry nor can 8 and 9. A box centre must be one of $\{3, 4, 5, 6, 7\}$. Also it is easy to prove that a quasi-magic square with centre 3 or 7 must have one of the following patterns.

67	67	2		34	34	8
1	3	9		9	7	1
8	45	45		2	56	56

And this is about as far as one can get by treating quasi-magic squares in isolation. But when nine of them are arranged as boxes in a quasi-magic sudoku square it is possible to prove much more. Indeed, we have two interesting and surprising facts.

(i) Numbers 3 and 7 can occur at most once each as box centres in a quasi-magic sudoku square; and if they both occur, they must occupy box centres in the same row or in the same column.

(ii) Although there exist quasi-magic squares where the 5 is in an edge position, in a quasi-magic sudoku square all the 5s occur either in box centres or in box corners, never in box edges.

The only proofs I know are long and complicated. I hope that by presenting them here someone will possibly see something I have missed and produce a shorter and more elegant proof. Anyway, as a consequence of the no-5-on-a-box-edge rule, the number of possible patterns with box centre 3 or 7 is halved to just these four (plus symmetries).

6	7	2		7	6	2		4	3	8		3	4	8
1	3	9		1	3	9		9	7	1		9	7	1
8	4	5		8	4	5		2	6	5		2	6	5

And since they map to each other in pairs under the action of the symmetry operation $x \mapsto 10 - x$ there are really only two.

Distribution of 5 in a quasi-magic sudoku square

Theorem 1 *In a quasi-magic sudoku square with $\Delta = 2$, each row block and each column block has the number* 5 *in one of the box centre positions.*

Proof. Suppose not. By symmetry we can assume that the box centres in row 1 are either $(3, 7, 4)$, or $(3, 4, 6)$.

Consider the case $(3, 7, 4)$ first. From what we already know about boxes with 3 or 7 in the centre, one of the following four patterns must occur in rows 0–2. Each one blatantly leads to a contradiction.

67	67	2	8	1	56	·	·	·
1	3	9	34	7	56	·	4	·
8	45	45	34	9	2	·	·	·

67	67	2	34	9	2	·	·	·
1	3	9	34	7	56	·	4	·
8	45	45	8	1	56	·	·	·

8	1	7	56	56	2	·	·	·
5	3	6	1	7	9	·	4	·
4	9	2	8	34	34	·	·	·

8	1	7	34	9	2	·	·	·
5	3	6	34	7	56	·	4	·
4	9	2	8	1	56	·	·	·

That leaves just the case $(3, 4, 6)$ to worry about. By symmetry and what we already know about boxes with a 3 in the centre, there are only two possible patterns for rows 0–2:

67	67	2	·	·	·	·	·	·
1	3	9	·	4	·	·	6	·
8	45	45	·	·	·	·	·	·

and

8	1	6	·	·	·	·	·	·
5	3	7	·	4	·	·	6	·
4	9	2	·	·	·	·	·	·

But in the first case (left-hand diagram) there is nowhere for a 2 to go in box $(0, 3)$ without generating a contradiction. So see this, try putting 2 in cell $(1, 3)$ and see what happens. Then try putting 2 in cell $(2, 3)$. Then try putting 2 in cell $(2, 4)$. By symmetry you needn't bother to try putting 2 in cell $(1, 5)$ or $(2, 5)$.

That leaves only the second case (right-hand diagram) to consider. Unfortunately this is the troublesome one. Suppose cell $(4,4)$ is 3 or 7. Then one of the following four patterns must occur in the top six rows.

A

8	1	6	·	·	·	·	·	·
5	3	7	·	4	·	·	6	·
4	9	2	·	·	·	·	·	·
139	·	·	67	67	2	·	·	·
·	·	·	1	3	9	·	·	·
·	·	·	8	5	4	·	·	·

B

8	1	6	2379	·	·	·	·	·
5	3	7	·	4	·	·	6	·
4	9	2	·	·	·	·	·	·
·	·	·	8	1	67	·	·	·
·	·	·	45	3	67	·	·	·
·	·	·	45	9	2	·	·	·

C

8	1	6	·	·	·	·	·	·
5	3	7	·	4	·	·	6	·
4	9	2	·	·	·	·	·	·
12679	·	·	8	3	4	·	·	·
·	·	·	1	7	9	·	·	·
·	·	·	56	56	2	·	·	·

D

8	1	6	·	·	·	·	·	·
5	3	7	129	4	·	·	6	·
4	9	2	·	·	·	·	·	·
·	·	·	8	1	56	·	·	·
·	·	·	34	7	56	·	·	·
·	·	·	34	9	2	·	·	·

In each case we obtain a contradiction.

Pattern A. Cell $(3,0)$ is either 1, 3 or 9; each leads to a contradiction.

Pattern B. Cell $(0,3)$ is either 2, 3, 7 or 9; 2 and 3 lead to contradictions; 7 implies box $(6,3)$ middle column $= \{5,6,7\}$, which exceeds 17; and 9 implies box $(6,3)$ left column $= \{2,3,7\}$, which is less than 13.

Pattern C. Cell $(3,0)$ is one of $\{1,2,6,7,9\}$; all lead to contradictions.

Pattern D. Cell $(1,3)$ is one of $\{1,2,9\}$; all lead to contradictions.

Thus we have shown that cell $(4,4)$ is not 3 or 7. By symmetry the same is true of cell $(7,4)$. Similarly, cells $(4,7)$ and $(7,7)$ cannot be 3 or 7. Hence we can assume that $(4,4)=5$, $(7,4)=6$, $(4,7)=4$ and $(7,7)=5$.

8	1	6	·	·	·	·	·	·
5	3	7	·	4	·	·	6	·
4	9	2	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·
·	67	·	·	5	·	·	4	·
·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·
·	47	·	·	6	·	·	5	·
·	·	·	·	·	·	·	·	·

Now suppose $(4,1) = 7$. Then $(3,0)$ is one of $\{1,2,3,6,9\}$ all of which lead to contradictions. Hence we can assume $(4,1) = 6$. Then $(3,0)$ is one of $\{1,2,3,7,9\}$ all of which lead to contradictions. □

We use Theorem 1 to prove that interesting property of 5.

Theorem 2 *In a quasi-magic sudoku square with $\Delta = 2$, the number 5 never occurs in a box edge position.*

Proof. Suppose not. By symmetry we can assume that cell $(1,0) = 5$. Hence no box centre in row 1 has 5 in it. This contradicts Theorem 1. □

Box centres 3 and 7 in a quasi-magic sudoku square

Theorem 3 *In a quasi-magic sudoku square with $\Delta = 2$, the numbers 3 and 7 can each occur at most once each as a box centre.*

Proof. Suppose the theorem is false. By symmetry, it suffices to deal with 3 and we can assume that box centre 3s occur in cells (1,1) and (4,4).

Recalling the possible patterns with box centre 3 and using Theorem 2, we can assume that the part of the array consisting of the top left 36 cells contains one of the following patterns, denoted by L and R.

L

67	67	2			
1	3	9		x	
8	4	5			
			67	67	2
			1	3	9
			8	4	5

R

67	67	2			
1	3	9		y	
8	4	5			
			8	1	67
			4	3	67
			5	9	2

We consider seven cases: array L, $x = 5, 6$ or 7, or array R, $y = 4, 5, 6$ or 7.

Array L, $x = 5$. First, all of $(0,4) = 1$, $(0,5) = 1$ and $(2,5) = 1$ lead to contradictions. Hence $(2,4) = 1$ and $(1,3) = 2$. But then $(1,5) = 8$ for otherwise the middle row of box $(0,6)$ exceeds 17. Then $(0,4) = 9$, $(0,3) = 4$, $(0,5) = 3$ and $(2,5) = 6$. But now the right column of box $(6,3)$ sums to only 12.

Array L, $x = 6$. First, $(3,3) = 6$ and $(3,4) = 7$. Since $(0,3) = 9$, $(0,4) = 9$ and $(2,3) = 9$ lead to contradictions, we assume $(2,4) = 9$. Then $(0,4) = 1$, $(2,5) = 3$, $(2,3) = 2$, $(0,5) = 8$, $(1,3) = 7$, $(1,5) = 4$,

$(0,3) = 5$, $(1,7) = 5$, $(4,1) = 5$ and $(7,4) = 5$. Next, $(3,1) = 1$, $(3,2) = 1$ and $(5,2) = 1$ lead to a contradictions. So $(5,1) = 1$. Hence $(4,0) = 2$, $(5,7) = 2$, and $(3,6) = 1$ or 5, both of which lead to contradictions.

Array L, $x = 7$. The proof goes as for array L, $x = 6$ except that the 6 and 7 are swapped in cell pairs $\{(1,3),(1,4)\}$ and $\{(3,3),(3,4)\}$. This interchange does not affect the argument.

Array R, $y = 4$. This together with Theorem 2 implies $(0,4) = (1,5) = 8$, a contradiction.

Array R, $y = 5$. There is nowhere to put a 4 in column 4 since each of $(6,4) = 4$, $(7,4) = 4$ and $(0,4) = 4$ leads to a contradiction.

Array R, $y = 6$. This leads almost immediately to a contradiction.

Array R, $y = 7$. So does this. □

Box centres 3 and 7 together

Theorem 4 *In a quasi-magic sudoku square with* $\Delta = 2$, *the numbers* 3 *and* 7 *can occur together as box centres only in the same row or column.*

Proof. Suppose the theorem is false. By symmetry we can assume cell $(1,1) = 3$ and $(4,4) = 7$. Remembering the possible patterns with box centres 3 and 7, and using Theorem 2, we can assume that the top left 36 cells of the array look like the following, denoted by P and S.

P

67	67	2			
1	3	9		v	
8	4	5			
			5	6	2
			1	7	9
			8	34	34

S

67	67	2			
1	3	9		w	
8	4	5			
			8	1	5
			34	7	6
			34	9	2

We consider five cases: array P, $v = 4$ or 5, or array S, $w = 4, 5$ or 6.

Array P, $v = 4$. All of $(1,3) = 2$, $(2,3) = 2$ and $(2,4) = 2$ lead to contradictions.

Array P, $v = 5$. First, $(2,3) = 2$ leads to a contradiction. If $(1,3) = 2$, then $(1,5) = 8$ (otherwise the middle row of box $(0,6)$ exceeds 17); but now the left column of box $(6,3)$ exceeds 17. Hence $(2,4) = 2$. Then $(0,5) = 1$, $(1,3) = 4$, $(0,3) = 3$, $(0,4) = 9$, $(2,3) = 7$, $(2,5) = 6$ and $(1,5) = 8$. But by Theorem 3, $(1,7)$ can't be 7, so it must be 6 and this leads to a contradiction.

Array S, $w = 4$. Both $(2,3) = 2$ and $(2,4) = 2$ lead to contradictions. Hence $(1,3) = 2$. But now the middle row of box $(0,6)$ exceeds 17.

Array S, $w = 5$. All of $(1,3) = 2$, $(2,3) = 2$ and $(2,4) = 2$ lead to contradictions.

Array S, $w = 6$. Likewise. □

More properties of $\Delta = 2$ quasi-magic sudoku squares

We already know from Theorem 1 that every row block and column block must have a 5 in one of its box centres. Hence in a row block the only possible box centres are

$$\{3,5,6\},\ \{3,5,7\},\ \{4,5,6\},\ \{4,5,7\},\ \{3,4,5\} \text{ and } \{5,6,7\}.$$

We shall now show that the last two do not occur. It suffices to deal with $\{3,4,5\}$. By symmetry and Theorem 2, we need only consider the following row block pattern, and it is easy to see that there is nowhere for a 2 to go in the middle row.

67	67	2	·	·	·	·	·	·
1	3	9	·	4	·	·	5	·
8	4	5	·	·	·	·	·	·

Therefore the only allowed box centres in a row block or column block are

$$\{3,5,6\},\ \{3,5,7\},\ \{4,5,6\},\ \{4,5,7\},$$

and it is possible for any of them to occur.

If a box centre is 4 or 6, then $\{4,5,6\}$ either lies on a diagonal or occupies three positions in the form of a knight's move.

Also it is a fact that a 5 in a box centre forces $\{4,5,6\}$ on a diagonal of the box. However, the only proof I have is to ask the computer to solve four quasi-magic sudoku puzzles with just these starter digits (in the top left box):

(a)

·	4	·
·	5	·
·	·	·

(b)

4	6	·
·	5	·
·	·	·

(c)

4	·	6
·	5	·
·	·	·

(d)

4	·	·
·	5	6
·	·	·

Computer says no in each case.

When $\{4,5,6\}$ occurs on a diagonal, each of $\{1,2,3\}$ and $\{7,8,9\}$ occupies one of two broken diagonals into which the six remaining cells of the box are partitioned. In this case, after allowing for symmetries there is essentially only one admissible pattern.

$$\begin{array}{ccc} a & d & g \\ h & b & f \\ e & i & c \end{array} \qquad \begin{array}{l} \{a,b,c\} = \{4,5,6\} \\ \{d,e,f\} = \{1,2,3\} \\ \{g,h,i\} = \{7,8,9\} \end{array}$$

In the special case where there are no boxes with 3 or 7 in the centre, the nine box centres form a Latin square on $\{4,5,6\}$. Hence each box must have $\{4,5,6\}$ along a diagonal.

Summary

(i) The number 5 cannot occur in a box edge position (Theorem 2).

(ii) The number 3 cannot occur more than once as a box centre (Theorem 3). Nor can 7 by symmetry. And when 3 or 7 do occur, the boxes with them as centres must conform to one of the four patterns.

$$\begin{array}{ccc} 6 & 7 & 2 \\ 1 & 3 & 9 \\ 8 & 4 & 5 \end{array} \qquad \begin{array}{ccc} 7 & 6 & 2 \\ 1 & 3 & 9 \\ 8 & 4 & 5 \end{array} \qquad \begin{array}{ccc} 4 & 3 & 8 \\ 9 & 7 & 1 \\ 2 & 6 & 5 \end{array} \qquad \begin{array}{ccc} 3 & 4 & 8 \\ 9 & 7 & 1 \\ 2 & 6 & 5 \end{array}$$

(iii) Numbers 3 and 7 can both occur as box centres only in the same row or in the same column (Theorem 4).

(iv) The only allowed box centres in a row or column block are

$$\{3,5,6\},\ \{3,5,7\},\ \{4,5,6\} \text{ and } \{4,5,7\}.$$

(v) Only the following patterns occur in boxes with centre 4, 5 or 6. But note however that the only proof I have for the first one (with $4,5,6$ down the NW–SE diagonal) is not human-readable without considerable effort.

$$\begin{array}{ccc} 4 & \cdot & \cdot \\ \cdot & 5 & \cdot \\ \cdot & \cdot & 6 \end{array} \qquad \begin{array}{ccc} 5 & \cdot & \cdot \\ \cdot & 4 & \cdot \\ \cdot & \cdot & 6 \end{array} \qquad \begin{array}{ccc} 5 & 9 & 2 \\ 3 & 4 & 6 \\ 8 & 1 & 7 \end{array} \qquad \begin{array}{ccc} 5 & \cdot & \cdot \\ \cdot & 6 & \cdot \\ \cdot & \cdot & 4 \end{array} \qquad \begin{array}{ccc} 5 & 1 & 8 \\ 7 & 6 & 4 \\ 2 & 9 & 3 \end{array}$$

With all this information to hand you should now be able to solve with relative ease Puzzles B, C and D (as well as A) before advancing on to the more difficult E, F and G. Having only four starter digits Puzzle G looks truly fiendish!

					4			
9								
			5					
3		4		6	2			
			8	3		7		4
					7			
								7
			3					

Puzzle B

	4	5	6					
						6		
		2						
			3					
				4				
					9			
						7		
		7						
					7	8	3	

Puzzle C

	6							
	2	7						
		4	6					
				3				
					5	8		
						4	9	
							5	

Puzzle D

						9		
				8				
		9	2	5	7	3		
				1				
		2						

Puzzle E

				8				
					2			
8				7				9
			8					
				4				

Puzzle F

			4					
						4		
7								
	3							

Puzzle G

Solution 212.1 – Fibonacci numbers

Let $F_1, F_2, F_3, \ldots$ denote the Fibonacci sequence, 1, 1, 2, 3, 5, 8, 13, … and let $f(x) = \cos(\arctan(\sin(\operatorname{arccot} x)))$. Show that

$$f(f(\ldots f(x)\ldots)) \;=\; \sqrt{\frac{F_{2n-1}x^2 + F_{2n}}{F_{2n}x^2 + F_{2n+1}}},$$

with n iterations of f.

Ted Gore

Let $\alpha = \operatorname{arccot}\, x$. Then $x = \cot\alpha$ and $y = \sin\alpha = \dfrac{1}{\sqrt{x^2+1}}$. Let $\beta = \arctan y$, so that

$$f(x) \;=\; \cos\beta \;=\; \sqrt{\frac{x^2+1}{x^2+2}}.$$

Now let A_n be the proposition

$$f_n(x) \;=\; \sqrt{\frac{F_{2n-1}x^2 + F_{2n}}{F_{2n}x^2 + F_{2n+1}}}.$$

Then A_1 can easily be seen to be true. To save typing and to aid legibility I use the following assignments:

$$p = F_{2n-1}, \quad q = F_{2n}, \quad r = F_{2n+1}, \quad s = F_{2n+2}, \quad t = F_{2n+3}.$$

If A_n is true, then

$$f_n(x) \;=\; \sqrt{\frac{p\,x^2+q}{q\,x^2+r}}$$

and

$$f_{n+1}(x) \;=\; \sqrt{\frac{\dfrac{px^2+q}{qx^2+r}+1}{\dfrac{px^2+q}{qx^2+r}+2}} \;=\; \sqrt{\frac{(p+q)x^2+(q+r)}{(p+2q)x^2+(q+2r)}} \;=\; \sqrt{\frac{rx^2+s}{sx^2+t}},$$

so that A_{n+1} is true. Therefore by mathematical induction, A_n is true for all n. Also, as $n\to\infty$,

$$f_n(x) \to \sqrt{\frac{x^2+q/p}{x^2q/p+r/q}} \;=\; \sqrt{\frac{x^2+\phi}{\phi x^2+\phi^2}} \;=\; \frac{1}{\sqrt{\phi}}.$$

Also solved in a similar manner by **Steve Moon** and **Nick Hobson**.

Vector Calculus in 2-dimensional Euclidean Space

Dennis Morris

Conventional multilinear algebra recognizes two types of vectors, covariant vectors and contravariant vectors. Contravariant vectors are just normal vectors, but covariant vectors are linear functions that act on a vector. To add to the confusion, covariant vectors are also known as linear functions, linear functionals, co-vectors, vectors (sloppily), and 1-forms. Covariant vectors (1-forms) are written as row vectors:

$$[a \quad b].$$

Contravariant vectors (normal vectors) are written as column vectors:

$$\begin{bmatrix} x \\ y \end{bmatrix}.$$

The 1-form (covariant vector) acts upon the contravariant vector (normal vector), by matrix multiplication, to produce a real number:

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax + by \in \mathbb{R}.$$

This is the dot product of course (which is an invariant of the space).

Conventionally, the differential of a function, $f(a, b)$, is a 1-form (covariant vector):

$$df = \begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} \end{bmatrix}.$$

This will act upon a contravariant vector to produce a real number:

$$\begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{\partial f}{\partial a} x + \frac{\partial f}{\partial b} y \in \mathbb{R}.$$

Aside: The differential is sometimes presented, confusingly, as a 1-form with basis 1-forms da and db:

$$df = \frac{\partial f}{\partial a} da + \frac{\partial f}{\partial b} db \in \mathbb{R}.$$

Doing vectors within an algebraic field considerably simplifies all the above. An essential part of an algebra is the product of two elements of that algebra where both elements are of the same nature. So it is that, when we do vectors within an algebraic field, there is no distinction between different types of vectors. In particular, there is no distinction between contravariant vectors and covariant vectors, and we can call them all just vectors—which is lovely.

A function upon a space is a scalar field upon that space. In matrix form, the function (scalar field) is

$$\begin{bmatrix} f(a,b) & 0 \\ 0 & f(a,b) \end{bmatrix}.$$

The differential is formed by taking the partial derivatives. The matrix configuration identifies these partial derivatives as components of a vector, and so we do not need to keep them separate:

$$d\begin{bmatrix} f(a,b) & 0 \\ 0 & f(a,b) \end{bmatrix} = \left[\begin{bmatrix} \dfrac{\partial f}{\partial a} & 0 \\ 0 & \dfrac{\partial f}{\partial a} \end{bmatrix}, \begin{bmatrix} 0 & -\dfrac{\partial f}{\partial b} \\ \dfrac{\partial f}{\partial b} & 0 \end{bmatrix}\right] = \begin{bmatrix} \dfrac{\partial f}{\partial a} & -\dfrac{\partial f}{\partial b} \\ \dfrac{\partial f}{\partial b} & \dfrac{\partial f}{\partial a} \end{bmatrix}.$$

Of course, the differential is the gradient of the scalar field, $\mathrm{grad}(f)$, which is a vector, which is what we have.

A vector field is more than one function upon a space:

$$\begin{bmatrix} F_a(a,b) & F_b(a,b) \\ -F_b(a,b) & F_a(a,b) \end{bmatrix}.$$

The differential of a vector field is

$$\begin{aligned} d&\begin{bmatrix} F_a(a,b) & F_b(a,b) \\ -F_b(a,b) & F_a(a,b) \end{bmatrix} \\ &= \left[\begin{bmatrix} \dfrac{\partial F_a(a,b)}{\partial a} & \dfrac{\partial F_b(a,b)}{\partial a} \\ -\dfrac{\partial F_b(a,b)}{\partial a} & \dfrac{\partial F_a(a,b)}{\partial a} \end{bmatrix}, \begin{bmatrix} \dfrac{\partial F_b(a,b)}{\partial b} & -\dfrac{\partial F_a(a,b)}{\partial b} \\ \dfrac{\partial F_a(a,b)}{\partial b} & \dfrac{\partial F_b(a,b)}{\partial b} \end{bmatrix}\right] \\ &= \begin{bmatrix} \dfrac{\partial F_a(a,b)}{\partial a} + \dfrac{\partial F_b(a,b)}{\partial b} & \dfrac{\partial F_b(a,b)}{\partial a} - \dfrac{\partial F_a(a,b)}{\partial b} \\ \dfrac{\partial F_a(a,b)}{\partial b} - \dfrac{\partial F_b(a,b)}{\partial a} & \dfrac{\partial F_a(a,b)}{\partial a} + \dfrac{\partial F_b(a,b)}{\partial b} \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{div}(F) & \mathrm{curl}(F) \\ -\mathrm{curl}(F) & \mathrm{div}(F) \end{bmatrix}. \end{aligned}$$

Differentiating a scalar field twice with respect to both variables gives

$$\begin{bmatrix} \dfrac{\partial^2 f}{\partial a^2} + \dfrac{\partial^2 f}{\partial b^2} & 0 \\ 0 & \dfrac{\partial^2 f}{\partial a^2} + \dfrac{\partial^2 f}{\partial b^2} \end{bmatrix},$$

which is the Laplacian for a scalar field.

The second differential of a vector is

$$\begin{bmatrix} \dfrac{\partial^2 F_a}{\partial a^2} + \dfrac{\partial^2 F_a}{\partial b^2} & \dfrac{\partial^2 F_b}{\partial a^2} + \dfrac{\partial^2 F_b}{\partial b^2} \\ -\left(\dfrac{\partial^2 F_b}{\partial a^2} + \dfrac{\partial^2 F_b}{\partial b^2}\right) & \dfrac{\partial^2 F_a}{\partial a^2} + \dfrac{\partial^2 F_a}{\partial b^2} \end{bmatrix},$$

which is the Laplacian for a vector field.

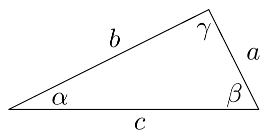
Problem 215.1 – Pythagoras's theorem

A triangle has sides of length a, b and c opposite angles α, β and γ respectively. Prove that

$$\operatorname{sgn}(\alpha + \beta - \gamma) \;=\; \operatorname{sgn}(a^2 + b^2 - c^2), \tag{1}$$

where $\operatorname{sgn}(x)$ is defined by

$$\operatorname{sgn}(x) \;=\; \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$



The theorem is the subject of a hand-written manuscript by Edsger W. Dijkstra, 'On a theorem of Pythagoras'. Dijkstra writes, 'The title of this little note could make one wonder why I would waste my time flogging a horse as dead as Pythagoras's theorem. ... I am in a paradoxical situation. I am convinced that of the people knowing the theorem of Pythagoras, almost no one can read the above [(1)] without being surprised at least once. Furthermore I think all those surprises are relevant. Yet I don't know of a single respectable journal in which I could flog this dead horse.'

The article is freely available on the Web.

Number boggle

Tony Forbes

X	A	I	J
N	A	R	K
S	A	R	D
Z	A	V	G

0	0	0	0
0	1	2	0
0	3	4	0
0	0	0	0

In the popular game BOGGLE, players are presented with a 4×4 array of randomly chosen letters, such as the one on the left, above, and the objective is to find words. The rules are (i) a word must appear in the grid on a non-self-intersecting path spelling out its letters in the correct order travelling one step at a time, up, down, left, right or diagonally; (ii) only valid English words (such as AARDVARK) are allowed; (iii) you score

$$n \; - \; 3 \; - \; \frac{92\,n^2 - 265\,n + 177}{15!} \prod_{k=4}^{16} (n - k)$$

points for each n-letter word that has not also been found by one of your opponents; (iv) the player with the highest score wins; (v) I come second in contests involving just me and my daughter Tamsin.

I have often wondered whether my chances of winning would significantly increase if the game were played with digits instead of letters. Here you are given a 4×4 array of randomly chosen digits and the objective is to find *numbers*.

To test this idea, see how many numbers you can find in the grid on the right. You may if you wish use the same scoring scheme as for words.

Problem 215.2 – Three angles

If $A + B + C = 180°$, show that

$$\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix} \; = \; 0.$$

Solution 204.1 – Jigsaw

(i) For which x and y is it possible to make an $x \times y$ jigsaw puzzle which has the same number of boundary pieces as interior pieces?

(ii) Estimate how long it takes to complete a jigsaw puzzle of n pieces, giving the answer as an order-of-magnitude approximation as n tends to infinity. Is the time determined by the difficulty of locating each piece? Assume that the pictures are of reasonably constant squareness and complexity, and that the pieces are of approximately constant size, say 2 cm × 2 cm, regardless of the value of n. Assume also that you have only one transportation device for moving pieces from the box to their correct places.

Steve Moon

(i) For an $x \times y$ jigsaw puzzle, the condition requires

$$2x + 2y - 4 = (x - 2)(y - 2),$$

the left-hand side being the number of edges. Therefore

$$y = \frac{4(x-2)}{x-4}.$$

Hence $y = 4k$ and $k = (x - 2)/(x - 4)$, an integer. By exhaustion, $x = 5$ or $x = 6$ only. For all higher values of x, $1 < k < 2$.

(ii) Assume the puzzle is done by an 'intelligent machine', and that the following hold.

(a) All pieces are randomly distributed within reach;

(b) they have no immediately obvious differences (e.g. picture, shape, size) and can be picked randomly;

(c) the machine can tell if they fit together—if not, it discards a selected piece and tries again from the remaining pieces;

(d) it takes unit time to try any combination of two pieces;

(e) the machine can discern where an available slot exists to fit a piece, and only tries to fit that slot—hence it must succeed eventually.

For an n-piece puzzle, on average the next piece will take $(n+1)/2$ time units to find (median of $1, 2, \ldots, n$). The next piece will take $n/2$, etc. Hence the expected total time T is given by

$$\begin{aligned} T &= \frac{n+1}{2} + \frac{n}{2} + \cdots + 1 \\ &= \frac{1}{2}\Big((n+1) + n + (n-1) + \cdots + 2\Big) \\ &= \frac{n\,(n+3)}{4}. \end{aligned}$$

As $n \to \infty$, $T \to n^2/4$. Hence $T \propto n^2$.

If the puzzle is easy (i.e. the right piece can be spotted by eye, and if sufficiently small, all pieces can be viewed simultaneously), then each piece takes unit time to find and fit. Then $T \propto n$.

As n increases, if there are k pieces to scan, on average you will find the right piece after $(k+1)/2$ time units (median of $1, 2, \ldots, k$). Also as n increases, the time to fit a piece once found is small in comparison to $(k+1)/2$. Summing will give another arithmetic progression with dominant term in n^2 as $n \to \infty$.

ADF—It would be interesting to examine in detail the consequences of omitting part of assumption (a) and the whole of assumption (d) in the list on the previous page. Do not assume that all pieces are necessarily within reach. And, because the puzzle under construction occupies a significant area, it is not necessarily the case that constant unit time is the appropriate measure for locating the correct position of a piece. This to me seems more in keeping with real life.

Imagine n so large that the finished puzzle occupies a significant fraction of the Earth's surface, say the whole of Asia. Now there is a real problem of transporting a piece from the box—which could occupy a few cubic kilometres in a convenient city in Europe—to the correct part of Asia where it belongs. Would the transportation of pieces from the box to their rightful places in the array add significantly to the time taken to solve the puzzle? Again to be realistic, we should assume that the number of vehicles available for moving pieces is constant with respect to n; otherwise one could argue that the pieces are within reach, contrary to the negation of assumption (a). Since the piece size is constant, even larger puzzles could involve extraterrestrial activity. However, I don't think that would add significantly to the time. But I might be wrong.

Solution 211.3 – Trigonometrical limit

Find n given that $0 < \lim_{\theta \to 0} \dfrac{\theta^n \sin^n \theta}{\theta^n - \sin^n \theta} < \infty$.

Basil Thompson

Write $f_n(\theta) = \dfrac{\theta^n \sin^n \theta}{\theta^n - \sin^n \theta}$. It is possible to find the answer by calculating values of $f_n(\theta)$ for $n = 1, 2, 3, 4, \ldots$ with $\theta \to 0$.

To prove the result, write $\sin\theta$ as a series,

$$\sin\theta \;=\; \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots.$$

Neglecting terms after θ^3, we have $\sin\theta \sim \theta(1 - \theta^2/6)$ and hence

$$f_n(\theta) \;\sim\; \frac{\theta^n \theta^n (1 - \theta^2/6)^n}{\theta^n - \theta^n (1 - \theta^2/6)^n}.$$

But $(1 - \theta^2/6)^n \sim 1 - n\theta/6$ as $\theta \to 0$. Therefore

$$f_n(\theta) \;\sim\; \frac{\theta^{2n}(1 - n\theta^2/6)}{\theta^n - \theta^n(1 - n\theta^2/6)} \;=\; \frac{\theta^{2n}(1 - n\theta^2/6)}{n\theta^{n+2}/6} \;=\; \frac{6\theta^{n-2}}{n} - \theta^n.$$

Hence $\lim_{\theta\to 0} f_n(\theta) = 6\theta^{n-2}/n$. So as $\theta \to 0$,

$$f_n(\theta) \;\to\; \begin{cases} \infty & \text{if } n < 2, \\ 3 & \text{if } n = 2, \\ 0 & \text{if } n > 2. \end{cases}$$

Also solved by **Ian Adamson**.

Problem 215.3 – Sine series

Compute

$$\sum_{n=1}^{\infty} \frac{1}{n^3 (\sin n)^2}.$$

On evaluating the partial sums up to the 354th term it appears that this simple-looking series is converging nicely to about 4.8. Now add the 355th term, and the sum jumps to about 29.4056. What is going on? What if you replace n^3 in the denominator by n^4?

Solution 212.7 – Dice

Throw n dice and remove any that have landed six-up. Repeat until no dice are left. What is the expected number of throws?

Nick Hobson

Let $\mathbb{E}(k)$ be the expected number of throws, starting with k dice, until all dice have been removed. Beginning with n dice, following the first throw:

◇ with probability $(1/6)^n$, no dice remain in play;

◇ with probability $(5/6)^n$, all dice remain in play;

◇ in general, with probability $(1/6)^{n-r}(5/6)^r\binom{n}{r}$, exactly r dice remain in play.

Under the reasonable assumption that the expected values are finite, we have

$$\mathbb{E}(n) = \sum_{r=0}^{n} \frac{5^r}{6^n}\binom{n}{r}(1+\mathbb{E}(r)).$$

Collecting terms in $\mathbb{E}(n)$ and rearranging we obtain the recurrence equation

$$\left(1-\frac{5^n}{6^n}\right)\mathbb{E}(n) = \frac{5^n}{6^n} + \sum_{r=0}^{n-1} \frac{5^r}{6^n}\binom{n}{r}(1+\mathbb{E}(r))$$

or, multiplying by 6^n and taking the $r=0$ term out of the summation,

$$\mathbb{E}(n) = \frac{1+5^n}{6^n-5^n} + \sum_{r=1}^{n-1} \frac{5^r}{6^n-5^n}\binom{n}{r}(1+\mathbb{E}(r)).$$

Calculating the recurrence as far as $n = 6$, we find that $\mathbb{E}(6) = 9438928992/677218157 = 13.9378$. The table below shows how slowly $\mathbb{E}(n)$ increases with n.

n	$\mathbb{E}(n)$
1	6
2	$96/11 \approx 8.7273$
3	$10566/1001 \approx 10.5554$
4	$728256/61061 \approx 11.9267$
5	$3698650986/283994711 \approx 13.0237$
6	$9438928992/677218157 \approx 13.9378$
7	14.7213
8	15.4069
9	16.0164

n	$\mathbb{E}(n)$
10	16.5648
16	19.0427
32	22.7601
64	26.5194
128	30.2998
256	34.0909
512	37.8873
1024	41.6864
2048	45.4869

Large numbers

Eddie Kent

Every now and then M500 tackles the subject of large numbers, finding them or even just naming them. For the more popular numbers it pays to be careful since the British and the Americans use different names for the same numbers, even though in 1974 Harold Macmillan committed Britain, and thus *The Times*, to the smaller, less logical system, because America has it for sheer size and braggadocio.

However, some numbers have names that are in not in dispute. Googol, for instance; this has a value fixed at 10^{100}. I saw an interesting idea recently. Since $10^{100} = 10^{50\times 2} = 100^{50}$, and since GOOGOL is GOOGO followed by L = 50, it follows that googo$n = 100^n$ whenever n has a value in Roman numerals. Thus googoiii $= 100^3 = 1,000,000$.

I have been wrestling with *Graham's number*. This is often cited as the largest ever used seriously in a mathematical proof. It was devised by Graham and Rothschild in 1971 [1] as an upper bound for a well-known problem in Ramsey Theory.

They were also able to show a lower bound for the same problem, of six, and Martin Gardner [2] commented that this 'would make Graham's number perhaps the worst smallest upper bound ever discovered'. However, Geoff Exoo of Indiana State University proved, in 2003 [3], that the lower bound is at least 11; not such a gap then.

I shall now do two things. First, I shall define Graham's number using Knuth's arrow notation [4]. Let

$$\begin{aligned} 3 \uparrow 3 &= 3^3 = 27, \\ 3 \uparrow\uparrow 3 &= 3 \uparrow (3 \uparrow 3) = 3 \uparrow 27 = 7,625,597,484,987, \\ 3 \uparrow\uparrow\uparrow 3 &= 3 \uparrow\uparrow (3 \uparrow\uparrow 3) = 3 \uparrow\uparrow 7,625,597,484,987 \\ &= 3 \uparrow (7,625,597,484,987 \uparrow 7,625,597,484,987), \end{aligned}$$

which is already too large to write down, and

$$3 \uparrow\uparrow\uparrow\uparrow 3 = 3 \uparrow\uparrow\uparrow (3 \uparrow\uparrow\uparrow 3),$$

which is even bigger. Then Graham's number is G_{64}, where G_n is defined recursively by

$$\begin{aligned} G_1 &= 3 \uparrow\uparrow\uparrow\uparrow 3, \\ G_n &= 3 \uparrow^{(G_{n-1})} 3. \end{aligned}$$

The 'exponent' in this last expression indicates that there are G_{n-1} upward arrows between the two threes.

Next, here is the problem, as interpreted by me. Consider an n-dimensional hypercube, and connect each pair of vertices to obtain a complete graph on 2^n vertices. Then colour each of the edges of this graph using only the colours red and black. What is the smallest value of n for which every possible such colouring must necessarily contain a single-coloured complete subgraph with four vertices that lies in a plane?

Graham's number is certain to be at least bigger than this n. I believe (but cannot find the reference) that it was Donald Knuth who pointed out that if every particle in the universe were turned into ink there wouldn't be enough to write the number down. Just as well, really, since there wouldn't be any paper.

References

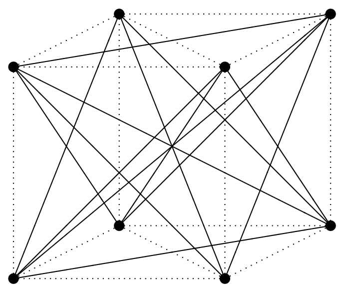
1. R. L. Graham and B. L. Rothschild, Ramsey's theorem for n-parameter sets, *Trans. Amer. Math. Soc.* **159** (1971) 257–292.

2. Martin Gardner, Mathematical games, *Scientific American*, Nov. 1977.

3. G. Exoo, A Ramsay problem on hypercubes, http://isu.indstate.edu/ge/GEOMETRY/cubes.html.

4. D. E. Knuth, Mathematics and computer science: coping with finiteness. Advances in our ability to compute are bringing us substantially closer to ultimate limitations, *Science* **194** (1976) 1235–1242.

For illustration, we prove that $n > 3$, where n is the parameter in the problem related to Graham's number. The picture shows a 2-colouring (solid black and dotted black) of the 28 lines joining pairs of vertices of a 3-dimensional cube. You can verify that none of the twelve planar configurations of four vertices has six connecting lines of the same colour. — **ADF**



Problem 215.4 – Four vertices

Four vertices of an n-dimensional hypercube are chosen such that they lie in a plane. In how many ways can this be done?

Solution 210.2 – Cosecs

Show that

$$\operatorname{cosec} 10^\circ + \operatorname{cosec} 50^\circ - \operatorname{cosec} 70^\circ \;=\; 6.$$

Are there other interesting identities of the same kind?

Nick Hobson

Using de Moivre's theorem, equating the imaginary part of $(\cos x + i \sin x)^3$ to $\sin 3x$ (or otherwise), we obtain:

$$\sin 3x \;=\; 3 \sin x - 4 \sin^3 x.$$

If $a = 10^\circ$, 50°, or -70°, then $\sin 3a = 1/2$. Hence, letting $s = \sin a$, we have

$$8s^3 - 6s + 1 \;=\; 0. \tag{1}$$

That is, $\sin 10^\circ$, $\sin 50^\circ$, and $-\sin 70^\circ$ are three distinct (and therefore the only) roots of (1). So $1/\sin 10^\circ$, $1/\sin 50^\circ$, and $-1/\sin 70^\circ$ are roots of $t^3 - 6t^2 + 8 = 0$; whence

$$\operatorname{cosec} 10^\circ + \operatorname{cosec} 50^\circ - \operatorname{cosec} 70^\circ = 6$$

by Viète's relations.

A similar identity,

$$\operatorname{cosec} 6^\circ - \operatorname{cosec} 42^\circ - \operatorname{cosec} 66^\circ + \operatorname{cosec} 78^\circ \;=\; 8,$$

is easily proved. We begin by considering

$$\sin 5x \;=\; 5 \sin x - 20 \sin^3 x + 16 \sin^5 x.$$

If $a = 6^\circ$, 30°, -42°, -66°, or 78°, then $\sin 5a = 1/2$. Hence, letting $s = \sin a$, we have

$$32s^5 - 40s^3 + 10s - 1 \;=\; 0. \tag{2}$$

That is, $\sin 6^\circ$, $\sin 30^\circ$, $\sin -42^\circ$, $\sin -66^\circ$, and $\sin 78^\circ$ are five distinct (and therefore the only) roots of (2). Dividing by $2s - 1$, we obtain

$$16s^4 + 8s^3 - 16s^2 - 8s + 1 \;=\; 0,$$

with roots $\sin 6^\circ$, $-\sin 42^\circ$, $-\sin 66^\circ$, and $\sin 78^\circ$. Thus $\operatorname{cosec} 6^\circ$, $\operatorname{cosec} -42^\circ$, $-\operatorname{cosec} - 66^\circ$, and $\operatorname{cosec} 78^\circ$ are the roots of

$$t^4 - 8t^3 - 16t^2 + 8t + 16 \;=\; 0,$$

whence

$$\operatorname{cosec} 6^\circ - \operatorname{cosec} 42^\circ - \operatorname{cosec} 66^\circ + \operatorname{cosec} 78^\circ \;=\; 8$$

by Viète's relations.

Here is an identity that is not of the same kind, but is quite intriguing:

$$\sin 18^\circ \sin 27^\circ \sin 39^\circ \;=\; \sin 15^\circ \sin 24^\circ \sin 57^\circ.$$

Solution 211.4 – Trigonometric product

Compute $\left(\sin x\ \cos \tfrac{1}{2}x\right)^{1/2} \left(\sin \tfrac{1}{2}x\ \cos \tfrac{1}{4}x\right)^{1/4} \ldots .$

Nick Hobson

We assume that the quantities $\sin(x)\cos(x/2)$, $\sin(x/2)\cos(x/4)$, … are all non-negative, so that the product is real. This is true, for instance, for $x \in [0, 2\pi]$. Let $P(n)$ be the partial product of the first n bracketed terms. In each bracketed term we apply the identity $\sin 2t = 2 \sin t \cos t$:

$$\begin{aligned} P(n) &= \left(2\sin\frac{x}{2}\cos^2\frac{x}{2}\right)^{1/2}\left(2\sin\frac{x}{4}\cos^2\frac{x}{4}\right)^{1/4}\ldots\left(2\sin\frac{x}{2^n}\cos^2\frac{x}{2^n}\right)^{1/2^n} \\ &= 2^{1-1/2^n}\left(\sin\frac{x}{2}\cos\frac{x}{4}\right)^{1/2}\left(\sin\frac{x}{4}\cos\frac{x}{8}\right)^{1/4}\ldots \\ &\qquad \left(\sin\frac{x}{2^{n-1}}\cos\frac{x}{2^n}\right)^{1/2^n}\left|\cos\frac{x}{2}\right|\left(\sin\frac{x}{2^n}\right)^{1/2^n}. \end{aligned}$$

We have assumed that $\sin(x/2^n)$ is non-negative. Note that $(\cos^2(x/2))^{1/2}$ $= |\cos(x/2)|$, as we must take the positive square root.

The terms of the form $(\sin(x/2^{k-1})\cos(x/2^k))^{1/2^{k-1}}$ are the squares of the corresponding terms in the original expression for $P(n)$. Hence, if $\sin(x)\cos(x/2)$ is non-zero, we can write

$$\begin{aligned} P(n) &= 2^{1-1/2^n}\left|\cos\frac{x}{2}\right|\left(\sin\frac{x}{2^n}\right)^{1/2^n}\left(\frac{P(n)}{(\sin(x)\cos(x/2))^{1/2}}\right)^2 \\ &= 2^{1-1/2^n}\left(\sin\frac{x}{2^n}\right)^{1/2^n}\frac{P(n)^2}{|\sin x|}. \end{aligned}$$

If $P(n)$ is non-zero, we can divide by $P(n)$ and rearrange:

$$P(n) \;=\; \frac{|\sin x|}{2^{1-1/2^n}(\sin(x/2^n))^{1/2^n}}.$$

Thus $\lim_{n\to\infty} P(n) = |\sin(x)|/2$. This holds unless $P(n) = 0$ for some n.

Balls

ADF

In M500 **207** we printed this interesting problem from Bob Newman:

> A and B play snooker. They play a complete frame which B wins with the minimum possible score, and B pots only one ball. *What colour is it?*

Nobody has sent the answer. So we shall now reveal all, even though only 16 months have elapsed. Thanks to **Jeremy Humphries** for the following material.

The smallest possible winning score 16. Here's one way to do it. (i) A pots the cue ball and all 15 reds on a single stroke. One must assume that B has been cooperative in helping A set this up. Anyway it's a foul—so the score is (0,4). (ii) B scores nothing (0,4). (iii) A pots yellow, green and brown (9,4). (iv) B pots blue (9,9). (v) A pots pink (15,9). (vi) B pots black (15,16).

And another. (i) A pots all the reds off a single foul stroke (0,4). (ii) B pots yellow and green (0,9). (iii) A pots brown, blue and pink (15,9). (iv) B pots black (15,16).

But you have noticed that in both of these examples the number of balls potted by B is nonsingular, and with only a superficial knowledge of the game it seems that there is no way to avoid this situation. However, there exists a little-known snooker rule, which says:

> When only the black remains on the table, the frame ends with the first score or foul, unless the scores are then level, in which case the black is re-spotted.

Now the solution to the problem is clear. (i) A pots all the reds off a single foul stroke (0,4). (ii) B's visit scores nothing (0,4). (iii) A pots yellow, green and brown (9,4). (iv) B pots BLUE (9,9). (v) A pots pink (15,9). (vi) A fouls the black (15,16) and the frame ends under the above rule.

Problem 215.5 – Pins

Norman Graham

What is the probability of a pin of unit length crossing a crack if dropped at random on to a floor consisting of (I) infinitely long parallel floorboards of width a, or (II) square blocks of side a?

Letters to the Editor

What's next?

Tony,

With regard to the sequence 1, 11, 21, 1211, ... [M500 **213** 22].

Consider a subsequence of four digits, a, b, c, d. In order to get a digit 4 in the next number in the sequence, all four digits must be the same. Next we split the sequence into pairs of digits so that the first of each pair is the multiplier and the second is the digit being replicated. If a is a multiplier then b and d must be different, otherwise the two sequences of bs would be consecutive and would have been combined in the previous step. On the other hand, if a is a digit being replicated, then c cannot be the same, as the same argument then applies. Either way, one of the four digits in any subsequence must differ and hence you can never exceed three digits.

I am giving some thought to the growth situation. If we simply consider the growth factor of arbitrary combinations of digits, I get something a bit less than 1.3, but this does not take into account the relative proportions of these combinations. I suspect this is more subtle and needs further contemplation ....

Can I wish you and M500 a Happy New Year,

Bill Purvis

Rain

Many thanks, rather late, for M500 **208**, with another giant opus by Sebastian Hayes.

Although I have nothing useful to say about anything, I do remember that a few years ago someone asked the rain question in the *New Scientist*, and after some calculations of dubious validity, it was decided that the best way to go was at a brisk walk, leaning forward at such an angle that the rain only fell on one's upper surface, depending on the angle of the rain and the feasibility of adopting the appropriate attitude. It will be interesting to see if anyone can come up with anything more rigorous. First, assume the pedestrian to be a perfect sphere. ...

On the same page, 19, the curiosity of the diagram is not the concentric circles, which seem to my pea-sized brain to be an obvious consequence, but that I can't help seeing the background as light grey, apart from the small central circle. The straight line diagram on page 18 does not cause this illusion.

Best wishes,

Ralph Hancock

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