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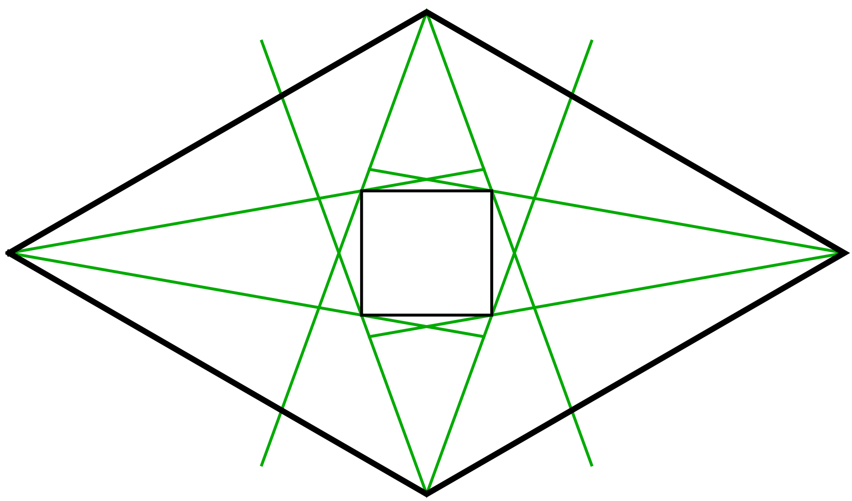
M500 298



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M500 Revision Weekend 2021 In the light of the continuing coronavirus pandemic, the M500 committee has reluctantly decided that at this point we should not be taking bookings for the 2021 Revision Weekend, planned for the 7th – 9th May. We will review the decision in early spring 2021, and keep you informed. We have thoroughly researched the possibility of running an electronic weekend equivalent but have decided it would not be possible to do so.

Decomposition of K_{10} into Petersen graphs

Reinhardt Messerschmidt

Many textbooks, for example [1], [2], [3], use the following problem as an example of how linear algebra can be used to solve problems in graph theory:

> Can the edges of K_{10} (the complete graph on 10 vertices) be coloured red, green and blue in such a way that the resulting red, green and blue spanning subgraphs are each isomorphic to the Petersen graph?

The answer to the question is 'no', which is where [1], [2], [3] stop. We will go one step further and identify the graph(s) that the blue graph can be isomorphic to, given that the red and green graphs are each isomorphic to the Petersen graph. This is not a new result, but I could find only one reference, at [5], and the solution given there was not clear to me. I will attempt to present a clearer solution.

Terminology and notation

Unless otherwise stated, the terminology and notation of [3] will be used.

Given a colouring of the edges of K_{10}, let R, G, B denote the resulting red, green and blue spanning subgraphs, and let A_R, A_G, A_B denote their adjacency matrices. If u, v are vertices of K_{10}, then v is respectively a *red neighbour*, *green neighbour* or *blue neighbour* of u if the edge uv is red, green or blue.

Let P denote the Petersen graph, and let X denote the circulant graph on 10 vertices with connection set $\{\pm 1, 5\}$. Two drawings of X are shown in Figure 1.

Main result

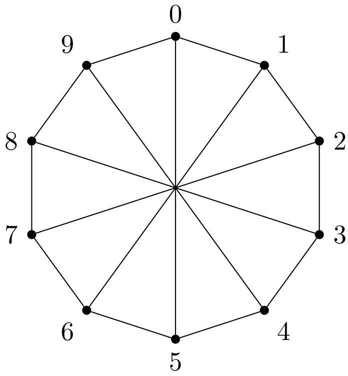
We claim that:

(i) there exists a colouring such that $R \cong P$, $G \cong P$, $B \cong X$;

(ii) X is unique, in the sense that if $R \cong P$, $G \cong P$ then $B \cong X$.

Prerequisites

We will need the following results from section 8.8 of [3]:



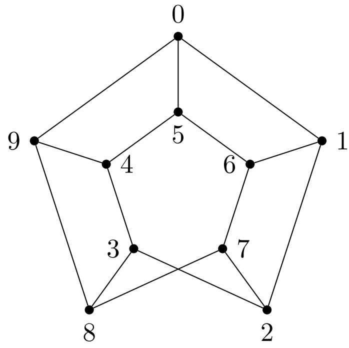


Figure 1: Two drawings of X

Proposition 1 *If H is a k-regular graph with eigenvalue $-k$, then H has a bipartite component.*

Note that H itself is not necessarily bipartite. For example, if H is the disjoint union of K_4 and $K_{3,3}$, then H is 3-regular and has -3 as an eigenvalue, but it is not bipartite, because the K_4 component has a triangle.

Proposition 2 *If H is a connected k-regular graph, then k is a simple eigenvalue of H with eigenvector $\mathbf{1}$.*

Proposition 3 *If H is a bipartite graph and λ is an eigenvalue of H with eigenvector $\boldsymbol{z} = [\boldsymbol{a},\ \boldsymbol{b}]^T$, where $\boldsymbol{a}$, $\boldsymbol{b}$ are the values of $\boldsymbol{z}$ on the two parts of H, then $-\lambda$ is an eigenvalue of H with eigenvector $[\boldsymbol{a},\ -\boldsymbol{b}]^T$.*

We will also need the following properties of the Petersen graph:

(i) its spectrum is 3, $1^{(5)}$, $(-2)^{(4)}$;

(ii) any two adjacent vertices have no common neighbours;

(iii) any two non-adjacent vertices have exactly one common neighbour.

A consequence of (ii) and (iii) is that the Petersen graph has no triangles and no 4-cycles.

Vertex	Neighbours		
	Red	Green	Blue
0	3, 4, 6	2, 7, 8	1, 5, 9
1	3, 8, 9	4, 5, 7	0, 2, 6
2	5, 6, 8	0, 4, 9	1, 3, 7
3	0, 1, 5	6, 7, 9	2, 4, 8
4	0, 7, 8	1, 2, 6	3, 5, 9
5	2, 3, 7	1, 8, 9	0, 4, 6
6	0, 2, 9	3, 4, 8	1, 5, 7
7	4, 5, 9	0, 1, 3	2, 6, 8
8	1, 2, 4	0, 5, 6	3, 7, 9
9	1, 6, 7	2, 3, 5	0, 4, 8

Table 1: A colouring for which $R \cong P$, $G \cong P$, $B \cong X$

Existence

If the edges of K_{10} are coloured according to Table 1, then the red and green graphs are as in Figure 2, and the blue graph is as in Figure 1.

Uniqueness

Suppose that $R \cong P$, $G \cong P$. The objective is to show that $B \cong X$.

Step 1. Since K_{10} is 9-regular and R, G are each 3-regular, the graph B is 3-regular.

Step 2. This step is as in [1], [2], [3]. Let $\mathcal{E}_R$, $\mathcal{E}_G$ be the eigenspaces of R, G with respect to the eigenvalue 1, and let $\mathcal{S}$ be the span of $\mathcal{E}_R \cup \mathcal{E}_G$. Note that $\mathcal{S} \subseteq \mathbf{1}^\perp$ and $\dim \mathcal{E}_R = \dim \mathcal{E}_G = 5$; therefore

$$\dim \mathcal{S} \;\le\; \dim(\mathbf{1}^\perp) \;=\; 9, \qquad \dim \mathcal{E}_R + \dim \mathcal{E}_G \;=\; 10;$$

therefore $\dim \mathcal{S} < \dim \mathcal{E}_R + \dim \mathcal{E}_G$. This implies that there exists a nonzero $\boldsymbol{x} \in \mathcal{E}_R \cap \mathcal{E}_G$, because if $\mathcal{E}_R \cap \mathcal{E}_G = \{\mathbf{0}\}$ then $\dim \mathcal{S} = \dim \mathcal{E}_R + \dim \mathcal{E}_G$. Since $A_R + A_G + A_B = J - I$, we have

$$\begin{aligned} A_B \boldsymbol{x} &= (J - I - A_R - A_G)\boldsymbol{x} \\ &= \mathbf{0} - \boldsymbol{x} - 1\boldsymbol{x} - 1\boldsymbol{x} \\ &= (-3)\boldsymbol{x}; \end{aligned}$$

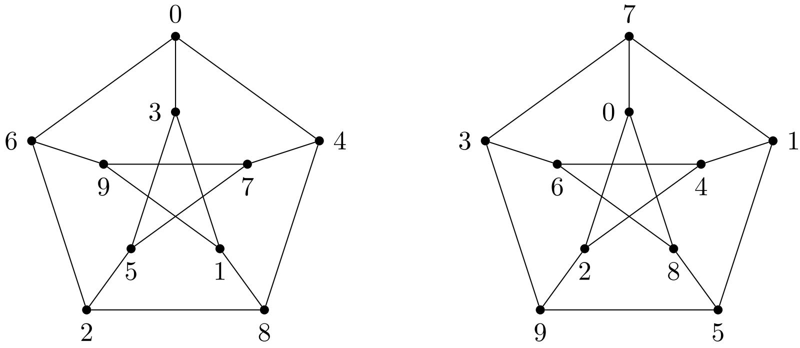


Figure 2: The red and green graphs resulting from the colouring in Table 1

therefore -3 is an eigenvalue of B. (The Petersen graph does not have -3 as an eigenvalue, so we can already conclude that $B \not\cong P$.) It follows by proposition 1 that B has a bipartite component.

Step 3. Since B is 3-regular, each of its components has at least 4 vertices. One of the following therefore holds:

(i) B has exactly two components B_1, B_2, with $B_1 \cong K_4$ and B_2 a bipartite graph on 6 vertices;

(ii) B is a connected bipartite graph on 10 vertices.

We will show that (i) leads to a contradiction. (I believe that [5] omits this necessary step.)

Suppose (i) holds. Let 0, 1, 2, 3 be the vertices of B_1, i.e. the edges 01, 02, 03, 12, 13, 23 are blue. Let 4, 5, 6 be the red neighbours of 0. The edge 01 is blue; therefore 0, 1 are not adjacent in R; therefore 0, 1 have exactly one common red neighbour. This implies that exactly one of 14, 15, 16 is red. The other two are green, because the three blue neighbours of 1 are 0, 2, 3. We may assume that 14 is red and 15, 16 are green.

Similarly, one of 24, 25, 26 is red and the other two are green. The red edge is in fact one of 25, 26, because if both 25, 26 are green then 15261 is a green 4-cycle. We may assume that 25 is red and 24, 26 are green. Similarly, 36 is red and 34, 35 are green.

The edge 45 is blue, because if 45 is red then 0450 is a red triangle, and if 45 is green then 3453 is a green triangle. Similarly, 46 and 56 are blue. It

Figure 3: The two possibilities for $\overline{B}$

follows that 4564 is a triangle in B_2, which contradicts the fact that B_2 is bipartite.

Step 4. So far, we know that B is a connected 3-regular bipartite graph on 10 vertices, and there exists a nonzero vector $\boldsymbol{x}$ such that $A_R\boldsymbol{x} = A_G\boldsymbol{x} = 1\boldsymbol{x}$ and $A_B\boldsymbol{x} = (-3)\boldsymbol{x}$. Let V_1, V_2 be the two parts of B. Since B is regular, the two parts have the same size, i.e. $|V_1| = |V_2| = 5$. We will show that every $u \in V_1$ has exactly one red neighbour in V_2. Since $|V_2| = 5$ and u has exactly three blue neighbours in V_2, it will also follow that u has exactly one green neighbour in V_2.

Let $\boldsymbol{y}$ be the vector that is 1 on V_1 and -1 on V_2. By propositions 2 and 3, the vector $\boldsymbol{y}$ is a simple eigenvector of B with eigenvalue -3; therefore $\boldsymbol{x}$, $\boldsymbol{y}$ are scalar multiples of each other. We may assume that $\boldsymbol{x} = \boldsymbol{y}$. Let a, b be the number of red neighbours of u in V_1, V_2 respectively. Since $A_R\boldsymbol{y} = 1\boldsymbol{y}$, we have $1 \cdot a + (-1) \cdot b = 1$. Since $a, b \geq 0$ and $a + b = 3$, we have $a = 2$ and $b = 1$, and so we are done.

Similarly, every vertex in V_2 has exactly one red neighbour and exactly one green neighbour in V_1.

Step 5. The graph B can be viewed as a spanning subgraph of $K_{5,5}$. Let $\overline{B}$ be its complement with respect to $K_{5,5}$. In other words, $\overline{B}$ consists of all non-blue edges with one endpoint in V_1 and one endpoint in V_2. It follows that $\overline{B}$ is a 2-regular bipartite graph; therefore $\overline{B}$ is a disjoint union of cycles of even length; therefore $\overline{B}$ is isomorphic to one of the two graphs in Figure 3.

Suppose, for a contradiction, that $\overline{B}$ is isomorphic to the left-hand side graph in Figure 3. By step 4, we may assume that 03, 69, 25 are red and 05, 36, 29 are green. Suppose 06 is red. The edge 39 is then green, because if 39 is red then 39603 is a red 4-cycle. The edge 26 is red, because if 26 is green then 26392 is a green 4-cycle. The edge 59 is green, because if 59 is red then 59625 is a red 4-cycle. The edge 02 is green, because if 02 is red then 0260 is a red triangle. It follows that 02950 is a green 4-cycle, which is a contradiction. If 06 is green, then a contradiction can be derived in a similar way.

Step 6. The complement, with respect to $K_{5,5}$, of the right-hand side graph in Figure 3 is the graph X in Figure 1.

Generalization

We can go even further and ask:

> In how many ways can the edges of K_{10} be coloured red, green and blue so that the resulting red, green and blue spanning subgraphs are each a 3-regular graph?

It is reported in [4] that, for an appropriate definition of two colourings being 'different', there are 8999 ways.

References

[1] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Springer, New York, 2011.

[2] D. Cvetković, P. Rowlinson and S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press, Cambridge, 2010.

[3] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer, New York, 2001.

[4] G. B. Khosrovshahi, Ch. Maysoori and B. Tayfeh-Rezaie, *A note on 3-factorizations of* K_{10}, J. Combin. Des., 9 (2001), pp. 379-383.

[5] `https://mathoverflow.net/questions/87886` (16 November 2020).

Problem 298.1 – Vectors

Tony Forbes

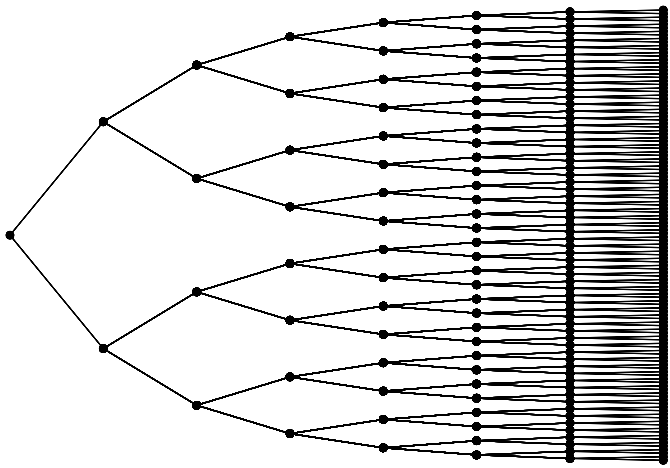
Given an integer $n \geq 2$, show how to construct a set of n mutually orthogonal linearly independent vectors of dimension n that includes the all-ones vector. Here's an example when $n = 4$:

$$\{(1,1,1,1),\quad (1,1,-1,-1),\quad (1,-1,1,-1),\quad (-1,1,1,-1)\}.$$

Solution 208.2 – Binary tree

Imagine the picture below extended to infinity left, right, up and down. It is clear that there are infinitely many nodes; but what sort of infinity? Are the nodes countable or uncountable?

What about the simpler case, where the diagram extends to infinity only to the right?



Peter Fletcher

The number of nodes in each layer from left to right is obviously

$$1, 2, 4, 8, 16, \ldots \;=\; 2^0, 2^1, 2^2, 2^3, 2^4, \ldots, 2^n, \ldots.$$

If we take logs to base 2 of the number of nodes in each layer and add 1, we shall get $1, 2, 3, 4, 5, \ldots, n + 1, \ldots$.

Therefore we can put the numbers of nodes layer by layer in one-to-one correspondence with every element of $\mathbb{N}$ and the total number of nodes is countably infinite.

Georg Cantor

Was ist los mit dem folgenden Argument? Weisen Sie einem absteigenden Pfad 0 und einem aufsteigenden Pfad 1 zu. Dann erzeugt jeder Pfad eine eindeutige unendliche Folge von Nullen und Einsen. Nach meinem diagonalen Argument muss es also unzählige Wege geben.

Solution 295.2 – States

What is the probability of winning a game of *Hangman* where the words are restricted to the names of six-letter USA states. Assume only one life. Assume also that you and your opponent always play sensibly.

(At the start you choose a 6-letter USA state, S, say, and maybe draw six dashes, _ _ _ _ _ _. Then $(*)$ your opponent chooses a letter, α, say. If α does not occur in S, the game ends and you win. Otherwise you reveal the position(s) of α in S. If S is identified, the game stops and you lose. Otherwise and the game continues from $(*)$.) See also M500 **276**.

Ted Gore

I approached this question in an intuitive way. I first decided on a minimum number of responses that player 2 should consider making and chose {a n} since these letters allow player 2 to identify 4 and 3 of the states respectively (based on the position of the chosen letter in the state name) and together they cover all the states.

I grouped the states together in subsets AH = {Alaska, Hawaii}, KN = {Kansas, Nevada} and O = {Oregon}

Player 2 can win by choosing 'a' if player 1 has chosen from AH; by choosing 'a' or 'n' if player 1 has chosen from KN ; by choosing 'n' if player 1 has chosen O.

I chose to assume that the loser of each game would pay the winner 1 pound. In Game Theory terms this makes the game a zero sum game.

The game can be represented by a Game Theory tableau where $1, -1$ is a win for player 1 and $-1, 1$ is a win for player 2.

	a	n
AH	$-1, 1$	$1, -1$
KN	$-1, 1$	$-1, 1$
O	$1, -1$	$-1, 1$

Looking at this, player 1 is bound to lose by choosing KN and it can be removed from the tableau leaving the following

	a	n
AH	$-1, 1$	$1, -1$
O	$1, -1$	$-1, 1$

A pure Nash equilibrium occurs when the result for a particular row and column cannot be improved on for either player by moving to a different row/column. There is no pure equilibrium in this game. There is a mixed strategy Nash equilibrium. Both players should choose one of their options at random with 0.5 probability. They will each win half of the games. There are easy ways to calculate these probabilities for a 2×2 game but they are not suitable for larger games.

There is a way to reduce the number of rows or columns in a game depending on the idea of dominance.

Row A strictly dominates row B if each outcome for player 1 using choice A is greater than the corresponding outcome using choice B.

Row A weakly dominates row B if each outcome for player 1 using choice A is greater than or equal to the corresponding outcome using choice B.

A column may dominate other columns in a similar fashion.

A strictly dominated row can be eliminated without affecting the equilibria. A weakly dominated row can be eliminated but may reduce the number of Nash equilibria (and may therefore result in one or other of the players missing a better strategy).

My intuitive decision to group states together in the six-letter game is equivalent to removing weakly dominated strategies.

A better method for solving games is to use linear programming. I found a solution to this game using the linear programming software in Octave with player 2 considering using all 14 letters that appear in the state names. The solution was the same as the one above. In this case the removal of weakly dominated rows and columns did not affect the final result.

For the game with states with eight letter names I started with {a i n r o l} for player 2 since each of these identifies three states. This starts as a 6×6 game.

By iterative application of the removal of weakly dominated rows and columns we arrive at the following.

	n	r	o
D	$1, -1$	$-1, 1$	$1, -1$
K	$-1, 1$	$1, -1$	$1, -1$
Ok	$1, -1$	$1, -1$	$-1, 1$

The solution to this game is that each player should employ a mixed strategy, choosing each of their options at random with 1/3 probability. Player

1 will win 2/3 of the games.

In this case the elimination of rows and columns has resulted in some solutions being lost.

There are better mixed strategies for player 2.

The linear programming solution to the 6×6 game with player 2 considering just the letters {a i n r o l} produced the following solution.

State	Probability	Letter	Probability
D	1/5	a	1/5
K	2/5	n	2/5
M	1/5	r	1/5
Ok	1/5	o	1/5

Using this strategy player 1 would win 3/5 of the games. It is obviously better for player 2 to adopt this strategy rather than the one for the 3×3 game.

The game in which player 2 considers using all eleven letters that occur in 2 or 3 state names {a i n r o l e k m s u} gives the following solution. Note that player 2 actually only needs to use six of the letters.

State	Probability	Letter	Probability
D	2/11	n	2/11
I	1/11	r	3/11
K	3/11	o	2/11
M	2/11	l	1/11
Ok	2/11	e	1/11
V	1/11	k	2/11

This results in player 1 winning 6/11 of the games, once again an improvement for player 2.

The game in which player 2 considers using all 19 letters gives the same result as above. No further improvement is possible for player 2.

A computer simulation confirmed the expected number of games won using the above probabilities.

6-letter states: Alaska, Hawaii, Kansas, Nevada, Oregon;
7: Alabama, Arizona, Florida, Georgia, Indiana, Montana, Vermont, Wyoming;
8: Arkansas, Colorado, Delaware, Illinois, Kentucky, Maryland, Michigan, Missouri (M), Nebraska, Oklahoma, Virginia.

Solution 169.4 – Functional inequality

> The function f takes a positive integer, n, as an operand, and must produce a positive integer result; that is, the function is undefined unless both n and $f(n)$ are positive integers. If, for any positive integer, n, it is always true that $f(n+1) > f(f(n))$, prove that $f(n) = n$ must follow as a consequence.

David Sixsmith

I must admit I am a little unsure about the solution to 169.4 in M500 **296**. The solution states that powers are the only 'nonlinear functions that have inputs and outputs both positive integers'. But I don't quite understand this. For example, $f(n) = n!$ is a counter-example.

This seems a really nice and quite tricky problem. Here is an alternative solution. We have this function f defined from the natural numbers $\mathbb{N}$ to itself, and such that:

$$f(n+1) \; > \; f(f(n)) \qquad \text{for all } n. \tag{A}$$

We can rewrite this as

$$f(n) \; > \; f(f(n-1)) \qquad \text{for } n > 1. \tag{B}$$

We claim first that:

$$f(n) \; \geq \; n \qquad \text{for all } n. \tag{C}$$

We prove (C) by induction on $f(n)$. When $f(n) = 1$ (C) asserts that $f(n) = 1$ implies that $n = 1$. To see this suppose that $f(n) = 1$ and $n > 1$. Then by (B) we have $1 = f(n) > f(f(n-1))$. This is impossible as it asserts that the right-hand side is a natural number less than one. This contradiction completes the proof in the case $f(n) = 1$.

Now suppose we have proved (C) for $f(n) = 1, 2, \ldots, m$ for some $m \geq 1$. We need to prove that (C) holds when $f(n) = m+1$, i.e. that $n \leq m+1$. We can suppose $n > 1$, since otherwise there is nothing to prove. By (B) we have $m+1 = f(n) > f(f(n-1))$. Thus $f(f(n-1)) \leq m$. Hence we can apply induction, twice, and deduce first that $f(n-1) \leq m$, and then that $n-1 \leq m$. Hence $n \leq m+1$ as required. This completes the inductive proof of (C).

Finally we need to show that (C) implies that $f(n) = n$ for all n. Note first by (A) and (C) that

$$f(n+1) \; > \; f(f(n)) \; \geq f(n) \qquad \text{for all } n. \tag{D}$$

In other words, f is strictly increasing. Now, suppose by way of contradiction that there is n such that $f(n) > n$, so that $f(n) \geq n+1$. Then, by (A), and because f is strictly increasing,

$$f(n+1) \;>\; f(f(n)) \;\geq\; f(n+1).$$

This is a contradiction which completes the proof.

Solution 293.2 – Graphs with integer eigenvalues

For $i = 1, 2, \ldots,$ define a graph G_i as follows.
Let $n_i = (i-1)^2 + 1$. The vertices of G_i are $1, 2, \ldots, n_i$. For the edges, write down the pairs $\{a, b\}$, $1 \leq a < b \leq n_i$ in lexicographical order and remove the last $i(i-1)/2$ items from the list. The remaining $n_i(n_i-1)/2 - i(i-1)/2$ pairs form the edges of G_i.
Prove that the adjacency matrix of G_i has integer eigenvalues, or find a counter-example. For the first few, we have the following.

i	n_i	edges	eigenvalues
1	1	0	0
2	2	0	$0, 0$
3	5	7	$3, -2, -1, 0, 0$
4	10	39	$8, -3, -1, -1, -1, -1, -1, 0, 0, 0$
5	17	126	$15, -4, -1$ eleven times$, 0, 0, 0, 0$
6	26	310	$24, -5, -1$ nineteen times$, 0, 0, 0, 0, 0$

Tommy Moorhouse

For each natural number $i > 0$ a graph G is constructed with n_i vertices by deleting the edges connecting a subset of i vertices. For $i > 3$ the construction is that of the smash product $K_{n_i - i} \vee N_i$ (see M500 Problem 294.6). An adjacency matrix of a smash product $G \vee H$ where G has n vertices and H has m vertices can be constructed as follows. Set an adjacency matrix for G in the top left block of an $(n+m) \times (n+m)$ matrix, and set an adjacency matrix for H in the bottom right block. Then fill the top right and bottom left blocks with 1s. To avoid notational clutter we denote the blocks of 1s by U, keeping in mind that they have different shapes in general. That is,

$$\begin{pmatrix} K_{n\times n} & U_{n\times m} \\ U_{m\times n} & N_{m\times m} \end{pmatrix} \to \begin{pmatrix} K & U \\ U & N \end{pmatrix}$$

Suppose that $\Lambda \equiv K_{n_i - i} \vee N_i$ has an eigenvector v, which we write in block form as $(t_n, b_m)^T$, so that $\Lambda v = \lambda v$. We write $\sum t$ for $(\sum_{k=1}^{n} t_k)U_{m\times 1}$ and $\sum b$ for $(\sum_{k=1}^{m} b_k)U_{n\times 1}$. We have

$$\begin{pmatrix} K & U \\ U & N \end{pmatrix}\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} Kt + \sum b \\ \sum t \end{pmatrix} = \begin{pmatrix} \lambda t \\ \lambda b \end{pmatrix}.$$

Here we have used the unique adjacency matrix for N_m consisting of an $m \times m$ block of zeros. The bottom block tells us that b is a scalar multiple of $U_{n\times 1}$, so that $\lambda \sum b = m \sum t$.

Now we consider the eigenvectors of K_n. It can be shown (see for example M500 Solution 282.5) that the eigenvalues of K_n are $n-1$ and -1 ($n-1$ times). The eigenvector associated with $n-1$ is $U_{n\times 1}$. If $K_n t = -t$ then $\sum t = 0$ (check this). So we take t to be an eigenvector of K_n with $Kt = \mu t$ and find that

$$(\lambda - \mu)t = \sum b = \lambda^{-1} m \sum t.$$

We have two cases to consider. First, take $\mu = n-1$ and $t = U_{n\times 1}$. Then $\sum t = n$ and

$$\lambda(\lambda - \mu) = mn.$$

The construction of G tells us that $n = n_i - i$ and $m = i$. The solutions for λ (after some straightforward algebra) are

$$\lambda = i(i-2),\ -(i-1),$$

both integers.

Next consider $Kt = -t$, with $\sum t = 0$. Then $b = 0_{m\times 1}$ and $\lambda + 1 = 0$, so that $\lambda = -1$ for each such eigenvector. The number of nonzero eigenvalues of Λ is the number of linearly independent rows in Λ, which is $n_i - i + 1 = n + 1$. The remaining $i-1$ eigenvalues are zero. Thus by choosing t to be an eigenvector of K_n we have found all the eigenvalues of G, which are all therefore integers.

Problem 298.2 – Three 7-cycles

Take a K_7, a complete graph on 7 vertices. Choose 7 of the 21 edges of the K_7 such that, together with their incident vertices, they form a 7-cycle. From the remaining 14 edges of the K_7, choose another 7 such that, together with their incident vertices, they too form a 7-cycle. Do the remaining 7 edges together with their incident vertices form a 7-cycle?

Solution 192.1 – Root 33

If $\theta = 2\pi/33$, show that

$$\cos\theta + \cos 2\theta + \cos 4\theta + \cos 8\theta + \cos 16\theta \;=\; \frac{1+\sqrt{33}}{4}.$$

Peter Fletcher

Let

$$\begin{aligned} S &= \cos(\theta) + \cos(2\theta) + \cos(4\theta) + \cos(8\theta) + \cos(16\theta) \\ &= \Re\Big(\exp(\mathrm{i}\theta) + \exp(2\mathrm{i}\theta) + \exp(4\mathrm{i}\theta) + \exp(8\mathrm{i}\theta) + \exp(16\mathrm{i}\theta)\Big). \end{aligned}$$

If we write down the Gaussian sum per `https://mathworld.wolfram.com/GaussianSum.html` with $p = -2$ and $q = 33$, we shall find that S is included as part of the expansion of the sum.

This particular Gaussian sum is:

$$\sum_{r=0}^{32} \exp\left(\frac{2\pi \mathrm{i} r^2}{33}\right) \;=\; \sqrt{33}$$

since $33 \equiv 1 \pmod 4$.

If we write out all 33 terms, we find that we can group them as follows, remembering that $\exp\big(\mathrm{i}(\theta + 2\pi)\big) = \exp(\mathrm{i}\theta)$:

$$\begin{aligned} \sqrt{33} \;=\; 1 + 4\Bigg[&\exp\left(\frac{2\pi\mathrm{i}}{33}\right) + \exp\left(\frac{2\pi\mathrm{i}\cdot 4}{33}\right) + \exp\left(\frac{2\pi\mathrm{i}\cdot 16}{33}\right) \\ &+ \exp\left(\frac{2\pi\mathrm{i}\cdot 25}{33}\right) + \exp\left(\frac{2\pi\mathrm{i}\cdot 31}{33}\right)\Bigg] \\ + 2\Bigg[&\exp\left(\frac{2\pi\mathrm{i}\cdot 3}{33}\right) + \exp\left(\frac{2\pi\mathrm{i}\cdot 9}{33}\right) + \exp\left(\frac{2\pi\mathrm{i}\cdot 12}{33}\right) \\ &+ \exp\left(\frac{2\pi\mathrm{i}\cdot 15}{33}\right) + \exp\left(\frac{2\pi\mathrm{i}\cdot 22}{33}\right) + \exp\left(\frac{2\pi\mathrm{i}\cdot 27}{33}\right)\Bigg]. \end{aligned}$$

Since the LHS of this equation is obviously real, we know that the imaginary parts on the RHS must cancel out. If we put a '$\Re$' in front of the RHS and keep the real parts unchanged, the sum will still be $\sqrt{33}$ whatever we do to the imaginary parts.

Now

$$\exp\left(\frac{2\pi\mathrm{i}\cdot 25}{33}\right) \;=\; \cos\left(\frac{2\pi\cdot 25}{33}\right) + \mathrm{i}\sin\left(\frac{2\pi\cdot 25}{33}\right)$$

and

$$\cos\left(\frac{2\pi\cdot 25}{33}\right) \;=\; \cos\left(\frac{2\pi\cdot(33-25)}{33}\right) \;=\; \cos\left(\frac{2\pi\cdot 8}{33}\right),$$

so that

$$\Re\left[\exp\left(\frac{2\pi\mathrm{i}\cdot 25}{33}\right)\right] \;=\; \Re\left[\exp\left(\frac{2\pi\mathrm{i}\cdot 8}{33}\right)\right].$$

Similarly,

$$\Re\left[\exp\left(\frac{2\pi\mathrm{i}\cdot 31}{33}\right)\right] \;=\; \Re\left[\exp\left(\frac{2\pi\mathrm{i}\cdot 2}{33}\right)\right]$$

and inside the first pair of square brackets we have 1, 2, 4, 8 and 16 multiplying $2\pi\mathrm{i}/33$ inside the exponentials. Thus we can write $\sqrt{33}$ as $1+4S+2\Re(T)$, where T is the sum of the remaining complex exponentials inside the second pair of square brackets.

Of the terms in T, we can write

$$\Re\left[\exp\left(\frac{2\pi\mathrm{i}\cdot 27}{33}\right)\right] \;=\; \Re\left[\exp\left(\frac{2\pi\mathrm{i}\cdot 6}{33}\right)\right]$$

and

$$\Re\left[\exp\left(\frac{2\pi\mathrm{i}\cdot 22}{33}\right)\right] \;=\; \cos\left(\frac{4\pi}{3}\right) \;=\; -\frac{1}{2}.$$

We can now identify the remaining five terms in T as forming a geometric sum. Referring to `https://mathworld.wolfram.com/ExponentialSumFormulas.html`, if we let $N=6$ and $x=\exp(2\pi\cdot 3/33)$, we can write

$$\Re(T) \;=\; \Re\left[\left(\sum_{n=0}^{6}\exp\left(\frac{2\pi\mathrm{i}\cdot 3n}{33}\right)-1\right)-\frac{1}{2}\right].$$

From the link to Wolfram, we can write

$$\sum_{n=0}^{6}\exp\left(\frac{2\pi\mathrm{i}\cdot 3n}{33}\right) \;=\; \frac{1-\exp\left(\dfrac{2\pi\mathrm{i}\cdot 18}{33}\right)}{1-\exp\left(\dfrac{2\pi\mathrm{i}\cdot 3}{33}\right)}$$

$$= \;\frac{1-\cos(12\pi/11)-\mathrm{i}\sin(12\pi/11)}{1-\cos(2\pi/11)-\mathrm{i}\sin(2\pi/11)}$$

$$= \frac{\big(1+\cos(\pi/11)+\mathrm{i}\sin(\pi/11)\big)\big(1-\cos(2\pi/11)+\mathrm{i}\sin(\pi/11)\big)}{\big(1-\cos(2\pi/11)\big)^2+\sin^2(2\pi/11)}.$$

The numerator of this last expression is

$$\begin{aligned} &1-\cos(2\pi/11)+\cos(\pi/11) \\ &\qquad -\cos(\pi/11)\cos(2\pi/11)-\sin(\pi/11)\sin(2\pi/11)+\mathrm{i}\Im(\mathrm{expr}) \\ &= 1-\cos(2\pi/11)+\cos(\pi/11)-\cos(2\pi/11-\pi/11)+\mathrm{i}\Im(\mathrm{expr}) \\ &= 1-\cos(2\pi/11)+\mathrm{i}\Im(\mathrm{expr}). \end{aligned}$$

The denominator is

$$\begin{aligned} &1-2\cos(2\pi/11)+\cos^2(2\pi/11)+\sin^2(2\pi/11) \\ &= 2-2\cos(2\pi/11). \end{aligned}$$

We can now write down

$$\Re(T) = \left(\frac{1}{2}-1\right)-\frac{1}{2} = -1$$

and

$$\sqrt{33} = 1+4S-2 = 4S-1.$$

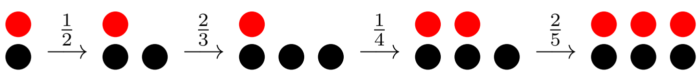
Therefore

$$S = \cos(\theta)+\cos(2\theta)+\cos(4\theta)+\cos(8\theta)+\cos(16\theta) = \frac{1+\sqrt{33}}{4}.$$

Problem 298.3 – Balls

Let $n \geq 1$ be an integer. Take a sufficiently large bucket and put in it a red ball and a black ball. Perform the following step repeatedly. Remove a ball, chosen uniformly at random, duplicate it, and then return the two balls to the bucket.

What's the probability that at some stage the bucket contains n red balls and n black balls? Here's a typical successful path (and hence one contribution to the probability computation) when $n = 3$.



The answer when $n = 1$ is 1. The balls in the top rows are red.

Three aces

This is a card trick that was presented to us at the 2020 M500 Winter Weekend by Angela and Paul.

Take a standard deck of cards and remove the four aces. Deal the rest of the deck face down into three piles until the first pile has 14 cards and the other two 15 cards each. Place the remaining four cards face down in a fourth pile. Invite your victim to:

(i) shuffle the first pile;
(ii) put one of the aces face down on top of it;
(iii) shuffle the second pile;
(iv) choose a random number (0 to 15) of cards from the second pile and place them face down on top of the ace on the first pile;
(v) put another ace face down on top of the second pile (which is not necessarily non-empty);
(vi) shuffle the third pile;
(vii) choose a random number (0 to 15) of cards from the third pile and place them face down on top of the ace on the second pile;
(viii) put another ace face down on top of the third pile (which is not necessarily non-empty);
(ix) shuffle the fourth pile and place it on top of the third pile.

Now gather the three piles to form a single stack with the first pile at the bottom, the second pile next, and the third pile at the top. Deal the single stack face down alternately into two new piles, A and B, starting with A.

Pick up the cards of A (so that A is now empty) and deal them face down alternately on to A and B, starting with B.

Pick up the cards of A (so that A is now empty) and deal them face down alternately on to A and B, starting with B.

Pick up the cards of A (so that A is now empty) and deal them face down alternately on to A and B, starting with B.

Point out to the victim that he/she was responsible for the shuffling, the random number choices and the ace placements in steps (i) to (ix), above. Point out also that three aces were buried amongst 51 cards and that pile A now contains three cards. Turn them over. Observe the looks of astonishment on the faces of your audience. Why?

Solution 192.2 – 10 degrees

Let $x = 1 + 4\sin 10^\circ$. Show that

$$x = \sqrt{11 - 2\sqrt{11 + 2\sqrt{11 - 2x}}}.$$

Peter Fletcher

We have

$$1 + 4\sin\left(\frac{\pi}{18}\right) = \sqrt{11 - 2\sqrt{11 + 2\sqrt{11 - 2x}}}.$$

Squaring both sides,

$$1 + 8\sin\left(\frac{\pi}{18}\right) + 16\sin^2\left(\frac{\pi}{18}\right) = 11 - 2\sqrt{11 + 2\sqrt{11 - 2x}}.$$

Using $2\sin^2(\theta) = 1 - \cos(2\theta)$, this becomes

$$1 - 4\sin\left(\frac{\pi}{18}\right) + 4\cos\left(\frac{\pi}{9}\right) = \sqrt{11 + 2\sqrt{11 - 2x}}.$$

From WolframAlpha, we find (from entering '$\cos(B) - \sin(A)$' and '$\cos(B) + \sin(A)$' separately) that

$$\cos(B) \pm \sin(A) = 2\sin\left(\frac{A}{2} - \frac{B}{2} \pm \frac{\pi}{4}\right)\sin\left(\frac{A}{2} + \frac{B}{2} \pm \frac{\pi}{4}\right)$$

so that using the $-$ version,

$$1 + 4\sin\left(\frac{5\pi}{18}\right) = \sqrt{11 + 2\sqrt{11 - 2x}}.$$

Here, the second sine is $\sin(-\pi/6) = -\sin(30^\circ) = -1/2$. A similar thing happens the other times we use one of these identities below.

Squaring both sides,

$$1 + 8\sin\left(\frac{5\pi}{18}\right) + 16\sin^2\left(\frac{5\pi}{18}\right) = 11 + 2\sqrt{11 - 2x}.$$

Using the double-angle formula again, this becomes

$$-1 + 4\sin\left(\frac{5\pi}{18}\right) + 4\cos\left(\frac{4\pi}{9}\right) = \sqrt{11 - 2x}$$

and using the + version of the identity from WolframAlpha, we get

$$-1+4\sin\left(\frac{7\pi}{18}\right) \;=\; \sqrt{11-2x}.$$

Squaring both sides,

$$1-8\sin\left(\frac{7\pi}{18}\right)+16\sin^2\left(\frac{7\pi}{18}\right) \;=\; 11-2x.$$

Using the double-angle formula again, this becomes

$$-1-4\sin\left(\frac{7\pi}{18}\right)+4\cos\left(\frac{2\pi}{9}\right) \;=\; -\,x.$$

Finally, using the − version of the identity from WolframAlpha, we get

$$-1-4\sin\left(\frac{\pi}{18}\right) \;=\; -\,x$$

or

$$x \;=\; 1+4\sin\left(\frac{\pi}{18}\right)$$

and the given expression is satisfied by x.

Solution 293.9 – Sin 105 degrees

What (if anything) is wrong with this argument? We have

$$\sin 45^\circ \;=\; \frac{1}{\sqrt{2}} \qquad \text{and} \qquad \sin 60^\circ \;=\; \frac{\sqrt{3}}{2}.$$

Hence

$$\sin 105^\circ \;=\; \sin 45^\circ+\sin 60^\circ \;=\; \frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2} \;=\; \frac{1+\sqrt{3}}{2\sqrt{2}}.$$

Chris Pile

The 'argument' seems to be a piece of mathematical legerdemain where the required result appears, as if by magic, after some misleading deceptions.

The word 'Hence' implies that $\sin(A+B)=\sin A+\sin B$, and the result of the calculation is obtained by optical illusion rather than addition! In fact, $1/\sqrt{2}+\sqrt{3}/2=(\sqrt{2}+\sqrt{3})/2$, which is greater than 1.

I can remember some trig. identities, including $\sin(A + B) = \sin A \cos B + \cos A \sin B$. Therefore

$$\begin{aligned}\sin 105^\circ &= \sin(45^\circ + 60^\circ) = \sin 45^\circ \cos 60^\circ + \cos 45^\circ \sin 60^\circ \\ &= \frac{1}{\sqrt{2}} \cdot \frac{1}{2} + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2\sqrt{2}},\end{aligned}$$

as expected. In the diagram, $ABEG$ is a rectangle, $|AB| = \sqrt{3}$, $|AG| = \sqrt{3} + 1$ and $|BC| = |DE| = |EF| = 1$. [Exercise for reader: fill in the remaining lengths and angles—TF.] Hence

$$\sin 105^\circ = \sin 75^\circ = \frac{|AG|}{|AF|} = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{2}{2\sqrt{2}} + \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{|AC|}{|AF|} + \frac{|GF|}{|AF|}.$$

Therefore

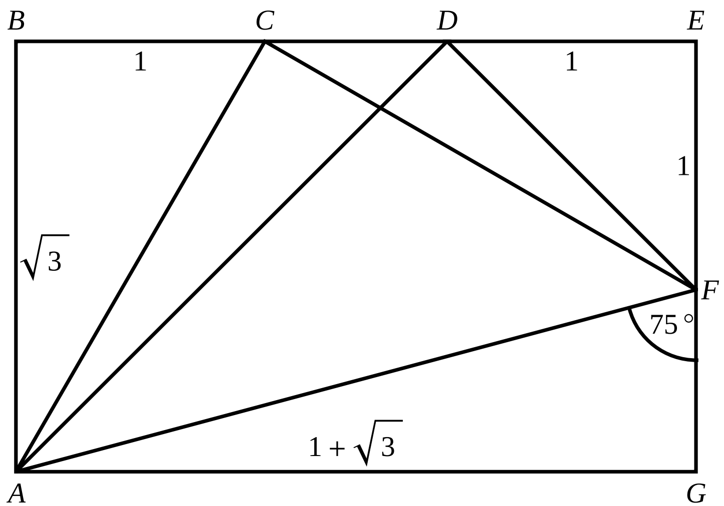
$$\sin 105^\circ = \sin 45^\circ + \sin 15^\circ; \quad \sin(45^\circ + 60^\circ) = \sin 45^\circ + \sin(60^\circ - 45^\circ).$$

This is true for any angle A, $\sin(A + 60^\circ) = \sin A + \sin(60^\circ - A)$, because $\cos 60^\circ = 1 - \cos 60^\circ = 1/2$.

Another integral relationship can be found from pentagonal geometry:

$$\sin 54^\circ = \frac{1}{\sqrt{5}-1} \quad \text{and} \quad \sin 18^\circ = \frac{1}{\sqrt{5}+1}.$$

Hence $\sin 54^\circ - \sin 18^\circ = 1/2 = \sin 30^\circ$.



Peter Michael Neumann

Judith Furner

It was with great sadness that the M500 committee heard of the death of Peter Neumann. I first met him in 1987, when I was organizing the M500 Revision Weekend at Aston University, and Peter came to speak on the Saturday evening. He gave us the most fascinating talk on Galois Theory, ending with the suggestion that some as yet undiscovered Galois papers might be secreted somewhere in Paris. We were entranced. Had he said that he had an aeroplane outside, to take us there, I think that we would have obediently followed his lead, and made our way to Paris to assist the search.

Subsequently I joined the British Society for the History of Mathematics, where Peter was a shining light, and I always enjoyed an interesting conversation with him. He was a stalwart support of the Open University, of the M500 Society, and a great promulgator of the joys of mathematics. On one occasion I invited him to Café Scientifique in Brighton, and he once again gave a fascinating talk.

Peter was unfailingly courteous and kind, interested in people and particularly students of mathematics everywhere. He is a great loss to the mathematical world and we regret his passing.

Dorothy Leddy

Peter was a major international figure in algebra, the history of mathematics and in mathematics education. He was a Tutorial Fellow at The Queen's College, Oxford, where he had gained his B.A. and his D.Phil, and a lecturer in the Mathematical Institute in Oxford, retiring in 2008, whereupon he was elected Emeritus Fellow. Peter gave great service to the College in many roles throughout his years there. He always remained deeply committed to it and its life, and he was much loved and respected by Fellows, staff and students. He was a founder member of the lively Oxford Mathematics Forum based at The Queen's College, and supported it heartily to the very end of his life.

Peter's work was in the field of group theory. In all his work he demonstrated enormous precision and attention to detail. This can be clearly seen when in 2011 he published the first full English edition of the mathematical writings of French mathematician Évariste Galois. In 1987 Peter won the Lester R. Ford Award of the Mathematical Association of America for his review of Harold Edwards' book *Galois Theory*. He was the first Chairman of the United Kingdom Mathematics Trust, from 1996 to 2004, and in the 2008 New Year Honours he was appointed Officer of the Order of the British Empire (OBE) for services to education. Peter was President of the Mathematical Association from 2015 to 2016. He also served as President of the British Society for the History of Mathematics, whose Neumann Prize is named in his honour. These are just a few of Peter's many accolades and appointments. He died on 18th December, 2020.

04.00

Problem 298.4 – Sum

Show that $$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{5\cdot 6\cdot 7}+\frac{1}{9\cdot 10\cdot 11}+\dots \;=\; \frac{\log 2}{4}.$$

Problem 298.5 – Trisected rhombus

Tony Forbes

A rhombus has small internal angle 60° and big diagonal 2 m. The angle at each vertex is trisected by lines shown in green/grey as on the front cover. Show that the thing in the middle is not square and that its sides are

$$\frac{2}{3}\left(2-\cos 20^\circ-\sqrt{3}\sin 20^\circ\right)\text{m} \quad\text{and}\quad \frac{2}{3}\left(\sqrt{3}-\sqrt{3}\cos 20^\circ+\sin 20^\circ\right)\text{m},$$

approximately 0.311941 m and 0.29765 m.