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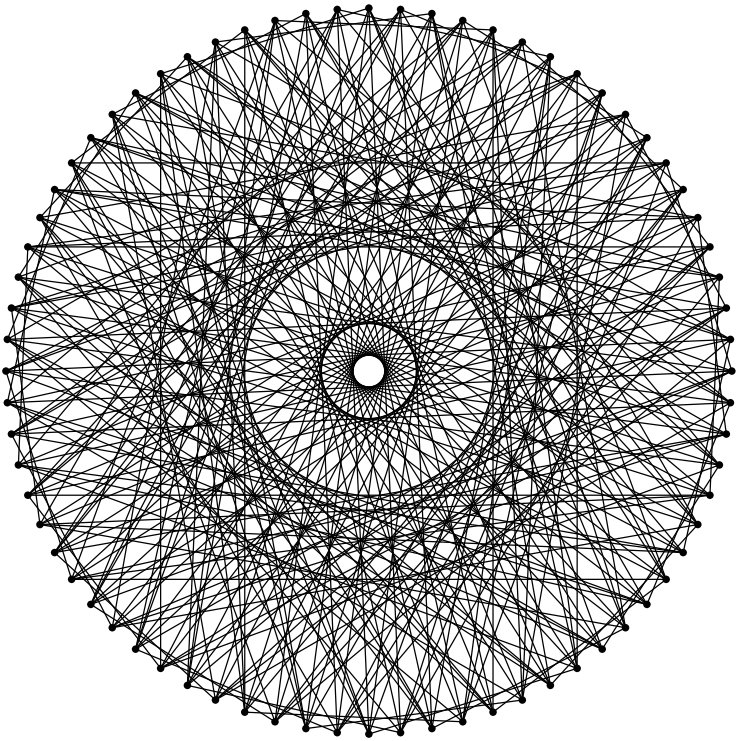

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M500 Revision Weekend 2021 In the light of the coronavirus pandemic, the M500 committee has reluctantly decided that we cannot run the 2021 Revision Weekend, planned for the 7th–9th May. The Revision Weekend 2022 is in the early stages of planning and we will keep you informed.

The Boundary Element Method

Alan Davies

1 Introduction

The *Boundary Element Method* (BEM) is a numerical method for solving problems in engineering and the physical sciences. We shall describe it as a method for solving a boundary-value problem (BVP) which involves a partial differential equation subject to boundary conditions. As well as the BEM there are two alternative approaches: the *Finite Difference Method* (FDM) and the *Finite Element Method* (FEM). The FDM requires a mesh of points over the region of interest and a set of algebraic equations is set up using finite difference approximations to the partial derivatives (Smith 1985). The FEM also sets up a set of algebraic equations at discrete points in the domain but where the unknown function is approximated in a piecewise manner by polynomials (Davies 2011). The BEM is similar to the FEM but there is enough difference to warrant a different name and the BEM name was coined by Brebbia and Dominguez (1977). It is in the setting up of the equations where the two methods differ. In the BEM the boundary-value problem is first recast as an integral equation over the boundary of the region, the method used to be called the *Boundary Integral Equation Method*. The given conditions are then used to find the solution on the interior of the region. There are two distinct parts to the process: (i) solution for the values on the boundary, (ii) recovery of the internal values.

Both these processes require the solution of a set of algebraic equations. The purpose of this article is to explain the background to the BEM and to show how to set up the equations. We shall not consider the equation solution process; any book on numerical linear algebra will have a host of suitable solution techniques, e.g. Jennings and McKeown (1992).

The BEM is applicable to any boundary-value problem described by a partial differential equation. However, in this article we shall consider the particular case of *Laplace's equation*,

$$\nabla^2 u \;=\; 0; \tag{1}$$

other partial differential equations are treated in a similar manner.

2 Preliminaries

The BEM comprises a variety of mathematical techniques; we summarise two of them in this section. We assume that all functions are sufficiently differentiable for the operations to exist.

2.1 Green's theorem

The two theorems attributed to George Green (1793–1841) are usually known as the first and second forms of *Green's theorem*. The proof of the theorems can be found in any number of undergraduate texts on vector calculus. However, it is well worthwhile reading Green's essay (1828) whose rather understated title hides a seventy-two page piece of work, which is surprisingly easy to read given that it is not far off two hundred years old. The mathematical notation in formulae and equations is remarkably similar to that which we use today. On page 8 we find the definition of the Laplace operator, on pages 10–12 we find Green's theorem and the first use of the term *potential function* appears on page 6. We can also recognize, on page 14, a *Green's function*. Einstein said that the material on page 11 was twenty years ahead of its time.

We shall require only the second form in two dimensions: if A is the plane region bounded by the closed curve C then

$$\int_A \left(v\nabla^2 u - u\nabla^2 v\right)\, dA \;=\; \oint_C \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds, \tag{2}$$

where $\mathbf{n}$ is the unit normal to the curve C pointing out of the region A.

There is a special case which we shall need: if $v \;\equiv\; 1$ then

$$\int_A \nabla^2 u \, dA \;=\; \oint_C \frac{\partial u}{\partial n}\, ds. \tag{3}$$

2.2 Fundamental solution

A *Fundamental solution*, u^*, of Laplace's equation is a function that depends only on the distance, r, from the origin, has a singularity there and is a solution of the equation

$$\nabla^2 u^* \;=\; -\,\delta\,(r)\,, \tag{4}$$

where $\delta\,(r)$ is the *Dirac delta function*.

Using plane polar coordinates, independent of the polar angle θ, u^* is given by

$$\left(r^2\frac{d^2}{dr^2} + r\frac{d}{dr}\right) u^* \;=\; -\,\delta(r).$$

A solution to the homogeneous equation

$$r^2 \frac{d^2 u^*}{dr^2} + r \frac{du^*}{dr} \;=\; 0$$

and which has a singularity at $r = 0$, is given by

$$u^* \;=\; k \ln r,$$

where k is a constant and we have ignored an additive constant.

Suppose that the region D_a is the disc centre the origin, radius a and circumference C_a. Using equation (4)

$$\int_{D_a} \nabla^2 u^* \, dA \;=\; -\int_{D_a} \delta(r) \, dA \;=\; -1.$$

So that, using (3),

$$\oint_{C_a} \frac{\partial u^*}{\partial n} \, ds \;=\; -1.$$

On C_a, $\partial/\partial n \;\equiv\; \partial/\partial r \;\equiv\; d/dr$ and $ds = a d\theta$; hence

$$\int_0^{2\pi} \left. \frac{du^*}{dr} \right|_{r=a} a d\theta \;=\; -1;$$

i.e.

$$\int_0^{2\pi} \left. k \frac{d}{dr} (\ln r) \right|_{r=a} a d\theta \;=\; -1$$

and

$$k \, \frac{1}{a} \, 2\pi a \;=\; -1,$$

from which we find $k \;=\; -1/(2\pi)$ and

$$u^* \;=\; -\frac{1}{2\pi} \ln r. \tag{5}$$

3 Boundary representation of the solution

While we shall deal only with Laplace's equation, the process applies to any BVP provided that we can find an appropriate *reciprocal theorem* such as Green's theorem, equation (2). We seek a solution of the following BVP in the plane region D bounded by the closed curve C.

Laplace's equation (1)

$$\nabla^2 u \;=\; 0,$$

subject to the *Dirichlet* condition

$$u \;=\; g(s) \qquad \text{on } C_1 \tag{6}$$

and the *Neumann* condition

$$q \;\equiv\; \frac{\partial u}{\partial n} \;=\; h(s) \qquad \text{on } C_2, \tag{7}$$

where $C \;=\; C_1 + C_2$.

N.B. It is usual notation in the BEM to write q for the boundary flux $\partial u/\partial n$.

A third boundary condition, the so-called *Robin* condition, of the form

$$q + \sigma(s)u \;=\; f(s) \qquad \text{on } C_3,$$

is also possible and can be easily incorporated into the process. Doing so adds nothing to the understanding but simply requires a bit more algebra; so we shall not consider it here.

Suppose we wish to find the solution at a fixed point P, in BEM terminology this is called the *source point*, and that Q is any other point inside D, usually known as the *field point*. We surround the point P by a small disc, D_ϵ, radius ϵ and circumference C_ϵ. Also $\mathbf{R}$ is the position vector of Q relative to P. Details of the BVP and the geometry are shown in Figure 1.

We apply the second form of Green's theorem, equation (2), to the region $D - D_\epsilon$ to obtain

$$\int_{D-D_\epsilon} \left(u\nabla^2 u^* - u^*\nabla^2 u\right)\, dA \;=\; \oint_{C+C_\epsilon} \left(u\frac{\partial u^*}{\partial n} - u^*\frac{\partial u}{\partial n}\right)\, ds.$$

Now, both u and u^* satisfy Laplace's equation in $D - D_\epsilon$ so that we have

$$\oint_{C+C_\epsilon} \left(u\frac{\partial u^*}{\partial n} - u^*\frac{\partial u}{\partial n}\right)\, ds \;=\; 0.$$

We can write this in the form

$$\oint_{C} \left(u\frac{\partial u^*}{\partial n} - u^*\frac{\partial u}{\partial n}\right)\, ds \;=\; I_1 + I_2,$$

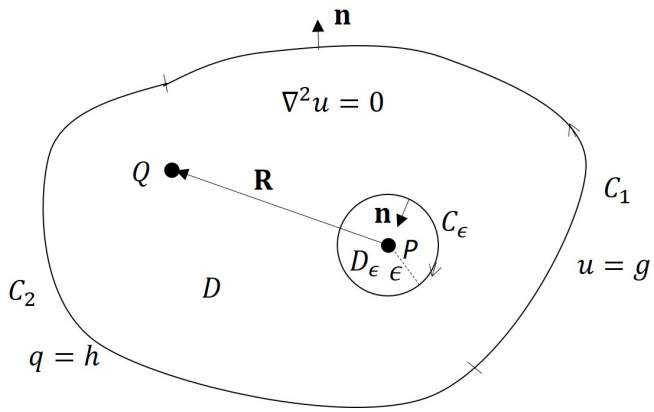

Figure 1: Boundary-value problem (1), (6), (7).

where

$$I_1 \;=\; \oint_{C_\epsilon} u^* q \, ds \quad \text{and} \quad I_2 \;=\; -\oint_{C_\epsilon} u q^* \, ds.$$

On C_ϵ, since $\mathbf{n}$ points out of $D - D_\epsilon$, see Figure 1, $\partial/\partial n \;=\; -\partial/\partial R$ so that

$$I_1 \;=\; \oint_{C_\epsilon} u^* q \, ds \;=\; -\int_0^{2\pi} \frac{1}{2\pi} \left(\ln R\right)_{R=\epsilon} q \epsilon d\theta.$$

Hence

$$|I_1| \;\leq\; 2\pi\epsilon \frac{1}{2\pi} \ln(\epsilon) \left|q(s)\right|_{\max}$$

and so $I_1 \to 0$ as $\epsilon \to 0$. Also,

$$I_2 \;=\; \oint_{C_\epsilon} u q^* \, ds \;=\; \int_0^{2\pi} \frac{1}{2\pi} \left(\frac{d \ln R}{dR}\right)_{R=\epsilon} \left(u_P + \phi(s)\right) \epsilon d\theta,$$

where, on C_ϵ,

$$u(s) \;=\; u_P + \phi(s) \quad \text{with} \quad \left|\phi(s)\right|_{\max} \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Since $\left(d(\ln R)/dR\right)_{R=\epsilon} \;=\; 1/\epsilon$ it follows that

$$I_2 \;=\; \int_0^{2\pi} \frac{1}{2\pi} u_P d\theta + \int_0^{2\pi} \frac{1}{2\pi} \phi(s) d\theta \;=\; u_P + I_3,$$

where $I_3 \to 0$ as $\epsilon \to 0$. Hence, as $\epsilon \to 0$, we obtain

$$u_P = \oint_C (u^* q - u q^*)\, ds \tag{8}$$

and we have the amazing result that the solution at any interior point depends only on the values of the function and its normal derivative on the boundary.

Unfortunately, we can't use equation (8) directly to find u_P because, for *properly-posed* problems, we can know only one of u or q at each point on the boundary. Our BVP is given by equations (1), (6) and (7) and is properly posed. So, before we can use (8), we must find the values of u on C_2 and the values of q on C_1.

Suppose now that P is a fixed point on the boundary, C, and that the angle between the left and right tangents at P is α_P. If C is smooth at P then $\alpha_P = \pi$. In a similar manner to that for which we found the equation for an interior value we can obtain the equation for a boundary point as:

$$\frac{\alpha_P}{2\pi} u_P = \oint_C (u^* q - u q^*)\, ds. \tag{9}$$

Equation (9) is a *boundary integral equation*. Even for the simplest geometry there is no analytic technique for its solution and we need to develop a numerical procedure to obtain an approximate solution. When we have an approximate solution then we can use equation (8) to find the solution at interior points. An equation similar to (8) and (9) can be found for a point P outside D and the three equations are often written as

$$c_P u_P = \oint_C (u^* q - q^* u) ds, \tag{10}$$

where

$$c_P = \begin{cases} 1 & P \in D, \\ \alpha_P/(2\pi) & P \in C, \\ 0 & P \notin D \cup C. \end{cases} \tag{11}$$

4 Approximate boundary solution

The procedure requires that we approximate the boundary geometry and the unknown functions using suitable *basis functions*. We shall consider the case when both the geometry and the function approximation is *piecewise linear*. In particular the boundary is approximated by a polygon and the functions by a set of *hat functions* as follows.

Suppose that the polygon has n edges and n vertices with position vectors $\{\mathbf{r}_i(s) : i = 1, 2, \ldots, n\}$, where s is a measure of the arc length around C_n . The edges are the so-called *elements* and the vertices the *nodes*. The idealization is shown in Figure 2, where $\mathbf{R}_{ij}$ is the position vector of any point in element $[j]$ relative to the node i.

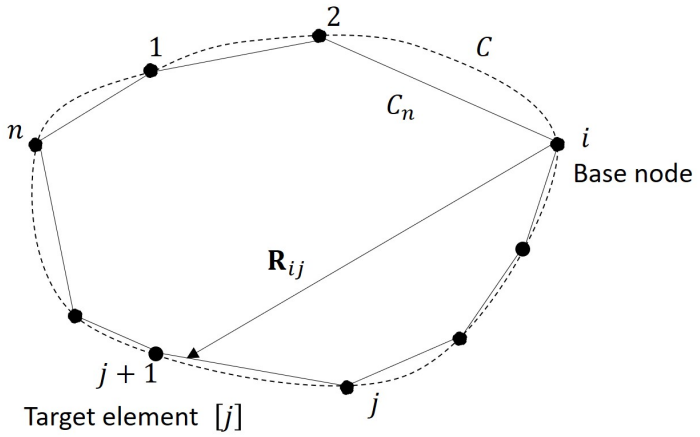

Figure 2: Boundary element idealization: nodes i, elements $[j]$; C is the boundary and C_n is the approximate polygonal boundary.

Choose the basis functions $\{\psi_j(s) : j = 1, 2, \ldots, n\}$ to be local to the nodes, i.e. $\psi_k(s)$ is non-zero only in elements which contain the node k, as shown in Figure 3.

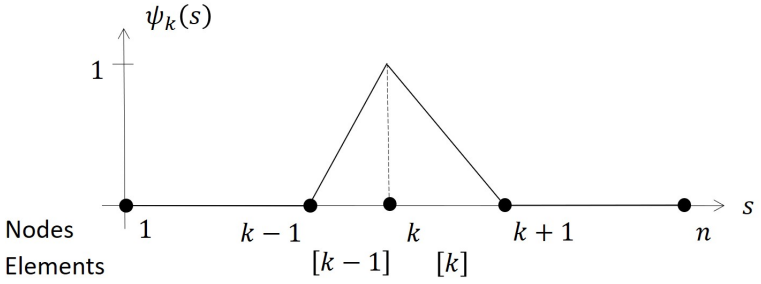

Figure 3: Basis function $\psi_k(s)$.

On the boundary we approximate u and q:

$$\tilde{u} = \sum_{j=1}^{n} \psi_j(s) U_j \quad \text{and} \quad \tilde{q} = \sum_{j=1}^{n} \psi_j(s) Q_j, \tag{12}$$

where U_j and Q_j are the approximate nodal values of u and q respectively.

We are now in a position to develop a numerical process for finding the approximate values U_j and Q_j with which we can find the boundary functions $\tilde{u}$ and $\tilde{q}$ from equation (12). These approximate values can be used in equation (9) to find approximate values for internal points in the region D. The technical details are straightforward but algebraically tedious; consequently we shall give an overview of the process without going into too much detail.

Choose the boundary point P to be, successively, the nodes $1, 2, \ldots, n$ and replace the curve C by C_n in equation (10) with the fundamental solution given by equation (5):

$$\begin{aligned} c_i U_i &= \frac{1}{2\pi} \oint_{C_n} \left(\sum_{j=1}^{n} \psi_j(s) Q_i \right) (-\ln R_i) ds \\ &\quad - \frac{1}{2\pi} \oint_{C_n} \sum_{j=1}^{n} \psi_j(s) U_j \left(-\frac{\partial}{\partial n} (\ln R_i) \right) ds, \end{aligned}$$

where $R_i = |\mathbf{R}_i(s)|$ and $\mathbf{R}_i(s)$ is the position vector of a point s, on the boundary C_n, relative to node i.

We can write this in the form, using equation (11),

$$\alpha_i U_i = \sum_{j=1}^{n} \left(\int_{[j]} \psi_j(s) \frac{\partial}{\partial n} (\ln R_{ij}) ds \right) U_j - \sum_{j=1}^{n} \left(\int_{[j]} \psi_j(s) \ln R_{ij} ds \right) Q_j,$$

$j = 1, 2, \ldots$ and $\alpha_i = 2\pi c_i$.

The integrals are taken only over those elements, $[j]$, in which $\psi_j(s)$ is non-zero and $R_{ij} = |\mathbf{R}_{ij}|$ with $\mathbf{R}_{ij}(s)$, the position vector of a point in the *target element* $[j]$ relative to the *base node* i, see Figure 2.

Now write these equations as

$$\sum_{j=1}^{n} H_{ij} U_j + \sum_{j=1}^{n} G_{ij} Q_j = 0 \qquad i = 1, 2, \ldots, n, \tag{13}$$

with

$$H_{ij} \;=\; \int_{[j]} \psi_j(s) \frac{\partial}{\partial n} \left(\ln R_{ij}\right) ds - \alpha_i \delta_{ij} \tag{14}$$

and

$$G_{ij} \;=\; -\int_{[j]} \psi_j(s) \ln R_{ij} ds. \tag{15}$$

The system of equations (13) is conveniently written in matrix form as

$$\mathbf{HU} + \mathbf{GQ} \;=\; \mathbf{0}, \tag{16}$$

where $\mathbf{U}$ and $\mathbf{Q}$ are vectors of the approximate boundary values of u and q respectively.

We can't solve this system immediately because the vectors $\mathbf{U}$ and $\mathbf{Q}$ contain both known and unknown values. We partition the system (16) so that the known U_i and Q_i values are associated with the superscripts 1 and 2 respectively to obtain the system of equations

$$\mathbf{Ax} \;=\; \mathbf{b}, \tag{17}$$

with

$$\mathbf{A} \;=\; [\mathbf{H}^2\mathbf{G}^1], \quad \mathbf{b} \;=\; -\left[\mathbf{H}^1\mathbf{G}^2\right] \begin{bmatrix} \mathbf{U}^1 \\ \mathbf{Q}^2 \end{bmatrix}$$

and the unknowns are given by the vector

$$\mathbf{x} \;=\; \begin{bmatrix} \mathbf{U}^2 \\ \mathbf{Q}^1 \end{bmatrix}.$$

We now have all the boundary nodal values U_i and Q_i so we can use the discrete form of equation (10) to find internal values. Suppose that we wish to obtain values at m internal points; then the approximate solutions are given by

$$U_k = \frac{1}{2\pi} \oint_{C_n} \left[\left(\sum_{j=1}^{n} w_j(s) U_j \right) \frac{\partial}{\partial n} \ln R_k - \left(\sum_{j=1}^{n} w_j(s) Q_j \right) \ln R_k \right] ds$$

$$k = 1, 2, \ldots, m.$$

We write this in matrix form as

$$\mathbf{U}_I \;=\; \breve{\mathbf{H}}\mathbf{U} + \breve{\mathbf{G}}\mathbf{Q} \tag{18}$$

where

$$\breve{H}_{kj} \;=\; \frac{1}{2\pi} \int_{[j]} \frac{\partial}{\partial n} (\ln R_{kj}) ds \quad \text{and} \quad \breve{G}_{kj} \;=\; -\frac{1}{2\pi} \int_{[j]} \ln R_{kj} ds. \tag{19}$$

5 Computational aspects

At this stage most of the *interesting* mathematics is complete, and we are left with the rather tedious work of evaluating the integrals in equations (14), (15) and (19) followed by the solution of the system of equations (17). We shall omit the details and give an overview of the process.

Evaluating the integrals

By virtue of the piecewise nature of the basis functions $\psi_k(s)$, see Figure 3, the integrals in equations (14), (15) and (19) require evaluations over pairs of adjacent elements. There are three possibilities.

(1) If the base node i is not in the target element $[j]$, see Figure 2, then the integrals are non-singular and can be evaluated using a standard Gauss quadrature.

(2) If the base node i is in the target element $[j-1]$ or $[j+1]$ then one integral is non-singular and can be evaluated using Gauss quadrature. The other is singular.

(3) If the base node i is in the target element $[j]$ then both integrals are singular.

There are some technicalities which won't concern us here. We shall use J to denote either $j \pm 1$ or j.

The integrals for G_{iJ} are

$$-\int_{[J]} \psi_J(s) \ln R_{iJ} ds$$

which, after a little algebra, require the evaluation of integrals of the form

$$\int_{-1}^{1} (1 \pm \xi) \ln(1-\xi) d\xi$$

and these can be integrated analytically.

The integral for H_{iJ},

$$\int_{[J]} \psi_J(s) \frac{\partial}{\partial n} \left(\ln R_{iJ} \right) ds \tag{20}$$

is more complicated. Since

$$\frac{\partial}{\partial n}\left(\ln R_{iJ}\right) \;=\; \left(\operatorname{grad} R{\cdot}\mathbf{n}\right)_{iJ} \;=\; \frac{1}{R_{iJ}^2}\mathbf{R}_{iJ}{\cdot}\mathbf{n}_J$$

the singularity is $\mathcal{O}(1/R)$ and such integrals usually require special treatment. However, we can see from Figure 4 that $\mathbf{n}_J$ is perpendicular to $\mathbf{R}_{iJ}$ so that

$$\mathbf{R}_{iJ}{\cdot}\mathbf{n}_J \;=\; 0.$$

It follows that the integral (20) is zero and that $H_{ii} \;=\; -\alpha_i$.

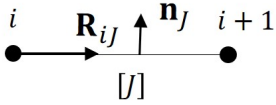

Figure 4: Base node i in target element $[J]$.

Further still, in the case of the Laplace operator, we don't need to calculate the parameter α_i. We notice that we can apply our approach to the problem whose unique solution is $u \equiv 1$. The equivalent boundary-value problem is

$$\nabla^2 u = 0 \quad \text{in } D, \qquad u = 1 \quad \text{on } C$$

and clearly $q = 0$ on C.

Hence $\mathbf{U} = [1 \;\; 1 \;\; \ldots \;\; 1]^T$ and $\mathbf{Q} = [0 \;\; 0 \;\; \ldots \;\; 0]^T$ so that equation (16) yields

$$\mathbf{HU} \;=\; \mathbf{0}.$$

Consequently it follows that

$$H_{ii} \;=\; -{\sum_{j=1}^{n}}' H_{ij} \qquad i = 1, 2, \ldots, n$$

and the diagonal terms are found from the sum of the off-diagonal terms, a very convenient result.

Finally, now that we have all the boundary values we can compute internal values using equation (18) with the coefficients given by equations (19). All the integrals are non-singular and can be evaluated using a standard Gauss quadrature.

6 Postscript

The question usually arises as to what are the advantages of the BEM compared with the FEM? The major difference is that the BEM reduces a partial differential equation in a region to an integral equation on the boundary of that region. Consequently there is much less work in developing the computational grid. This is particularly important for three-dimensional problems. However, if, instead of Laplace's equation (1), we have Poisson's equation

$$\nabla^2 u \;=\; f(\mathbf{r}),$$

the non-homogeneous term, $f(\mathbf{r})$, means that the integral equation developed in section 3 will have a term of the form

$$\int_D u^* f(\mathbf{r}) dA$$

and this will require a grid over the region D thus removing the set-up advantage of the BEM. There are other ways of dealing with this non-homogeneous term but we won't consider them here.

Finally we note that the system of equations developed in the FEM is sparse and symmetric allowing for very efficient equation solvers. In the BEM the system of equations (17) is neither sparse nor symmetric. It is smaller but requires much less efficient solvers.

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Exam question: Expand $(a+b+c)^2$.

Candidate's answer: (a + b + c) 2.

Problem 299.1 – Adjugate

Tony Forbes

The *adjugate*, $\operatorname{adj} M$, of a square matrix M is the matrix defined by

$$[\operatorname{adj} M]_{i,j} \;=\; (-1)^{i+j} \det M_{\bar{j},\bar{i}},$$

where $M_{\bar{j},\bar{i}}$ is the matrix you get from M by deleting row j and column i. For example, you can verify that

$$\operatorname{adj} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 8 & 9 \end{bmatrix} \;=\; \begin{bmatrix} -3 & 6 & -3 \\ 0 & -9 & 6 \\ 2 & 4 & -3 \end{bmatrix}.$$

Note the reversal of i and j on the right. So the 6 in row 1, column 2 on the right-hand side arises from deleting row 2 and column 1 on the left: $6 = (-1)^{1+2}(2 \cdot 9 - 8 \cdot 3)$.

The matrix M is not necessarily non-singular, but if it is, you might remember from your high-school days Cramer's rule for solving simultaneous equations:

$$\text{if} \quad M\mathbf{z} \;=\; \mathbf{b}, \quad \text{then} \quad \mathbf{z} \;=\; \frac{1}{\det M} \operatorname{adj}(M)\, \mathbf{b}.$$

Moreover, the adjugate might be useful to know if you want to try to invert a matrix. Indeed, we have the simple property

$$M\,(\operatorname{adj} M) \;=\; (\operatorname{adj} M)\,M \;=\; (\det M)\,I,$$

I being the appropriate identity matrix.

Now for the problem. Suppose x is a variable, A is a matrix that is independent of x, and $M = xI - A$. Show that

$$\frac{d(\det M)}{dx} \;=\; \operatorname{trace}(\operatorname{adj} M)\,.$$

This is actually a special case of Jacobi's formula,

$$\frac{d(\det M)}{dx} \;=\; \operatorname{trace}\left((\operatorname{adj} M)\,\frac{dM}{dx}\right),$$

and I am hoping that the simplifying assumption ($dM/dx = I$) will make it much easier to prove.

Solution 221.2 – Coefficients

Given n, find a simple formula for the coefficient of x^k in the polynomial

$$(x+n-1)\,(2x+n-2)\,\ldots\,((n-1)x+1)\,(nx).$$

Peter Fletcher

If we try the first few values of n, we get the following:

$$\begin{aligned}
y_1 &= x,\\
y_2 &= 2x^2+2x,\\
y_3 &= 6x^3+15x^2+6x,\\
y_4 &= 24x^4+104x^3+104x^2+24x,\\
y_5 &= 120x^5+770x^4+1345x^3+770x^2+120x,\\
y_6 &= 720x^6+6264x^5+16344x^4+16344x^3+6264x^2+720x,\\
y_7 &= 5040x^7+56196x^6+200452x^5+300167x^4\\
&\qquad\qquad +200452x^3+56196x^2+5040x.
\end{aligned}$$

The coefficients of the first and last terms are obviously $n!$, but what about the others? We can find how the other coefficients came to be by using $a_1, a_2, a_3, \ldots$ in place of $1, 2, 3, \ldots$ in a few expansions of the expression in the question. For example, going back to numbers, we find that the coefficient of x^6 in y_7 is

$$7!\left(\frac{1}{6}+\frac{2}{5}+\frac{3}{4}+\frac{4}{3}+\frac{5}{2}+\frac{6}{1}\right) \;=\; 7!\sum_{p=1}^{6}\frac{p}{7-p},$$

the coefficient of x^5 in y_7 can be written

$$7!\left(\frac{1}{6}\left(\frac{2}{5}+\frac{3}{4}+\frac{4}{3}+\frac{5}{2}+\frac{6}{1}\right)+\frac{2}{5}\left(\frac{3}{4}+\frac{4}{3}+\frac{5}{2}+\frac{6}{1}\right)+\frac{3}{4}\left(\frac{4}{3}+\frac{5}{2}+\frac{6}{1}\right)\right.$$
$$\left.+\frac{4}{3}\left(\frac{5}{2}+\frac{6}{1}\right)+\frac{5}{2}\cdot\frac{6}{1}\right) \;=\; 7!\sum_{p=1}^{5}\frac{p}{7-p}\sum_{q=p+1}^{6}\frac{q}{7-q},$$

and the coefficient of x^4 in y_7 is

$$7!\Bigg(\frac{1}{6}\left(\frac{2}{5}\left(\frac{3}{4}+\frac{4}{3}+\frac{5}{2}+\frac{6}{1}\right)+\frac{3}{4}\left(\frac{4}{3}+\frac{5}{2}+\frac{6}{1}\right)+\frac{4}{3}\left(\frac{5}{2}+\frac{6}{1}\right)+\frac{5}{2}\cdot\frac{6}{1}\right)$$
$$+\frac{2}{5}\left(\frac{3}{4}\left(\frac{4}{3}+\frac{5}{2}+\frac{6}{1}\right)+\frac{4}{3}\left(\frac{5}{2}+\frac{6}{1}\right)+\frac{5}{2}\cdot\frac{6}{1}\right)$$
$$+\frac{3}{4}\left(\frac{4}{3}\left(\frac{5}{2}+\frac{6}{1}\right)+\frac{5}{2}\cdot\frac{6}{1}\right)+\frac{4}{3}\cdot\frac{5}{2}\cdot\frac{6}{1}\Bigg)$$
$$=\ 7!\sum_{p=1}^{4}\frac{p}{7-p}\sum_{q=p+1}^{5}\frac{q}{7-q}\sum_{r=q+1}^{6}\frac{r}{7-r}.$$

The corresponding entries in Pascal's triangle for the above three coefficients are 6, 15 and 20, and if we count up the numbers of fractions being summed, the answers are also 6, 15 and 20.

How many nested sums do we need? In tables 1 and 2 we give the number of nested sums required with $n = 8$ and $n = 9$ respectively; we also give $n-k$ and $k-1$. From these tables it is clear that the number of nested sums in the calculation of the coefficient of x^k is $\min(n-k, k-1)$ for even n and odd n.

x^k	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x^1
No. of nested sums	0	1	2	3	3	2	1	0
$n-k$	0	1	2	3	4	5	6	7
$k-1$	7	6	5	4	3	2	1	0

Table 1: Numbers of nested sums with $n = 8$.

x^k	x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x^1
No. of nested sums	0	1	2	3	4	3	2	1	0
$n-k$	0	1	2	3	4	5	6	7	8
$k-1$	8	7	6	5	4	3	2	1	0

Table 2: Numbers of nested sums with $n = 9$.

The last question we need to answer is, how many terms are there in the first nested sum? We can answer this by looking at the three coefficients above for $n = 7$: the answer is clearly k, except that the coefficient of x^k is also the coefficient of x^{n-k+1}. This means we need to take $\max(k, n-k+1)$. This is true for even n and odd n.

To summarize, the coefficient of x^k for a general number n is

$$n! \underbrace{\sum_{p=1}^{\max(k,n-k+1)} \frac{p}{n-p} \sum_{q=p+1}^{\max(k,n-k+1)+1} \frac{q}{n-q} \sum_{r=q+1}^{\max(k,n-k+1)+2} \frac{r}{n-r} \cdots}_{\min(n-k,k-1) \text{ nested sums}}.$$

Now for a couple of checks.

If $n = 12$ and $k = 5$, $\min(12-5, 5-1) = \min(7, 4) = 4$, so we want four nested sums; also, $\max(5, 12-5+1) = \max(5, 8)$, so we want eight terms in the first one. Evaluating these sums in Maple gives 1 447 536 199 680, which is exactly what we get if we expand the expression in the question with $n = 12$.

Trying the nested sums with $n = 17$ and $k = 12$ means having five nested sums and 12 terms in the first one: the answer is 40 644 230 357 623 625 216, which is also correct.

Solution 293.8 – Roots

Let a, b, c, d denote the roots of the quartic $x^4 - x^3 - 4x^2 + 4x + 1$. Determine the quartic that has roots $a^2 - 2$, $b^2 - 2$, $c^2 - 2$, $d^2 - 2$.

Peter Fletcher

We can immediately write down

$$\begin{aligned} s_1 &= a + b + c + d = 1, \\ s_2 &= ab + ac + ad + bc + bd + cd = -4, \\ s_2 &= abc + abd + acd + bcd = -4, \\ s_4 &= abcd = 1. \end{aligned}$$

We now want to find the equivalent sums and products for our new quartic in terms of the above sums and products.

$$\begin{aligned} S_1 &= (a^2 - 2) + (b^2 - 2) + (c^2 - 2) + (d^2 - 2) \\ &= a^2 + b^2 + c^2 + d^2 - 8 \\ &= (a + b + c + d)^2 - 2(ab + ac + ad + bc + bd + cd) - 8 \\ &= 1 + 2 \cdot 4 - 8 = 1; \end{aligned}$$

$$S_2 = (a^2 - 2)(b^2 - 2) + (a^2 - 2)(c^2 - 2) + (a^2 - 2)(d^2 - 2)$$

$$
\begin{aligned}
&+ (b^2-2)(c^2-2) + (b^2-2)(d^2-2) + (c^2-2)(d^2-2) \\
&= a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2 - 6(a^2+b^2+c^2+d^2) + 24 \\
&= (ab+ac+ad+bc+bd+cd)^2 \\
&\quad - 2(a+b+c+d)(abc+abd+acd+bcd) + 2abcd \\
&\quad - 6(a+b+c+d)^2 + 12(ab+ac+ad+bc+bd+cd) + 24 \\
&= 16 + 2\cdot 4 + 2 - 6 - 12\cdot 4 + 24 \;=\; -4;
\end{aligned}
$$

$$
\begin{aligned}
S_3 &= (a^2-2)(b^2-2)(c^2-2) + (a^2-2)(b^2-2)(d^2-2) \\
&\quad + (a^2-2)(c^2-2)(d^2-2) + (b^2-2)(c^2-2)(d^2-2) \\
&= a^2b^2c^2 + a^2b^2d^2 + a^2c^2d^2 + b^2c^2d^2 \\
&\quad - 4(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) \\
&\quad + 12(a^2+b^2+c^2+d^2) - 32 \\
&= (abc+abd+acd+bcd)^2 - 2abcd(ab+ac+ad+bc+bd+cd) \\
&\quad - 4(ab+ac+ad+bc+bd+cd)^2 \\
&\quad + 8(a+b+c+d)(abc+abd+acd+bcd) - 8abcd \\
&\quad + 12(a+b+c+d)^2 - 24(ab+ac+ad+bc+bd+cd) - 32 \\
&= 16 + 2\cdot 4 - 4\cdot 16 - 8\cdot 4 - 8 + 12 + 24\cdot 4 - 32 \;=\; -4;
\end{aligned}
$$

$$
\begin{aligned}
S_4 &= (a^2-2)(b^2-2)(c^2-2)(d^2-2) \\
&= a^2b^2c^2d^2 - 2(a^2b^2c^2 + a^2b^2d^2 + a^2c^2d^2 + b^2c^2d^2) \\
&\quad + 4(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) \\
&\quad - 8(a^2+b^2+c^2+d^2) + 16 \\
&= a^2b^2c^2d^2 - 2(abc+abd+acd+bcd)^2 \\
&\quad + 4abcd(ab+ac+ad+bc+bd+cd) \\
&\quad + 4(ab+ac+ad+bc+bd+cd)^2 \\
&\quad - 8(a+b+c+d)(abc+abd+acd+bcd) + 8abcd \\
&\quad - 8(a+b+c+d)^2 + 16(ab+ac+ad+bc+bd+cd) + 16 \\
&= 1 - 2\cdot 16 - 4\cdot 4 + 4\cdot 16 + 8\cdot 4 + 8 - 8 - 16\cdot 4 + 16 \;=\; 1.
\end{aligned}
$$

Therefore, since $S_1 = s_1$, $S_2 = s_2$, $S_3 = s_3$ and $S_4 = s_4$, our new quartic is exactly the same as the one we were given in the question.

Solution 295.5 – Graphs with girth at least 5

Given $k \geq 3$ and sufficiently large v, kv even, it is often not too difficult to find a k-regular graph with v vertices and girth at least 5 (i.e. contains no triangles or 4-cycles). Show that there is no such graph that has a cyclic automorphism of order v (i.e. can be drawn with v-fold rotational symmetry).

Reinhardt Messerschmidt

Suppose G is a regular graph of degree $k \geq 3$ with v vertices and a cyclic automorphism ϕ of order v. We will use $\sim$ to denote adjacency in G, and we will use $[a]$ to denote the residue class modulo v that an integer a belongs to.

Since ϕ is a cyclic permutation of order v, we can label G's vertices $[0], [1], \ldots, [v-1]$ in such a way that $\phi([a]) = [a+1]$. Since ϕ is an automorphism of G, we have $[a] \sim [b]$ if and only if $[a+1] \sim [b+1]$. It follows by induction that if m is a nonnegative integer then

$$[a] \sim [b] \iff [a+m] \sim [b+m]. \qquad (*)$$

The inverse ϕ^{-1} of ϕ is also an automorphism, and it satisfies $\phi^{-1}([a]) = [a-1]$. It follows that $(*)$ also holds for negative integers m.

Proposition 1 *G has a 4-cycle, and hence a girth of at most 4.*

Proof Since $k \geq 3$, we can choose distinct neighbours $[a]$,$[b]$ of $[0]$ that are not additive inverses of each other modulo v, i.e. $[a+b] \neq [0]$. Since $[0] \sim [a]$, we have $[b] \sim [a+b]$ by $(*)$. By interchanging the roles of a and b, we also have $[a] \sim [a+b]$. The following sequence is therefore a 4-cycle in G:

$$[0] \sim [a] \sim [a+b] \sim [b] \sim [0]. \qquad \square$$

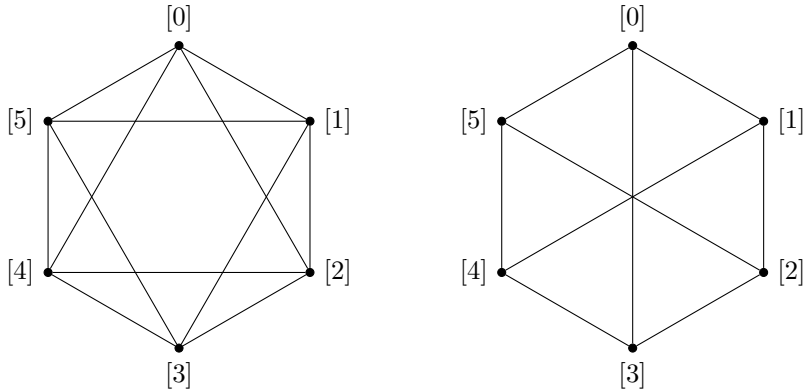
Triangles

The graph G has a triangle if and only if there exist neighbours $[a], [b], [c]$ of $[0]$, not necessarily distinct, such that $[a+b+c] = [0]$. If this condition holds, then the following sequence is a triangle in G:

$$[0] \sim [a] \sim [a+b] \sim [a+b+c] = [0].$$

For example, consider the two possible instances of G with $v = 6$ shown in the picture on page 19. The graph on the left has a triangle, because $[1], [4]$

are neighbours of [0] and $[1+1+4]=[0]$. The graph on the right does not have a triangle, because it is impossible to choose $a,b,c\in\{1,3,5\}$ such that $[a+b+c]=[0]$.



Problem 299.2 – Integral

Tony Forbes

Show that

$$\int_0^{\pi/2}\Big((\sin x)^{2/3}+(\cos x)^{2/3}\Big)^3\,dx \;=\; \frac{3\pi}{2}.$$

Problem 299.3 – Hair

Tony Forbes

This is another problem (see M500 **273** for the previous one) of possible interest to anyone concerned about maintaining ones hair at minimal cost.

Assume: a haircut costs £H after which your hair will be h metres long; hair grows at d metres per second; you wash your hair n times per second; hair shampoo costs £S per kilogram; a hair wash requires w kilograms per metre (for example, if $w=1$ and your hair length is 2 metres, you will need 2 kilograms of shampoo).

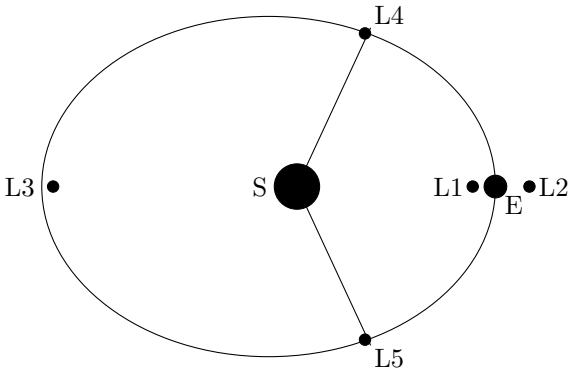
How often should you visit the haircutter to minimize the cost rate.

I put this in M500 because someone told me that keeping your hair short saves on shampoo costs.

The Lagrange points

Graham Lovegrove

1 Introduction



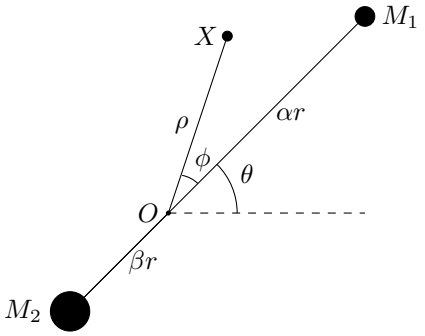
The Lagrange points are five points in a 2-body gravitational system (e.g. Sun/Earth) where an additional negligibly small mass can be perfectly sustained by the balancing gravitational forces of the main bodies. Although they are named after Lagrange, three of them were discovered by Euler a few years earlier. Lagrange published derivations of all of them in 1772 in his paper 'Essay on the Three Body Problem'. As is well known, the dynamics of three or more gravitating bodies of any size does not have a closed solution, although where the separations are large, such as the Solar System, approximations and perturbation theory suffice.

The purpose of this article is to demonstrate the existence of these points for general elliptical orbits of the main bodies in a simple way, using Lagrange's own beautiful formulation of dynamics via the Principle of Least Action. The readily available accounts of this topic seem overwhelmingly based on Cartesian coordinates and rotating frames, and mostly consider only circular orbits. A cursory look at Lagrange's own Essay reveals that he also analysed the three-body problem using Cartesian coordinates. However, a formulation using polar coordinates reveals the Lagrange points as natural symmetries in quite a simple way.

2 Lagrangian Mechanics

For the uninitiated, here is a brief introduction to Lagrangian mechanics. Lagrange noted that the Newtonian equations of motion can be derived from the minimization of the quantity $\int_0^t L dt$, where the integration is over a time interval and $L = T - V$, where T is the kinetic energy of the system, and V the potential energy. This provides a straightforward way of rewriting the equations of motion in any system of coordinates. If q_i $(i = 1, \ldots, n)$ are coordinates for the objects in the system and $\dot{q}_i$ are their derivatives with respect to time, then Lagrange defines the *generalized momenta* p_i to be $p_i = \partial L/\partial \dot{q}_i$ and the equations of motion are then $\dot{p}_i = \partial L/\partial q_i$.

3 Equations of motion



We take as the origin of coordinates the centre of gravity O of the main bodies. We know that for two bodies, motion is in a constant plane (a consequence of the conservation of angular momentum). Likewise, Lagrange points are located in the same plane of rotation. This is very simple to prove, but we shall not bother with it here. Let the masses of the two main bodies M_1 and M_2 be m_1 and m_2. The line joining them passes through O. We label the 'microbody' X and its mass μ. We choose the coordinates to be r, ρ, θ and ϕ, where r is the distance between M_1 and M_2, ρ is the distance OX, θ is the anticlockwise angle to OM_1 at time t from its position at time zero, ϕ is the anticlockwise angle from OM_1 to OX. Thus for X to be in a fixed configuration with respect to the main bodies, we would want ϕ to be constant. Additionally we set

$$\alpha = \frac{m_2}{m_1 + m_2} \quad \text{and} \quad \beta = \frac{m_1}{m_1 + m_2},$$

so that $|OM_1| = \alpha r$ and $|OM_2| = \beta r$. Note that $\alpha+\beta = 1$ and $\alpha m_1 = \beta m_2$. Recalling that for two masses m, m' separated by distance R, the potential energy is $U = -Gmm'/R$, it is clear that the Lagrangian for this system is

$$\begin{aligned} L \;=\; T - V \;&=\; \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}\mu(\dot{\rho}^2 + \rho^2(\dot{\theta}+\dot{\phi})^2) \\ &\quad + \frac{Gm_1m_2}{r} + \frac{G\mu m_1}{|XM_1|} + \frac{G\mu m_2}{|XM_2|}. \end{aligned}$$

By the cosine rule we have that

$$|XM_1| \;=\; \sqrt{\alpha^2 r^2 + \rho^2 - 2\alpha r\rho\cos\phi}$$

and

$$|XM_2| \;=\; \sqrt{\beta^2 r^2 + \rho^2 + 2\beta r\rho\cos\phi}.$$

Therefore the generalized momenta are

$$\begin{aligned} p_r &= \frac{m_1 m_2}{m_1+m_2}\dot{r}, \\ p_\theta &= \frac{m_1 m_2}{m_1+m_2} r^2\dot{\theta} + \mu\rho^2(\dot{\theta}+\dot{\phi}), \\ p_\rho &= \mu\dot{\rho}, \\ p_\phi &= \mu\rho^2(\dot{\theta}+\dot{\phi}). \end{aligned}$$

Up to this point, we have been exact. Now, however, we make the assumption that the body X is so small that it has no effect on the dynamics of the two much larger bodies M_1 and M_2, so we shall forget the second term in the expression for p_θ, and in the equations of motion for those bodies. So now we can calculate the equations of motion for the system from the Lagrange equations $\dot{p}_i = \partial L/\partial q_i$:

$$\ddot{r} \;=\; r\dot{\theta}^2 - \frac{G(m_1+m_2)}{r^2}, \tag{1}$$

$$\frac{d(r^2\dot{\theta})}{dt} \;=\; 0, \tag{2}$$

$$\begin{aligned} \ddot{\rho} \;&=\; \rho(\dot{\theta}+\dot{\phi})^2 \\ &\quad - \frac{Gm_1(\rho - \alpha r\cos\phi)}{|XM_1|^3} - \frac{Gm_2(\rho+\beta r\cos\phi)}{|XM_2|^3}, \end{aligned} \tag{3}$$

$$\frac{d(\rho^2(\dot{\theta}+\dot{\phi}))}{dt} \;=\; \frac{\beta G m_2 r\rho\sin\phi}{|XM_2|^3} - \frac{\alpha G m_1 r\rho\sin\phi}{|XM_1|^3}, \tag{4}$$

where we have simplified slightly by factoring out the masses. The first two equations represent the motion of the two main bodies M_1 and M_2. In particular, equation (2) expresses the conservation of angular momentum in that subsystem. These have the familiar elliptical-orbit solution. The last two equations express the motion of the microbody X. Equation (4) represents the change in the angular momentum of X due to the gravitational pull of M_1 and M_2. If we seek a solution to the motion of X which is in some way constant with respect to the motion of M_1 and M_2, then we should require that ϕ is constant, i.e. $\dot{\phi} = 0$. We could also assume that the ratio $\rho : r$ is a constant value. This would imply the right-hand side of equation (4) should also be zero in order to agree with equation (2). A requirement that the angular momentum of X should be conserved leads to the same result. There are two obvious ways in which the right-hand side of equation (4) can be zero: the first is if $\sin\phi = 0$; the second is (remembering that $\alpha m_1 = \beta m_2$) that $|XM_1| = |XM_2|$. We consider these two cases in turn, but before that we should observe that conservation of angular momentum for X, together with $\dot{\phi} = 0$, implies that ρ is a constant multiple of r. We let $\rho = \gamma r$, where γ is a constant to be determined. Considering first the case $\phi = 0$, equation (3) becomes

$$\gamma\ddot{r} \;=\; \gamma r\dot{\theta}^2 - \frac{Gm_1(\gamma-\alpha)}{r^2|\alpha-\gamma|^3} - \frac{Gm_2(\gamma+\beta)}{r^2|\beta+\gamma|^3}.$$

Comparing this with equation (1), we see that they will be the same if

$$\gamma(m_1+m_2) \;=\; \frac{m_1(\gamma-\alpha)}{|\alpha-\gamma|^3} + \frac{m_2(\gamma+\beta)}{|\beta+\gamma|^3},$$

which is the same as

$$\gamma \;=\; \frac{\beta(\gamma-\alpha)}{|\gamma-\alpha|^3} + \frac{\alpha(\gamma+\beta)}{|\gamma+\beta|^3}. \tag{5}$$

This equation and the similar equation obtained from setting $\phi = \pi$ account for the three Lagrange points $L1$, $L2$ and $L3$ which are collinear with M_1 and M_2. Now we consider the other the other possibility for conservation of angular momentum of X, namely that $|XM_1| = |XM_2|$. Substituting this and $\rho = \gamma r$ and $\dot{\phi} = 0$ into (3), we get

$$\begin{aligned}\gamma\ddot{r} \;&=\; \gamma r\dot{\theta}^2 - \frac{Gm_1(\gamma r-\alpha r\cos\phi)}{|XM_1|^3} - \frac{Gm_2(\gamma r+\beta r\cos\phi)}{|XM_1|^3}\\ &=\; \gamma r\dot{\theta}^2 - \frac{G(m_1+m_2)\gamma r}{|XM_1|^3},\end{aligned}$$

where the terms involving $\cos\phi$ cancel because $\alpha m_1 = \beta m_2$. Comparing this with equation (1), we see that they will be the same if $|XM_1| = |XM_2| = r$, i.e. if M_1, M_2 and X form an equilateral triangle! The values of ϕ and γ that achieve this are obtained from the cosine rule conditions

$$\alpha^2 + \gamma^2 - 2\alpha\gamma\cos\phi \;=\; \beta^2 + \gamma^2 + 2\beta\gamma\cos\phi \;=\; 1,$$

which yields $\gamma = \sqrt{1-\alpha\beta}$ and $\cos\phi = \dfrac{\alpha-\beta}{2\sqrt{1-\alpha\beta}}$.

We have demonstrated that the existence of the Lagrange points arises naturally out of the Lagrange equations. It is well known that points $L1$, $L2$ and $L3$ are unstable, so that objects placed there will drift away in time, whereas $L4$ and $L5$ are stable. Whereas this is fairly easy to demonstrate for circular orbits, that is not so for general elliptical orbits, and so we shall not go into that here.

Problem 299.4 – Advent calendar

Bruce Roth

The picture shows a friend's homemade Advent calendar. All of the socks are regularly spaced out along the string. A chocolate teddy is placed in each. One bear is removed each day and eaten. The question is: Where is the position of greatest asymmetry? Both ends are at the same height and the bears are taken out in order from left to right. The string is inelastic and the socks are identical, as are the bears. The socks do not have negligible mass but the bears have a greater mass than the socks. The string can be said to have negligible mass (compared with the socks and bears). The most lopsided curve is regarded as the most asymmetrical.

LETTER

Things you can't buy in shops

Dear Eddie,

Many thanks for M500 **296**. I may be able to help Tony a bit with his list of unobtainable items. No, you can't buy a rewirable cloverleaf (C5) plug anywhere—the thing is just too small to make a rewirable version of. But you can buy cloverleaf plugs with slimmer wires than the bloody great cables usually supplied. Here is one from a UK supplier: Cloverleaf to bare ends power lead, 2.5A 2m, `https://cpc.farnell.com/pro-elec/pe01097/lead-mains-2-5a-c5-conn-to-bare/dp/PL13336?st=cloverleaf`. Add your own UK plug.

I really don't think Tony wants to get into electron microscopy. Not only are the machines huge, but to allow you to see the sample it has (depending on the type of electron microscope) to be either electroplated with gold or sliced unbelievably thin with an ultramicrotome.

Bananas free of radioactive potassium are all very well, but how are you going to get rid of the killer carbon-14 that seeps in from the atmosphere itself?

The matter of popups on websites is solved in a blatantly obvious way. Install [a proprietary advertisement-blocker] on your browser and you won't get any, and there is no need to send hostile messages to their originator.

Best wishes,

Ralph Hancock

TF: It seems I must wait a few more years for the inexpensive desktop electron microscope to become widely available. Meanwhile, here are some useful facts from *Wikipedia*:

C-14: abundance 10^{-12}, half-life 5730 y, activity 4456 Ci/kg,

K-40: abundance 0.000117, half-life $1.251{\cdot}10^9$ y, activity 0.007144 Ci/kg,

where the activity is calculated by the formula

$$\frac{\log 2}{3.7\cdot 10^{10}\ (\text{half-life/sec})}\cdot\frac{1000\,N_{\mathrm{A}}}{(\text{atomic mass})}\ \ \text{Ci/kg}.$$

The units of the factors are Ci/atom for the first and atoms/kg for the second. A typical (ordinary) banana contains 0.015 kg carbon and 0.0005 kg potassium; therefore the amount of radioactivity generated by these elements will be $6.68\cdot 10^{-11}$ Ci for C-14 and $4.18\cdot 10^{-10}$ Ci for K-40.

04.00

Problem 299.5 – Integral

Let a be a positive integer. Show that

$$I(a) \;=\; \int_0^1 \frac{\sqrt{1-x^{1/a}}}{\sqrt{1+x^{1/a}}}\,dx \;=\; \frac{1}{B} - \frac{\pi a B}{2},$$

where

$$B \;=\; \begin{cases} \dfrac{1}{2^a}\dbinom{a}{a/2} & \text{if } a \text{ is even,} \\[2ex] \dfrac{-1}{2^{a-1}}\dbinom{a-1}{(a-1)/2} & \text{if } a \text{ is odd.} \end{cases}$$

What is interesting is the simplicity of this formula, and if you add them together in pairs and multiply, it gets even simpler. If $a > 1$, then

$$(I(a) + I(a+1))(I(a) + I(a-1)) \;=\; \frac{\pi}{2a}.$$

Front cover Graph, 72 vertices, 7-regular, girth 5.