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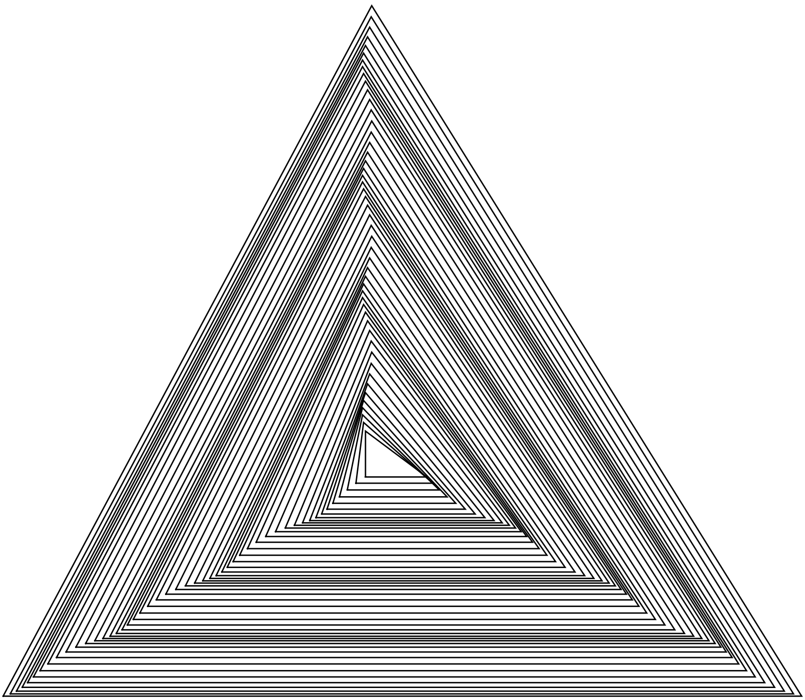
M500 251



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The May Weekend is a residential Friday to Sunday event. In 2013 it will provide revision and examination preparation for students taking undergraduate module examinations in June and study support for postgraduate modules starting in February. For full details and a booking form see m500.org.uk/may.

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Advice to authors We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file to tony@m500.org.uk.

Solution 248.3 – Integer triangles

A triangle has integer area and consecutive integer sides. Apart from $(3,4,5)$, is it the case that exactly one height must also be an integer?

Chris Pile

Let the triangle sides be $a = n-1$, $b = n$ and $c = n+1$. Then by Heron's formula the area, A, is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \quad \text{where} \quad s = \frac{a+b+c}{2} = \frac{3n}{2}.$$

So

$$A = \frac{\sqrt{3}n}{4}\sqrt{(n+2)(n-2)}, \quad n > 2.$$

For this to be an integer, n must be even and $n^2 - 4$ must be a multiple of three times a square.

The sides of the triangle differ by 1. Therefore the lengths of corresponding altitudes must in general differ by less than 1; i.e. there are only two, consecutive, integer values that the altitudes can take. As the side length increases, the triangle more closely resembles an equilateral triangle and the altitudes will differ by an amount approaching $\sqrt{3}/2$. This implies that only one can be an integer.

n	area	altitudes
4	6	3, 4
14	84	12 $(= 2 \cdot 6)$
52	1170	45
194	16296	168 $(= 2 \cdot 84)$
724	226974	627
2702	3161340	2340 $(= 2 \cdot 1170)$
10084	44031786	8733
37634	613283664	32592 $(= 2 \cdot 16296)$
140452	8541939510	121635
524174	118973869476	453948 $(= 2 \cdot 226974)$
1956244	1657092233154	1694157
7300802	23080317394680	6322680 $(= 2 \cdot 3161340)$
27246964	321467351292366	23596563
101687054	4477462600698444	88063572 $(= 2 \cdot 44031786)$

Steve Moon

Let the side lengths be x, $x+1$ and $x+2$. Then

$$A \;=\; \frac{x+1}{2}\sqrt{\frac{3(x+3)(x-1)}{4}}.$$

But A is an integer. Hence x cannot be even as all the factors in the numerator are then odd. For the heights we can write

$$h_i \;=\; \frac{x+1}{x+i}\sqrt{\frac{3(x+3)(x-1)}{4}}, \qquad i=0,\ 1,\ 2.$$

Clearly h_1 must be an integer as the square root is that of a perfect square for all x generating a triangle of integer area.

Now for all integers $x>1$, $\gcd(x,x+1)=\gcd(x+1,x+2)=1$. Suppose h_0 is an integer. Then we have

$$h_0 \;=\; (x+1)\sqrt{\frac{3(x+3)(x-1)}{4x^2}}.$$

As no factor of x divides $x+1$, we require

$$\sqrt{\frac{3(x+3)(x-1)}{4x^2}} \;=\; M$$

for some positive integer M. Then

$$3(x+3)(x-1)\geq 4x^2 \;\Rightarrow\; -x^2+6x+9\geq 0 \;\Rightarrow\; -(x-3)^2\geq 0,$$

which only holds for $x=3$. Hence h_0 is an integer only when $x=3$. (So for the right-angled triangle (3, 4, 5) we have at least two integer sides.)

Now suppose h_2 is an integer. Then

$$h_2 \;=\; (x+1)\sqrt{\frac{3(x+3)(x-1)}{4(x+2)^2}} \;=\; (x+1)N$$

for some positive integer N as no factor of $x+2$ divides $x+1$. Then

$$3(x+3)(x-1)\geq 4(x+2)^2 \;\Rightarrow\; -(x+5)^2\geq 0,$$

which is clearly false. So h_2 is never an integer for any x as defined.

Tony Forbes

Let $n = 2k$. Then $A^2 = 3k^2(k^2 - 1)$, and for A to be an integer we require $3(k^2 - 1)$ to be a square divisible by 3, $9m^2$, say. Thus

$$A = 3km, \qquad k^2 = 3m^2 + 1. \tag{1}$$

The three triangle heights are

$$h_1 = \frac{2A}{n} = 3m, \quad h_2 = \frac{2A}{n-1} = \frac{3mn}{n-1} \quad \text{and} \quad h_3 = \frac{2A}{n+1} = \frac{3mn}{n+1}.$$

Clearly h_1 is an integer. Now suppose $n \ge 8$. From (1) we have

$$3m = \sqrt{3k^2 - 3} = \sqrt{\frac{3n^2}{4} - 3} < \frac{\sqrt{3}\,n}{2} < n - 1.$$

Therefore, since $\gcd(n, n \pm 1) = 1$, neither $h_2 = 3mn/(n-1)$ nor $h_3 = 3mn/(n+1)$ can be an integer. We can show by trial that the only possible triangle with $n < 8$ is the Pythagorean triple $(3, 4, 5)$.

Although we do not need it for the problem as stated, the standard method for solving the Diophantine equation $k^2 = 3m^2 + 1$ gives all possible triangles with consecutive integer sides and integer area. We ignore the trivial solution, $k_0 = 1$, $m_0 = 0$, because it yields $(1, 2, 3)$, which is not really a triangle. The fundamental solution is $k_1 = 2$, $m_1 = 1$, leading to $(3, 4, 5)$. Thereafter solutions are defined by

$$\begin{bmatrix} k_i \\ m_i \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^i \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Problem 251.1 – Increasing digits

Jeremy Humphries

How many positive integers have the property that their digits increase when read from left to right? For example, 3, 26, 1357, but not 10, 43, 778, 34592.

Problem 251.2 – Thirteen boxes

Tony Forbes

For reasons which are of no interest to anyone other than me I would like to know how big h must be in order to pack thirteen $17 \times 6 \times 6$ cuboids into a $33 \times 23 \times h$ box.

Solution 248.4 – Integral

Compute

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2+z^2)^{3/2}}\,dx\,dy\,dz.$$

Steve Moon

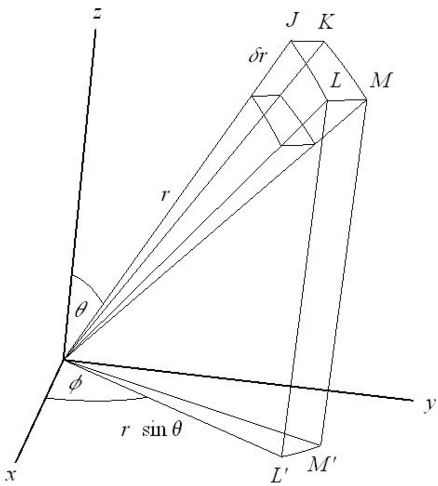
The integral can be thought of as the integral of the function $e^{-(x^2+y^2+z^2)^{3/2}}$ over all space. With mutually orthogonal cartesian axes x, y, z, each extending over $(-\infty,\infty)$, the infinitesimal volume element δv is the cuboid $\delta x\,\delta y\,\delta z$, which becomes $dx\,dy\,dz$ in the triple integral.

As set, I think this problem is relatively intractable. The symmetry of the function $e^{-(x^2+y^2+z^2)^{3/2}}$ indicates that spherical polar, rather than cartesian, coordinates will be useful. The key results are

$$\begin{aligned} r^2 &= x^2+y^2+z^2,\\ \delta V &= LM\cdot JL\cdot\delta\,r\\ &= L'M'\cdot JL\cdot\delta\,r\\ &= r(\sin\theta)\,\delta\phi\cdot r\,\delta\theta\cdot\delta r\\ &= r^2(\sin\theta)\,\delta\theta\,\delta\phi\,\delta r \end{aligned}$$

and

$$\begin{aligned} 0 \;\le\; r &\;\le\; \infty,\\ 0 \;\le\; \phi &\;\le\; 2\pi,\\ 0 \;\le\; \theta &\;\le\; \pi. \end{aligned}$$



For an integral over all space, it does not matter how we compile v from different forms of δv. So the integral becomes

$$I \;=\; \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} r^2e^{r^3}(\sin\theta)d\phi\,d\theta\,dr,$$

which is separable. Thus

$$I \;=\; 2\pi\int_0^{\infty} r^2e^{r^3}\int_0^{\pi}(\sin\theta)d\theta\,dr \;=\; 4\pi\int_0^{\infty} r^2e^{r^3}\,d\theta\,dr \;=\; \frac{4\pi}{3}.$$

Problem 251.3 – Four towns

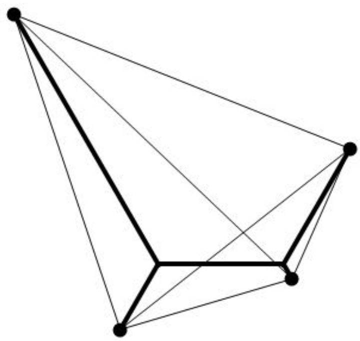
Dick Boardman

Four towns lie at the corners of a quadrilateral with integer sides and integer diagonals (no two the same). They are each connected to a single point such that the sum of the four distances is minimum. Find solutions where all of the individual lengths are integers.

Problem 251.4 – Four more towns

Tony Forbes

This is like Problem 251.3 – Four towns. However, we now require the towns to be serviced by a road network of minimum length. This can often be achieved by the creation of two junctions, called *Steiner points*, where three roads meet at 120°. A typical layout is shown on the right.



Find a solution where the lengths of the five road segments and the six distances between the towns are distinct integers.

Problem 251.5 – Continued fractions

Tony Forbes

Show that

$$\begin{aligned} 1/(1+2/(2+3/(3+4/(4+5/(5+\ldots))))) &= \frac{1}{e-1}, \\ 2/(1+3/(2+4/(3+5/(4+6/(5+\ldots))))) &= 1, \\ 3/(1+4/(2+5/(3+6/(4+7/(5+\ldots))))) &= \frac{4}{3}, \end{aligned}$$

and if possible explain in a simple manner how shifting the numerator sequence to the left converts the transcendental number $1/(e-1)$ to an integer and then to a rational. If you do one more shift, you get

$$4/(1+5/(2+6/(3+7/(4+8/(5+\ldots))))) = \frac{21}{13}.$$

Solution 248.1 – Two theorems

What's wrong with the following?

Theorem 1 $\lim_{x\to 0} \dfrac{\sin x}{x} = 1.$

Proof Since $\sin 0 = 0$ we use l'Hôpital's rule:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x)} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Theorem 2 $\dfrac{d(\sin x)}{dx} = \cos x.$

Proof Using the definition of the derivative we have

$$\begin{aligned}\frac{d(\sin x)}{dx} &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{(\sin x)(\cos h) + (\cos x)(\sin h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{(\cos x)(\sin h)}{h} = (\cos x) \lim_{h\to 0} \frac{\sin h}{h} = \cos x.\end{aligned}$$

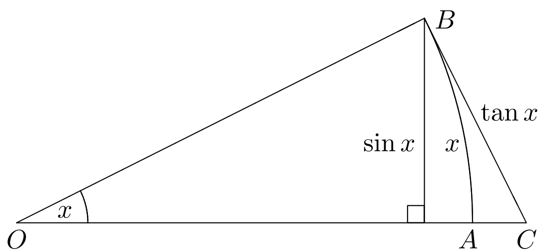
Steve Moon

I looked at this for some time before the penny dropped. The proof set out in Theorem 1 relies on showing (or here, knowing) that $d(\sin x)/dx = \cos x$. However, Theorem 2, which sets out a proof of this, relies in its final step on $\lim_{h\to 0}(\sin h)/h = 1$. This is the subject of Theorem 1, which we have implicitly assumed in Theorem 2 but clearly it is not yet proved. Hence, to rectify this, in order to rely on the argument set out in Theorem 2, we need an alternative proof of $\lim_{x\to 0}(\sin x)/x = 1$.

The one I recall is a geometric proof. In the diagram, $\angle AOB$ is x, $0 < x < \pi/2$, and BC is tangent to the circle at B. Then $\sin x \le x$. Also $x \le \tan x = |BC|$. Thus $\sin x \le x \le \tan x$. Hence

$$\cos x \le \frac{\sin x}{x} \le 1.$$

Now as $x \to 0$, $\cos x \to 1$; so $\lim_{x\to 0}(\sin x)/x = 1$ by the squeeze rule.



Sebastian Hayes

Reflections on Problem 248.1

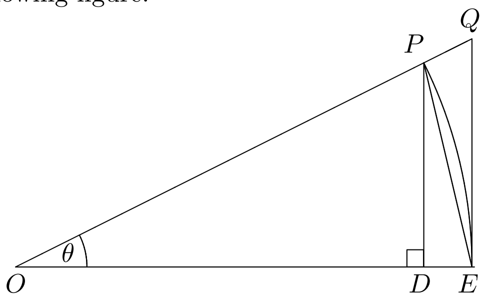
What is wrong with these two theorems? The argument is circular. Theorem 1 uses Theorem 2 in the final step and Theorem 2 uses Theorem 1. But this set me thinking. How would I convince myself that the limit of $(\sin\theta)/\theta = 1$ as $\theta \to \infty$?

For most people, including quite a few mathematics students, the relation is true since for anything less than 0.087 radians (about 5 degrees), $\sin\theta$ and θ are the same to three decimal places. But how is $\sin\theta$ calculated? Presumably by using the expansion

$$\sin\theta \;=\; \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots .$$

But I strongly suspect that the person or persons (Maclaurin? Euler?) who first concocted this series did so by using the 'fact' that the second derivative of $\sin x$ was $-\sin x$ and then expanding $\sin x$ as a power series. So we are not out of the woods yet.

As a mathematical fundamentalist or vegan, I view sines and cosines as essentially ratios between line segments rather than infinite series. Consider the following figure.



$OP = OE = r$

$PD = r\sin\theta$

arc $PE = r\theta$

$QE = r\tan\theta$

If $\angle POD = \theta$ in radians, PE, the arc subtended by the angle θ, is $r\theta$ and $r\theta > PD = r\sin\theta$. Also, $QE = r\tan\theta > r\theta > PD$. So

$$r\sin\theta \;<\; r\theta \;<\; r\tan\theta.$$

This inequality holds for any circle with $r > 0$ and all angles θ for which $\sin\theta$, $\cos\theta$ and $\tan\theta$ are defined. We take θ as positive (anti-clockwise from the x-axis) and, since we are only concerned with small angles, $0 < \theta < \pi/2$.

Dividing by $r\sin\theta$, which is positive and non-zero we have

$$1 \;<\; \frac{\theta}{\sin\theta} \;<\; \frac{1}{\cos\theta}.$$

But 1 has limit 1 since it is never anything else. If we can show that the limit of $1/(\cos\theta)$ is 1 as $\theta \to 0$, the expression $\theta/(\sin\theta)$ will be squeezed between two limits. What can at once be deduced from the diagram is:

1. $r\cos\theta$ must be smaller than the radius and so, for unit radius, $0 < \cos\theta < 1$;

2. As θ decreases, $\cos\theta$ increases, or if $\phi < \theta$, $\cos\phi > \cos\theta$.

Thus $1/(\cos\theta)$ is thus monotonic decreasing and has a lower limit of 1 which is sufficient to establish convergence. If we want to apply the canonical test, we have to find a δ such that, for any $\epsilon > 0$, whenever $0 < \theta < \delta$ we have $|(1 - 1/(\cos\theta)| < \epsilon$.

With $\delta < \cos^{-1} 1/(1+\epsilon)$ we should be home and, applying the 'sandwich principle' for limits, we have $\lim_{\theta\to 0}\theta/(\sin\theta) = 1$. Turning this on its head, we finally obtain

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} \;=\; 1.$$

Note, however, that θ is the *independent* variable — $\sin\theta$ depends on θ and not the reverse. From here we can find the derivative of $\sin\theta$ in the manner of Theorem 2 — all that is needed apart from the definition of the derivative is the 'double angle' formula $\sin(A + B) = \sin A\cos B + \cos A\sin B$, which can be easily proved geometrically for all angles A, B, where $0 < A < \pi/2$ and $0 < B < \pi/2$.

However, what's all this got to do with the well-known power series? Define a convergent power series $f(x)$ with the property that $d^2 f(x)/dx^2 = -f(x)$. Setting $A_0 = 0$, $A_1 = 1$ and equating coefficients we eventually end up with

$$f(x) \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots.$$

But I'm none too happy about identifying the above series with $\sin x$ (and its derivative with $\cos x$). For, if $\sin x$ and $\cos x$ are geometric relations between line segments, when there is no triangle, there can be no sine or cosine. For me, geometric $\sin x$ is undefined at $x = 0$ (and likewise at $x = \pi/2$ &c.) although the *limit* of $\sin x$ as $x \to 0$ is certainly 0. (It is distressing how often it seems necessary to point out, even to mathematicians, that the existence of a limit does not in any way guarantee that this limit is actually attained.)

All in all, I would feel a lot easier if the '$\sin x$ power series' were derived (or defined) recursively *term by term* along with a demonstration that the difference between $f(x)$ and $\sin x$ is *always decreasing* as we add more terms with limit zero. For $\sin x$ is, in my eyes, *itself* the limit of a power series as n increases without bound, i.e. if $0 < x < \pi/2$,

$$\lim_{n\to\infty} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right) \;=\; \sin x.$$

A more general point needs to be made. Practically all proofs in Analysis and Calculus depend on the assumption that the independent variable (in this case the angle θ) can be made arbitrarily small. This is quite legitimate if we restrict ourselves to pure mathematics. But Calculus was invented by Newton and Leibnitz to elucidate problems in physics and has been employed in applied areas ever since. Translated into physical terms, the basic assumption of Calculus and Analysis, is the presumption that space and time are 'infinitely divisible'. But I do not believe they are for both logical and observational reasons. There is a growing (but still minority) view amongst theoretical physicists that Space/Time is 'grainy', i.e. that there are minimal distances and minimal intervals of time just as there are minimal transfers of energy (quanta). If this proves to be the case, Calculus and a lot else besides constitutes a very misleading model of a reality that is, at bottom, discrete. The great majority of differential equations are, in any case, unsolvable analytically and increasingly the trend is to slog things out iteratively with high-speed computers taking things to the level of precision required by the conditions of the problem and then stopping. Dreadful to say so, but it seems that Calculus's reign, like that of the dinosaurs, is drawing to a close and that the future will go to algorithmic methods, genetic or otherwise.

Problem 251.6 – Integer

Tony Forbes

Show that for positive integer n,

$$\frac{\left(10^{9n10^{n-1}-1} - 1\right)\left(100^n - 10^n + 10\right) + 9}{\left(10^n - 1\right)^2}$$

is an integer.

Solution 210.1 – Determinant

Compute

$$\begin{vmatrix} 4 & a+b+c+d & a^2+b^2+c^2+d^2 & a^3+b^3+c^3+d^3 \\ a+b+c+d & a^2+b^2+c^2+d^2 & a^3+b^3+c^3+d^3 & a^4+b^4+c^4+d^4 \\ a^2+b^2+c^2+d^2 & a^3+b^3+c^3+d^3 & a^4+b^4+c^4+d^4 & a^5+b^5+c^5+d^5 \\ a^3+b^3+c^3+d^3 & a^4+b^4+c^4+d^4 & a^5+b^5+c^5+d^5 & a^6+b^6+c^6+d^6 \end{vmatrix}.$$

Steve Moon

Assuming there is a relatively elegant expansion for this determinant, I gave up trying to find it by brute force; any material simplification was well hidden in the lengthy algebra.

To simplify, hopefully, I considered the determinants of smaller $n \times n$ matrices, symmetric about the main diagonal, with all elements the sum of powers of n parameters, each freely interchangeable. Hence any solution is expected to reflect this interchangeability. The number in the top-left position is of course the sum of the zeroth powers of the parameters.

For $n = 2$,

$$\begin{vmatrix} 2 & a+b \\ a+b & a^2+b^2 \end{vmatrix} = 2(a^2+b^2) - (a+b)^2 = (a-b)^2.$$

For $n = 3$,

$$\begin{vmatrix} 3 & a+b+c & a^2+b^2+c^2 \\ a+b+c & a^2+b^2+c^2 & a^3+b^3+c^3 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^4+b^4+c^4 \end{vmatrix} = (a-b)^2(a-c)^2(b-c)^2,$$

which meets the pairwise interchangeability condition. Even for this small example the algebra is tedious, and is omitted.

So for $n = 4$, we deduce that the answer to the problem is

$$(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2,$$

and this is confirmed by MAPLE.

Finally I evaluated the determinant for $n = 5$:

$$(a-b)^2(a-c)^2(a-d)^2(a-e)^2(b-c)^2(b-d)^2(b-e)^2(c-d)^2(c-e)^2(d-e)^2,$$

again confirmed by MAPLE.

Tony Forbes

Curiously, if you generalize the thing slightly by adding t to each exponent, you also get a nice formula. For example, in the case $n = 3$ the determinant becomes

$$\begin{vmatrix} a^t + b^t + c^t & a^{t+1} + b^{t+1} + c^{t+1} & a^{t+2} + b^{t+2} + c^{t+2} \\ a^{t+1} + b^{t+1} + c^{t+1} & a^{t+2} + b^{t+2} + c^{t+2} & a^{t+3} + b^{t+3} + c^{t+3} \\ a^{t+2} + b^{t+2} + c^{t+2} & a^{t+3} + b^{t+3} + c^{t+3} & a^{t+4} + b^{t+4} + c^{t+4} \end{vmatrix}$$

$$= (abc)^t (a-b)^2 (a-c)^2 (b-c)^2.$$

Unfortunately I do not know of any way to obtain this result other than by getting MATHEMATICA to do the work.

Solution 234.2 – Series

Show that

$$1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \frac{1}{25} - \ldots \;=\; \frac{\pi(\sqrt{2}+1)}{8},$$

an interesting relation between π and $\sqrt{2}$.

Steve Moon

Let S_1 denote the series in question. From the power series for $\tan^{-1} x$,

$$\tan^{-1} x \;=\; 1 - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \ldots,$$

set $x = 1$ and recall that $\tan^{-1} 1 = \pi/4$. Thus

$$S_1 \;=\; \frac{\pi}{4} + \left(\frac{1}{3} - \frac{1}{5} + \frac{1}{11} - \frac{1}{13} + \frac{1}{19} - \frac{1}{21} + \ldots\right) \;=\; \frac{\pi}{4} + S_2,$$

say. So S_2 denotes the infinite series containing the ‘missed terms’.

We can generate the terms of S_2 as follows. We have

$$\int_0^1 \frac{x^2 - x^4}{1 - x^8} dx \;=\; \sum_{k=0}^{\infty} \int_0^1 (x^2 - x^4) x^{8k} dx \;=\; \sum_{k=0}^{\infty} \left[\frac{x^{8k+3}}{8k+3} - \frac{x^{8k+5}}{8k+5}\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{5} + \frac{1}{11} - \frac{1}{13} + \frac{1}{19} - \frac{1}{21} + \ldots,$$

where we have used $1/(1-x^8) = 1 + x^8 + x^{16} + \ldots$. To confirm that the integrand is finite throughout the range $0 \le x \le 1$ we need to check that $\lim_{x\to 1}(x^2 - x^4)/(1-x^8)$ exists. But by l'Hôpital's rule this is $\lim_{x\to 1}(2x - 4x^3)/(-8x^7) = 1/4$. So I believe we are OK to proceed. Therefore

$$S_1 = \frac{\pi}{4} + \int_0^1 \frac{x^2 - x^4}{1 - x^8}\,dx = \int_0^1 \frac{x^2}{(1+x^2)(1+x^4)}\,dx.$$

Splitting the integrand into partial fractions gives

$$S_1 = \frac{\pi}{4} + \frac{1}{2}\int_0^1 \frac{x^2+1}{1+x^4}\,dx - \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{8} + \frac{1}{2}\int_0^1 \frac{x^2+1}{1+x^4}\,dx,$$

since $\int_0^1 dx/(1+x^2) = \left[\tan^{-1} x\right]_0^1 = \pi/4$.

To deal with the other integral we use

$$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$$

and again split into partial fractions:

$$\frac{x^2+1}{1+x^4} = \frac{1}{2}\left(\frac{1}{x^2 + \sqrt{2}x + 1} + \frac{1}{x^2 - \sqrt{2}x + 1}\right).$$

Hence

$$\begin{aligned} S_1 &= \frac{\pi}{8} + \frac{1}{4}\int_0^1 \frac{1}{x^2 + \sqrt{2}x + 1} + \frac{1}{4}\int_0^1 \frac{1}{x^2 - \sqrt{2}x + 1} \\ &= \frac{\pi}{8} + \frac{1}{2}\int_0^1 \frac{1}{1 + (\sqrt{2}x + 1)^2} + \frac{1}{2}\int_0^1 \frac{1}{1 + (\sqrt{2}x - 1)^2} \\ &= \frac{\pi}{8} + \frac{\sqrt{2}}{4}\left(\tan^{-1}(\sqrt{2}+1) + \tan^{-1}(\sqrt{2}-1)\right). \end{aligned}$$

But $\tan \pi/8 = \sqrt{2} - 1$ and $\tan 3\pi/8 = \sqrt{2} + 1$. Hence

$$S_1 = \frac{\pi}{8} + \frac{\sqrt{2}}{4}\left(\frac{3\pi}{8} + \frac{\pi}{8}\right) = \frac{\pi}{8}(\sqrt{2}+1),$$

as required.

Problem 251.7 – Fourteen cubes

Can you fit 14 cubes of volume 2 in a $5 \times 5 \times 2\frac{1}{2}$ box?

Problem 251.8 – Six-week months

Tony Forbes

A week starts on Monday and ends on Sunday. A *six-week month* is a month that spans six weeks (September 2013 for example).

Show that a year has two, three or four six-week months, at least one of which must have 31 days. (This year has two—the other one being December.) So these things cannot be avoided, as I expect calendar designers already know.

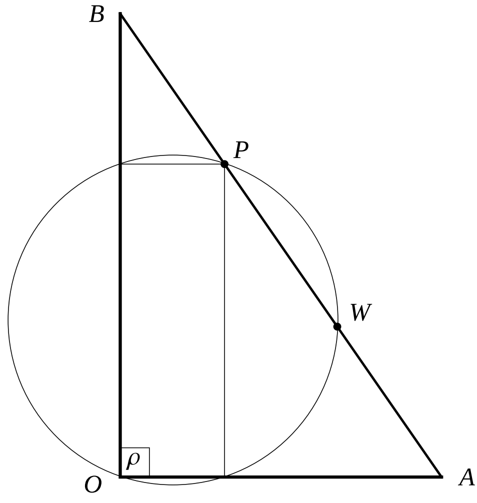
Show that when there are four six-week months in a year, they must be January, April, July and December.

Problem 251.9 – The Philo line

Dick Boardman

Philo of Byzantium (180–120 BC) examined the following problem: Given two lines meeting at angle ρ, say, and a fixed point between them, what is the shortest line through the fixed point and its intersections with both lines.

I can solve this using calculus. But how could Philo have proved the minimum using the tools available at that time? All I can think of is that he used a light ray and mirrors or that he used the tangent to a higher curve. Any ideas?



Tony Forbes — The illustration shows the special case where $\rho = 90$ degrees. The lines are OA and OB, the point is P, and the Philo line is AB. This suggests another problem. *Assuming that* $\rho = \pi/2$ *and that* P *has coordinates* $(1, p)$, *determine the coordinates of* W, *the other point of intersection between the Philo line and the circle with diameter* OP.

If you are like me, expect to be surprised and delighted by the answer.

Solution 249.4 – Radium

> Imagine you have constructed a sphere, 10 cm radius, of pure radium-226. You then leave it alone and return to it about 1600 years later. What would you expect to see? Recall that Ra-226 decays to radon-222 with half-life 1601 years, and Rn-222 decays to polonium-218 with half-life 3.8235 days. There are further steps: Po-218 to lead-214 and astatine-218, Pb-214 to bismuth-214, At-218 to Bi-214 and Rn-218 (half-life 0.036 seconds), and so on until Pb-206 is reached. Amongst the decay products are noble gases. Will they work their way out and escape gracefully from the surface of the sphere, or will there be sufficient accumulation in the centre to build up pressure and blow the thing apart?

Colin Davies

A few quick thoughts about the radium sphere.

I suspect that a sphere of pure radium-226, once assembled, would get extremely hot very quickly as a result of its very high radioactivity. Radium's melting temperature is apparently 700°C. It might melt, but before it melted it would probably collapse into a pancake under its own 23 kg weight.

Radium is the atomically heaviest of the group 2 metals and is therefore very reactive chemically. The hot sphere would probably react with almost all the gases (including nitrogen) in the atmosphere, so it would need to be left in a vacuum for the 1600-year duration of the experiment.

However, let's assume it can remain as a reasonably solid sphere in a vacuum. Then the radon-222 content would presumably be continually proportionate to the ratio of the half-lives of radon-222 to radium-226. That is presumably $3.8235/(1601 \cdot 365) \approx 6 \cdot 10^{-6}$. That is about 1 part in 100,000. So I don't think it would take up much room or have much effect.

You do not give the half-life of astatine-218, but as a halogen, it will be certain to react chemically with the radium to form $RaAt_2$. That reaction will be exothermic, and will increase the temperature even more.

It is not clear from your explanation how much stable noble gas will be produced, but as it will be produced evenly over the whole volume of the hot sphere I think it will form a sort of colloidal solution of a gas in a solid. In other words, the sphere would become a meringue, or possibly a stiff foam like a metallic pizza if it did collapse.

Barbara Lee

Dear Tony,

What are you trying to do, discover if there is a critical mass for radium? This was not discussed in our lectures at the Isotope School.

Radium-226 decays with the emission of an alpha particle, a gamma ray and a small amount of internal conversion where energy is transmitted to one of the orbital electrons.

Polonium-218 emits an alpha particle. Alpha particles are considered to be heavy in nuclear physics and their range in air is only about 4 cm; so the ones in your lump of radium won't be going anywhere.

Lead-214 emits mainly beta particles and some weak gamma rays. So does bismuth-214. And so on until you are left with lead-206.

As there is a lot of activity going on in your sphere I suspect that the critical mass might be reached before you finish constructing it. Please do not do this; my youngest son lives in Wolsey Drive and he might be exterminated by your big bang.

Yours sincerely,

Tony Forbes

Wikipedia provides the following details for Ra-226 and its decay products.

Nuclide	Half-life	Decay modes
Ra-226	1601 y	α to Rn-222
Rn-222	3.8235 d	α to Po-218
Po-218	3.10 m	99.98% α to Pb-214, 0.02% β^- to At-218
At-218	1.5 s	99.90% α to Bi-214, 0.10% β^- to Rn-218
Rn-218	35 ms	α to Po-214
Pb-214	26.8 m	β^- to Bi-214
Bi-214	19.9 m	99.98% β^- to Po-214, 0.02% α to Tl-210
Po-214	0.1643 ms	α to Pb-210
Tl-210	1.30 m	β^- to Pb-210
Pb-210	22.3 y	99.9999981% β^- to Bi-210, 0.0000019% α to Hg-206
Bi-210	5.013 d	99.99987% β^- to Po-210, 0.00013% α to Tl-206
Po-210	138.376 d	α to Pb-206
Hg-206	8.15 m	β^- to Tl-206
Tl-206	4.199 m	β^- to Pb-206
Pb-206	-	stable

From this information we can compute the numbers of atoms of the various elements in the sample arising from a single Ra-226 nucleus after 1601 years:

Ra-226: 0.5,	Rn-222: 3.269×10^{-6},	Po-218: 1.841×10^{-9},
At-218: 2.969×10^{-15},	Rn-218: 6.927×10^{-20},	Pb-214: 1.591×10^{-8},
Bi-214: 1.182×10^{-8},	Po-214: 1.626×10^{-15},	Tl-210: 1.544×10^{-13},
Pb-210: 0.00706,	Bi-210: 4.347×10^{-6},	Po-210: 0.000120,
Hg-206: 9.325×10^{-17},	Tl-206: 3.335×10^{-15},	Pb-206: 0.493,
He-4: 2.493.		

As we can see, the only significant quantities remaining are radium, lead, a little polonium and a rather large amount of helium (from alpha particles that have each acquired a couple of stray electrons). In terms of weight, from one gram of Ra-226 we get:

Ra-226: 0.5 g,	Rn-222: 3.211×10^{-6} g,	Po-218: 1.775×10^{-9} g,
At-218: 2.864×10^{-15} g,	Rn-218: 6.682×10^{-20} g,	Pb-214: 1.506×10^{-8} g,
Bi-214: 1.119×10^{-8} g,	Po-214: 1.539×10^{-15} g,	Tl-210: 1.434×10^{-13} g,
Pb-210: 0.00656157 g,	Bi-210: 4.038×10^{-6} g,	Po-210: 0.0001115 g,
Hg-206: 8.498×10^{-17} g,	Tl-206: 3.039×10^{-15} g,	Pb-206: 0.449092 g,
He-4: 0.0441441 g.		

Now one can add up all these masses to obtain a total of 0.999916 g. So, it looks as if our 10 cm sphere, weighing approximately 23 kg, will generate about $23 \times 0.000084\, c^2 \approx 1.7 \times 10^{14}$ joules during the life of the experiment—starting at about 4.8 kilowatts and decreasing roughly exponentially to about 2.4 kW.

There is no cause for concern. From the wording of the problem, I think it is clear we are dealing with an imaginary situation. In the past radium might have been obtainable by the shovelful from the local chemist but nowadays it is a little difficult to get hold of.

Americium-241, on the other hand, is readily available as the active ingredient of smoke alarms. However, a typical 0.9 microcurie's worth of Am-241 (half-life 432 y) in a smoke detection unit weighs less than 0.3 micrograms, and although the critical mass of Am-241 is understandably kept secret (figures in the public domain are presumably disinformation) we can assume you would probably need kilograms of the stuff. So if you are planning to build an Am-241 reactor, your biggest problem by far will be the disposal of the scrap plastic from several billion dismantled smoke alarms.

Solution 238.3 – Sums

Let

$$S_n(k) \;=\; \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^k,$$

where $k \geq 2$ is an integer. Prove that

$$\frac{S_1(k)}{1} - \frac{S_2(k)}{2} + \frac{S_3(k)}{3} - \cdots \pm \frac{S_k(k)}{k} \;=\; 0.$$

Reinhardt Messerschmidt

Let

$$S(k) \;=\; \frac{S_1(k)}{1} - \frac{S_2(k)}{2} + \frac{S_3(k)}{3} - \cdots \pm \frac{S_k(k)}{k}.$$

If k, m, r are integers such that $0 \leq r \leq m \leq k$, let

$$F(k,m,r) \;=\; \begin{cases} 0 & \text{if } m = r = 0, \\ \sum_{s=r}^{k} (-1)^{s+1} s^{m-r} \binom{k-r}{s-r} & \text{otherwise.} \end{cases}$$

Step 1. We first show that $S(k) = F(k, k-1, 0)$ for every $k \geq 2$. Since $\binom{n}{j} = \binom{n}{n-j}$, we have

$$S_n(k) \;=\; \sum_{j=0}^{n} (-1)^j \binom{n}{n-j} (n-j)^k.$$

Substituting $s = n - j$ gives

$$S_n(k) \;=\; \sum_{s=0}^{n} (-1)^{n-s} \binom{n}{s} s^k \;=\; \sum_{s=1}^{n} (-1)^{n-s} \binom{n}{s} s^k.$$

It follows that

$$\begin{aligned} S(k) \;&=\; \sum_{t=1}^{k} (-1)^{t+1} \frac{S_t(k)}{t} \;=\; \sum_{t=1}^{k} (-1)^{t+1} \frac{1}{t} \sum_{s=1}^{t} (-1)^{t-s} \binom{t}{s} s^k \\ &=\; \sum_{t=1}^{k} \sum_{s=1}^{t} (-1)^{2t-s+1} \binom{t}{s} \frac{s^k}{t} \;=\; \sum_{t=1}^{k} \sum_{s=1}^{t} (-1)^{s+1} \binom{t}{s} \frac{s}{t} s^{k-1}. \end{aligned}$$

We have $\binom{t}{s} \frac{s}{t} = \binom{t-1}{s-1}$; therefore

$$S(k) \;=\; \sum_{t=1}^{k} \sum_{s=1}^{t} (-1)^{s+1} \binom{t-1}{s-1} s^{k-1}.$$

Changing the order of summation gives

$$S(k) \;=\; \sum_{s=1}^{k} (-1)^{s+1} s^{k-1} \sum_{t=s}^{k} \binom{t-1}{s-1}.$$

We can show by induction that $\sum_{t=s}^{k} \binom{t-1}{s-1} = \binom{k}{s}$ for every $k \geq s$; therefore

$$S(k) \;=\; \sum_{s=1}^{k} (-1)^{s+1} \binom{k}{s} s^{k-1} \;=\; \sum_{s=0}^{k} (-1)^{s+1} \binom{k}{s} s^{k-1} \;=\; F(k, k-1, 0).$$

Step 2. Next, we show that F satisfies the boundary condition

$$F(k, m, m) \;=\; 0 \tag{1}$$

for every k, m such that $0 \leq m < k$. The case $m = 0$ follows from the definition of F. If $m > 0$, then

$$F(k, m, m) \;=\; \sum_{s=m}^{k} (-1)^{s+1} \binom{k-m}{s-m} s^{m-m} \;=\; \sum_{s=m}^{k} (-1)^{s+1} \binom{k-m}{s-m}.$$

We can show by induction that the last sum vanishes for every $k > m$.

Step 3. We now show that F satisfies the recurrence relation

$$F(k, m, r) - rF(k, m-1, r) \;=\; (k-r)F(k, m, r+1) \tag{2}$$

for every k, m, r such that $0 \leq r < m < k$.

Suppose $m = 1$. Since $0 \leq r < m$, we must have $r = 0$. The left-hand side of (2) then becomes

$$F(k, 1, 0) - 0 \cdot F(k, 0, 0) \;=\; \sum_{s=0}^{k} (-1)^{s+1} \binom{k}{s} s \;=\; \sum_{s=1}^{k} (-1)^{s+1} \binom{k}{s} s.$$

We have $\binom{k}{s} s = \binom{k-1}{s-1} k$, therefore

$$F(k, 1, 0) - 0 \cdot F(k, 0, 0) \;=\; k \sum_{s=1}^{k} (-1)^{s+1} \binom{k-1}{s-1}.$$

We can show by induction that the last sum vanishes for every $k \geq 2$. By (1), the right-hand side of (2) is

$$(k-0)F(k, 1, 1) \;=\; k \cdot 0 \;=\; 0;$$

therefore (2) holds for $m = 1$.

Suppose $m > 1$, then

$$\begin{aligned} F(k,m,r) - rF(k,m-1,r) &= \sum_{s=r}^{k} (-1)^{s+1} \binom{k-r}{s-r} \left(s^{m-r} - rs^{m-r-1}\right) \\ &= \sum_{s=r+1}^{k} (-1)^{s+1} \binom{k-r}{s-r} (s-r) s^{m-r-1}. \end{aligned}$$

We have $\binom{k-r}{s-r}(s-r) = \binom{k-r-1}{s-r-1}(k-r)$; therefore

$$\begin{aligned} F(k,m,r) - rF(k,m-1,r) &= (k-r) \sum_{s=r+1}^{k} (-1)^{s+1} \binom{k-r-1}{s-r-1} s^{m-r-1} \\ &= (k-r)F(k,m,r+1). \end{aligned}$$

Step 4. We now show that

$$F(k,m,r) = 0 \tag{3}$$

for every k, m, r such that $0 \le r \le m < k$. The case $m = r$ follows from (1). If $m \ge 1$, then

$$\begin{aligned} &F(k,m,m-1) \\ &= (m-1)F(k,m-1,m-1) + (k-m+1)F(k,m,m) && \text{by (2)} \\ &= (m-1) \cdot 0 + (k-m+1) \cdot 0 && \text{by (1)} \\ &= 0. && (4) \end{aligned}$$

If $m \ge 2$, then

$$\begin{aligned} &F(k,m,m-2) \\ &= (m-2)F(k,m-1,m-2) + (k-m+2)F(k,m,m-1) && \text{by (2)} \\ &= (m-2) \cdot 0 + (k-m+2) \cdot 0 && \text{by (4)} \\ &= 0. \end{aligned}$$

Repeating this process a sufficient number of times gives (3).

Step 5. Finally, substituting $m = k-1$ and $r = 0$ into (3) gives

$$S(k) = F(k,k-1,0) = 0.$$

Letter

Arabic numbers

Many thanks to Ralph Hancock [M500 **249**, p. 11] for pointing out my error in suggesting that the Arabs write their numerals in ascending order unlike us. There are no less than three distinct issues here: (1) writing anything, sentences, names, numbers &c. from *right to left* or from *left to right*, (2) writing numerals in *ascending* or *descending* order, (3) the order in which we deal with numerals when performing addition/subtraction and so on. My main source, Gillings, writes

> Today most nations write from *left to right*, and our numbers are so written also; but the values of the digits in our 'Hindu-Arabic' decimal system increase in place value from *right to left*. So, if we have to perform an addition or subtraction, we begin with the units column on the right, and work toward the left through the tens, hundreds and so on.
>
> Conversely, the Egyptians wrote their words and numbers from *right to left*. Of necessity, however, the Egyptian mathematicians, like ourselves, had to start adding in the opposite direction to that in which they were accustomed to write, so the place value of the Egyptians' digits increases from *left to right*, and the Egyptian system therefore runs widdershins to ours.
>
> R. Gillings, *Mathematics in the Time of the Pharaohs*

The muddle about left to right and right to left has been confounded because renderings of ancient Egyptian texts naturally reverse the orientation with respect to sentences but often print the numerals in imitation hieroglyphs in the order in which they appear in the papyrus with a modern numeral written underneath each one (so that people can see the hieroglyphs). This gives the erroneous impression that the Egyptians wrote their numerals in ascending order which they did not according to Gillings.

But how/why did writing numbers in descending order ever come about in the first place? Presumably because this mimicked the way in which large quantities were assessed by State officials. Confronted with, say, a confused mass of prisoners or pottery imports, an official would start by working out the thousands or hundreds, then pass to the tens and finally to the units. He would call out the amounts as he worked them out and the scribe would record the numerals in the order in which he heard them, i.e. largest amount first. Moreover, in this way, a visiting official could get a rough idea of the size simply by glancing at the first hieroglyphic numeral (which, remember, is a different pictogram for each power of ten). But, when it came to actual operations with numbers, the scribes like everyone else had to proceed the other way (though some mental arithmetic experts say they add the higher columns first).

The confusion demonstrates the basic conflict between two very different functions of numbers: (1) as devices for the compact recording of data and (2) as a means of drawing original conclusions from given data. The first process is a movement from the unknown (or very roughly known) to the known, the second a movement from the known into the unknown which, if the reasoning is valid, eventually transforms it into part of the known. As professional or amateur mathematicians we tend to think that numbers were invented for the purpose of getting out precise solutions to mathematical equations, but the recording function of numbers was by far the more important for millennia and arguably still is.

Sebastian Hayes

Please note that in M500 **249**, 'Arabic numbers', page 11, 'which can be transliterated "asalif wa-khamsaman'a wa-sita" and "means thousand and-five hundred and-six"' (How Ralph Hancock's original submission got so mangled to create this mess (which then remained undetected by this magazine's Editor and proof-readers) is a bit of a mystery. Someone, ICL mathematician Robert Vaughan, I think, once told me that the process of proof correction is not necessarily convergent. But whether that is relevant here I'm not so sure. Anyway, my fault. Apologies.) should of course read 'which can be transliterated as "alif wa-khamsamān'a wa-sita" and means "thousand and-fivehundred and-six"' — TF.

Mathematics in the kitchen – IX

Robin Whitty

This question concerns tea, in particular the preparation of the hot water for the brewing of the said beverage from tea-bags individually in cups.

Either (1) you boil a full cylinder of water and make a sequence of cups of tea, at times t, $t+k$, $t+2k$, $\dots$, reboiling the remaining volume each time;

or (2) you boil one cupful of water for each of your cups.

Is (1) is more energy efficient than (2) when $k=0$ (i.e. boiling one big kettle is better than boiling lots of little ones)? Maybe this depends on the parameters of the cylinder. If so, is there a value of k below which (1) is more energy efficient than (2)?

[Please do not try this at home. Boiling water is a hazardous substance capable of causing severe burns. — TF]

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Front cover: $(n-1, n, n+1)$ triangles; see page 1.