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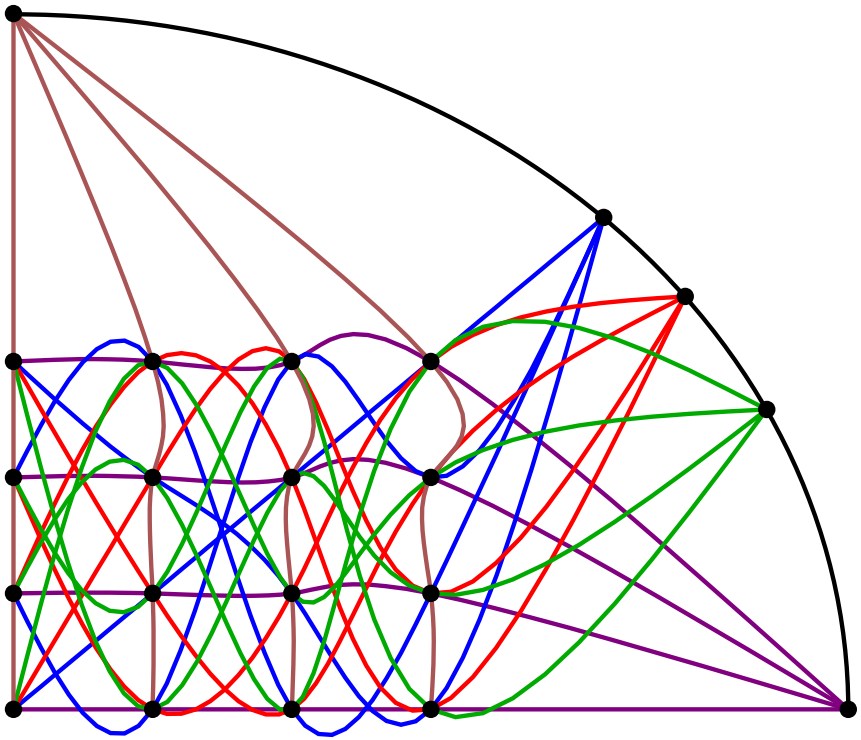

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Tiling a torus

Tommy Moorhouse

Consider the unit square with corners $(0,0), (0,1), (1,0)$ and $(1,1)$. We identify the opposite sides in pairs to make a torus T. A (closed) line on this torus is the image of a line from $(0,0)$ to (m,n) where m and n are relatively prime positive integers. We will call this image line $L(m,n)$. Thus the image line starts at the origin and wraps around the torus T an integer number of times before it ends at $(1,1)$, which is again the origin under the torus identifications. The construction is shown for two different lines in Figure 1. A line wrapping around the unit torus is shown in Figure 2.

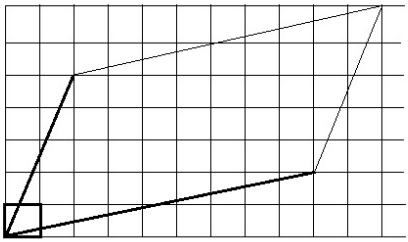


Figure 1: Unit torus T with lines

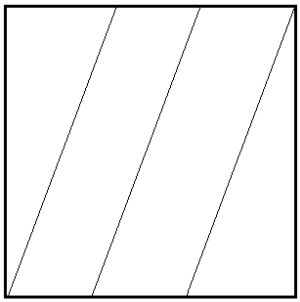


Figure 2: Line wrapped on torus T

Two distinct lines on the torus give rise to a tiling of the torus, in this case a regular division of the surface into four-sided tiles. How many tiles are there for two distinct lines $L(m,n)$ and $L(\hat{m},\hat{n})$? The answer is related to the number of times the lines intersect on the torus.

The triangle-halving deltoid envelope

Robin Whitty

In an article in 1972 [1] (and see [2] for further developments), Dunn and Pretty determined the collection of all straight lines bisecting the area of a triangle as a certain 'deltoid' envelope. More specifically, they observed that the general problem could be reduced to bisecting a right isosceles triangle at the origin, and that in this triangle the bisecting lines were the tangents to a deltoid curve traced by three hyperbolae. The hyperbolae in question were those whose asymptotes were the pairs of edges of the triangle (we are only concerned with the positive, or right-facing branches of the hyperbolae, suitably rotated).

Figure 1 shows Dunn and Pretty's deltoid envelope for an arbitrarily specified triangle ABC (whose edges have been produced, the more easily to picture them as asymptotes). There are six area-bisecting lines which are easy to identify, plotted as grey lines. These are the three triangle medians, joining vertices to midpoints of opposite sides; and the three lines parallel to the triangle edges and dividing the altitudes in the ratio $1/\sqrt{2} : 1-1/\sqrt{2}$. We observe that these six lines are, indeed, tangent to the plotted hyperbolae.

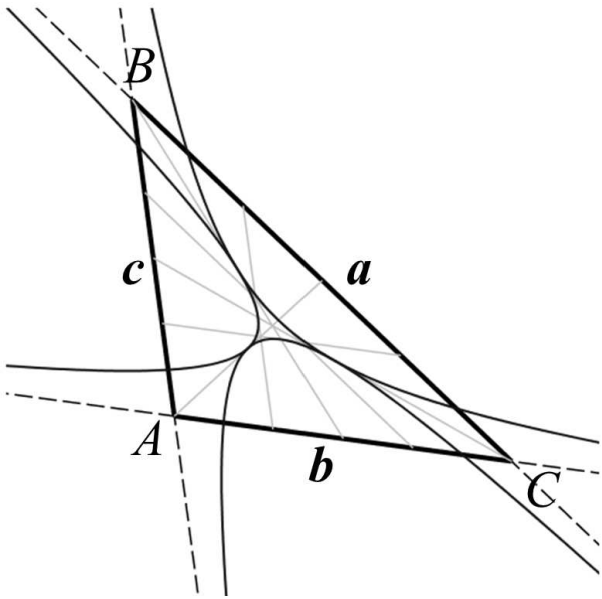


Figure 1

The purpose of this note is to record how we plotted the hyperbolae in Figure 1. There is a grown-up way we could have done it, which is to carry out Dunn and Pretty's affine transformation to a triangle placed at the origin, and then apply the reverse transformation to their hyperbolae. We would like to see this approach described in a future issue of M500! Still, it is a useful exercise to acquire the necessary ingredients from scratch, so to speak, for a given triangle. And this is what we will do here.

A hyperbola may be specified in terms of various subsets of its parameters: foci, vertices, centre, eccentricity, asymptotes, etc. Our aim will be to give values to those parameters necessary to write down each hyperbola in parametric form thus

$$\begin{aligned} x &= x_0 + p\cosh(t)\cos\theta - q\sinh(t)\sin\theta, \\ y &= y_0 + p\cosh(t)\sin\theta + q\sinh(t)\cos\theta, \end{aligned}$$

with t ranging over a half-circle, $-\tau/4 \le t \le \tau/4$.[1] The angle θ gives the rotation from the hyperbola symmetric about the horizontal axis. This is more explicit if we write these equations in terms of a rotation matrix:

$$(x, y) = (x_0, y_0) + (p\cosh(t), q\sinh(t)) \times \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \qquad (1)$$

We will start by finding θ, the angle of rotation. Of course we need this angle to correctly orient our hyperbola. In addition it includes the angle between one asymptote of the hyperbola and the midline between the asymptotes, and the tangent of this latter angle is q/p. So θ is doubly useful to us!

From now on we will base our calculations on the triangle ABC in Figure 1 and restrict our attention to the hyperbola whose asymptotes are edges $\mathbf{a}$ and $\mathbf{c}$. We will regard these edges as free vectors, oriented clockwise around the triangle, so that $\mathbf{a} = -B + C$, and so on, treating A, B and C as position vectors. The two other hyperbolae may be obtained in an identical manner.

We start with a right-facing half-hyperbola symmetrical about the horizontal axis. We rotate by a quarter circle clockwise, $-\tau/4$: the hyperbola is now downwards facing, about the vertical axis. We rotate back, anticlockwise, by the angle between the vertical axis and triangle edge $\mathbf{c}$. This is obtained from the dot product of the unit vectors in the directions of $\mathbf{c}$ and

[1] As usual, we define $\tau = 2\left(\int_{-\infty}^{\infty} \exp(-x^2)\,dx\right)^2$.

of the vertical axis as

$$\cos^{-1}\frac{\mathbf{c}\cdot(0,1)}{|\mathbf{c}|}. \tag{2}$$

Finally we must rotate anticlockwise by an angle, call it ϕ, which is half the angle between $\mathbf{c}$ and $-\mathbf{a}$. Now $\cos(2\phi) = -\mathbf{c}\cdot\mathbf{a}/(|\mathbf{c}||\mathbf{a}|)$. So, using the appropriate half-angle formula,

$$\tan(2\phi/2) = \sqrt{\frac{1-\cos(2\phi)}{1+\cos(2\phi)}} = \sqrt{\frac{1+\mathbf{c}\cdot\mathbf{a}/(|\mathbf{c}||\mathbf{a}|)}{1-\mathbf{c}\cdot\mathbf{a}/(|\mathbf{c}||\mathbf{a}|)}},$$

giving

$$\phi = \tan^{-1}\sqrt{X_{ac}} \quad \text{where} \quad X_{ac} = \frac{|\mathbf{a}||\mathbf{c}| + \mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}||\mathbf{c}| - \mathbf{a}\cdot\mathbf{c}}. \tag{3}$$

To sum up, the angle of rotation of the hyperbola whose asymptotes are $\mathbf{c}$ and $\mathbf{a}$ may be written

$$\theta_{ac} = -\frac{\tau}{4} + \cos^{-1}\frac{\mathbf{c}\cdot(0,1)}{|\mathbf{c}|} + \tan^{-1}\sqrt{X_{ac}}. \tag{4}$$

Moreover, as we said earlier with reference to equation (1), $\tan\phi = q/p$, so we can record

$$q = p\sqrt{X_{ac}}. \tag{5}$$

This means that there effectively remains just one unknown on the right-hand side of equation (1) and that is p. To find p we return to this equation, using the identity (5) and some shorthand, $X = x - x_0, Y = y - y_0, \mathbf{C} = \cos\theta_{ac}, \mathbf{S} = \sin\theta_{ac}$:

$$(X,Y) = (\cosh(t), \sinh(t)) \times \begin{pmatrix} p & 0 \\ 0 & p\sqrt{X_{ac}} \end{pmatrix} \times \begin{pmatrix} \mathbf{C} & \mathbf{S} \\ -\mathbf{S} & \mathbf{C} \end{pmatrix}. \tag{6}$$

Inverting the matrices:

$$(X,Y)\begin{pmatrix} \mathbf{C} & -\mathbf{S} \\ \mathbf{S} & \mathbf{C} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1/\sqrt{X_{ac}} \end{pmatrix} = p\,(\cosh(t), \sinh(t)).$$

Now apply the identity $\cosh^2(t) - \sinh^2(t) = 1$ and equate coordinates,

$$(X\mathbf{C} + Y\mathbf{S})^2 - \frac{(Y\mathbf{C} - X\mathbf{S})^2}{X_{ac}} = p^2$$

or as a vector length,[2]

$$p = \left| \begin{pmatrix} 1 & 0 \\ 0 & 1/\sqrt{-X_{ac}} \end{pmatrix} \begin{pmatrix} \mathbf{C} & \mathbf{S} \\ -\mathbf{S} & \mathbf{C} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \right|. \tag{7}$$

This is the distance between the centre of the hyperbola and its vertex. The centre, which is (x_0, y_0) in equation (1), is the point of intersection of the asymptotes. This is point B for our present purposes. To evaluate p we need to locate a point (x, y) lying on the hyperbola in order to give values to $X = x - x_0$ and $Y = y - y_0$. We will take that point on the bisecting line parallel to edge $\mathbf{b}$ which is tangent to the hyperbola. This point also lies on the median line from vertex B.[3] It therefore divides the median line in the ratio $1/\sqrt{2} : 1 - 1/\sqrt{2}$. The median line, as a free vector, may be written as $1/2(\mathbf{a} - \mathbf{c})$. So the required point, lying on the hyperbola asymptotic to $\mathbf{a}$ and $\mathbf{c}$, is $B + 1/(2\sqrt{2})(\mathbf{a} - \mathbf{c})$. And this gives us the final ingredients for our plot:

$$(X, Y) = \frac{1}{2\sqrt{2}}(\mathbf{a} - \mathbf{c}).$$

References

[1] Dunn, J. A. and Pretty, J. E., Halving a triangle, *Math. Gaz.* Vol. 56, No. 396, 1972, 105–108.

[2] Berele, A. and Catoiu, S., Bisecting the perimeter of a triangle, *Mathematics Magazine*, Vol. 91, No. 2, 2018, 121–133.

Problem 300.1 – Friends

There are a finite number of people. For any two distinct persons, a and b, there is a unique person c who is friends with both a and b. Show that there exists one person who is friends with every other person.

This is known in the literature on finite projective planes as the Friendship Theorem. What is wanted ideally is a proof that readers of this magazine can readily understand.

[2] The square root will be imaginary, so this is perhaps a notational convenience. Nevertheless, it seems there should be some direct link from equation (6) to the vector length. Another puzzle for M500 readers!

[3] This may presumably be confirmed retrospectively by differentiating the function we are in the process of specifying! It seems it should be obvious geometrically but this eludes us. Yet another M500 reader challenge.

Solution 223.4 – The arbelos

Also known as the 'shoemaker's knife' of Archimedes, this shape, the *arbelos*, is bounded by three touching circles of radii a, b and $a+b$, as in the upper diagram on page 8. A 'train' of circles C_n, $n = 1, 2, \ldots$, is constructed inside the arbelos as shown in the lower diagram. Show that their centres lie on an ellipse, and find the diameter d_n of each circle in terms of a, b and n. Also show that for each C_n, the centre is at a distance nd_n from the centreline.

Peter Fletcher

On the Wikipedia page

`https://de.wikipedia.org/wiki/Inversion_(Geometrie)`,

under the section headed **Beispiel Kreis**, it explains that the circle with centre (x_n, y_n) and equation

$$(x - x_n)^2 + (y - y_n)^2 \;=\; \frac{d_n^2}{4},$$

i.e. with diameter d_n per the question and not going through the origin, is inverted in the unit circle with centre the origin (equivalent to choosing the k in the question to be 1) to the circle

$$\left(x - \frac{x_n}{x_n^2 + y_n^2 - d_n^2/4}\right)^2 + \left(y - \frac{y_n}{x_n^2 + y_n^2 - d_n^2/4}\right)^2 = \left(\frac{d_n/2}{x_n^2 + y_n^2 - d_n^2/4}\right)^2;$$

so

$$x_n \mapsto \frac{x_n}{x_n^2 + y_n^2 - d_n^2/4}, \qquad y_n \mapsto \frac{y_n}{x_n^2 + y_n^2 - d_n^2/4}, \tag{1}$$

$$\frac{d_n}{2} \;\mapsto\; \frac{d_n/2}{x_n^2 + y_n^2 - d_n^2/4}.$$

If we use the equation in the question, the two tangent circles that go through $(2a, 0)$ and $(2(a+b), 0)$ are mapped to

$$x = \frac{1}{2a} \quad \text{and} \quad x = \frac{1}{2(a+b)}.$$

These are not mapped to circles because they do both go through the origin; or equivalently, they are mapped to circles with infinite radius.

The centres of the circles to which the C_i are all mapped, as we can see from the cover of M500 **203**, lie on the line halfway between these two,

$$x = \frac{1}{2}\left(\frac{1}{2a} + \frac{1}{2(a+b)}\right) = \frac{2a+b}{4a(a+b)}, \tag{2}$$

and the diameters are all the distance between these two lines,

$$\frac{1}{2a} - \frac{1}{2(a+b)} = \frac{b}{2a(a+b)}.$$

Note that the unlabelled circle in the question that would be C_0 has radius b and centre $(2a+b, 0)$: using Eqn. (1), it is mapped to

$$\left(\frac{2a+b}{(2a+b)^2 - b^2}, 0\right) = \left(\frac{2a+b}{4a(a+b)}, 0\right),$$

which agrees with Eqn. (2). The y coordinates of the centres of the inverses of $C_1, C_2, \ldots, C_n, \ldots$ are

$$\frac{b}{2a(a+b)}, \frac{2b}{2a(a+b)}, \ldots, \frac{nb}{2a(a+b)}, \ldots.$$

Thus for C_n we have

$$\begin{aligned}
\frac{2a+b}{4a(a+b)} &= \frac{x_n}{x_n^2 + y_n^2 - d_n^2/4}, \\
\frac{nb}{2a(a+b)} &= \frac{y_n}{x_n^2 + y_n^2 - d_n^2/4}, \\
\frac{b}{2a(a+b)} &= \frac{d_n}{x_n^2 + y_n^2 - d_n^2/4}.
\end{aligned}$$

Solving these simultaneously in Maple, we find

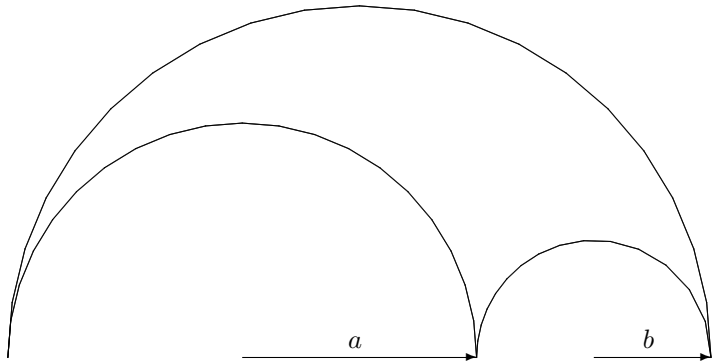
$$x_n = \frac{a(a+b)(2a+b)}{a(a+b) + n^2b^2}, \quad y_n = \frac{2nab(a+b)}{a(a+b) + n^2b^2}, \quad d_n = \frac{2ab(a+b)}{a(a+b) + n^2b^2}.$$

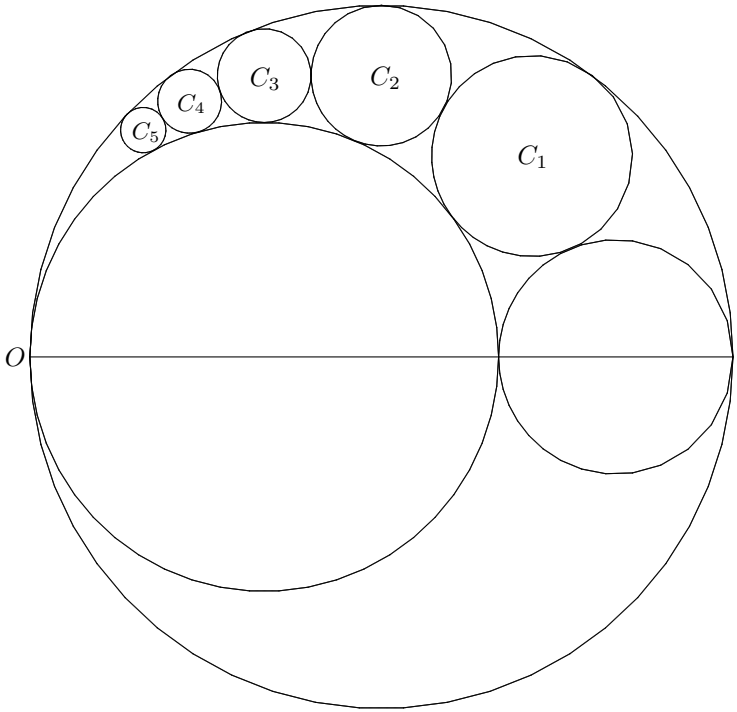
If we now solve $x = x_n$ for n and substitute the positive root for n in $y = y_n$, we get a horrible looking expression that we can simplify to

$$\frac{(x - (2a+b)/2)^2}{(2a+b)^2/4} + \frac{y^2}{a(a+b)} = 1,$$

which is the equation of an ellipse that goes through the centres of C_0, $C_{\pm 1}$, $C_{\pm 2}$, ….

We answered the second part of the question by finding d_n, the diameter of circle C_n. Finally, for the third part of the question, we can see from the expressions for y_n and d_n that $y_n = nd_n$, i.e. that the centre of C_n is nd_n from the x-axis.





Solution 228.5 – Square roots

Given that

$$(\sqrt{a}-\sqrt{a-1})^n = \sqrt{b}-\sqrt{b-1}, \tag{1}$$

show that for any positive integer n, b is a positive integer if a is a positive integer.

Steve Moon

Multiply both sides in turn by $\sqrt{b}+\sqrt{b-1}$ and $(\sqrt{a}+\sqrt{a-1})^n$ and simplify to produce

$$(\sqrt{a}+\sqrt{a-1})^n = \sqrt{b}+\sqrt{b-1}. \tag{2}$$

Now add (1) and (2), simplify and square to obtain an expression for b:

$$\begin{aligned} 2\sqrt{b} &= (\sqrt{a}+\sqrt{a-1})^n + (\sqrt{a}-\sqrt{a-1})^n, \\ b &= \frac{1}{4}\left((\sqrt{a}+\sqrt{a-1})^n + (\sqrt{a}-\sqrt{a-1})^n\right)^2. \end{aligned}$$

Consider the case for even $n = 2m$, m a positive integer. From the binomial expansion of the inner brackets, second and subsequent alternate terms in odd powers of $\sqrt{a}$ and $\sqrt{a-1}$ cancel out, leading to

$$\begin{aligned} b &= \frac{1}{4}\left(2(a^m + {}^nC_2 a^{m-1}(a-1) + {}^nC_4 a^{m-2}(a-1)^2 + \cdots + (a-1)^m)\right)^2 \\ &= \left(a^m + {}^nC_2 a^{m-1}(a-1) + {}^nC_4 a^{m-2}(a-1)^2 + \cdots + (a-1)^m\right)^2. \end{aligned}$$

All binomial coefficients ${}^nC_k$, $k = 0, 1, \ldots, n$ are positive integers. Hence for even n, if a is a positive integer then b is a positive integer.

Now for odd $n = 2m+1$, $m = 0, 1, \ldots$. Again in binomial expressions, second and subsequent terms in even powers of $\sqrt{a}$ and odd powers of $\sqrt{a-1}$ cancel. Then

$$\begin{aligned} b &= \left(a^m\sqrt{a} + {}^nC_2 a^{m-1}\sqrt{a}(a-1) + {}^nC_4 a^{m-2}\sqrt{a}(a-1)^2 + \ldots \right. \\ &\qquad \left. + {}^nC_{n-1}\sqrt{a}(a-1)^m\right)^2 \\ &= a\left(a^m + {}^nC_2 a^{m-1}(a-1) + {}^nC_4 a^{m-2}(a-1)^2 + \cdots + {}^nC_{n-1}(a-1)^m\right)^2 \end{aligned}$$

and hence, for odd positive n, if a is an integer then so is b.

If we relax the restriction on a, allowing it to take any integer values, not just positive ones, then (1) might not hold. For example, take $a = 0$, $n = 2$. Then

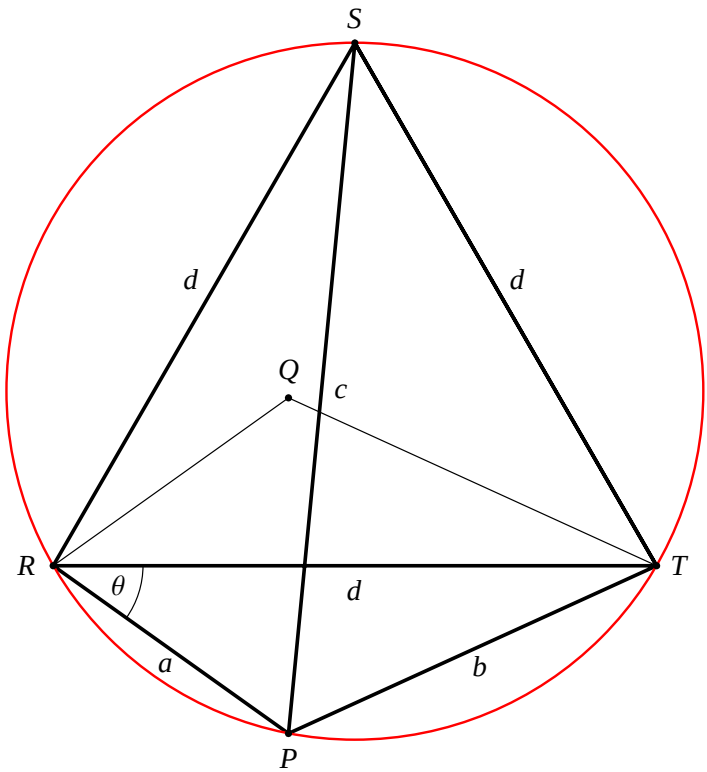
$$(\sqrt{a}-\sqrt{a-1})^n = (-\sqrt{-1})^2 = -1,$$

implying $\sqrt{b-1}-\sqrt{b} = 1$, which has no solutions in integer b.

Solution 266.3 – Equilateral triangle

There is an equilateral triangle. Point P is at distance a from one vertex and b from another vertex. What is the largest possible distance P can be from the third vertex?

Steve Moon



Let RST be the equilateral triangle, of side d. Point P is at distance a from R and b from T. Let the distance from P to the third vertex, S, be c. (There is a nearer point, Q, at distance a from R and b from T at the other point of intersection of circles of radius a and b centred on R and T respectively.)

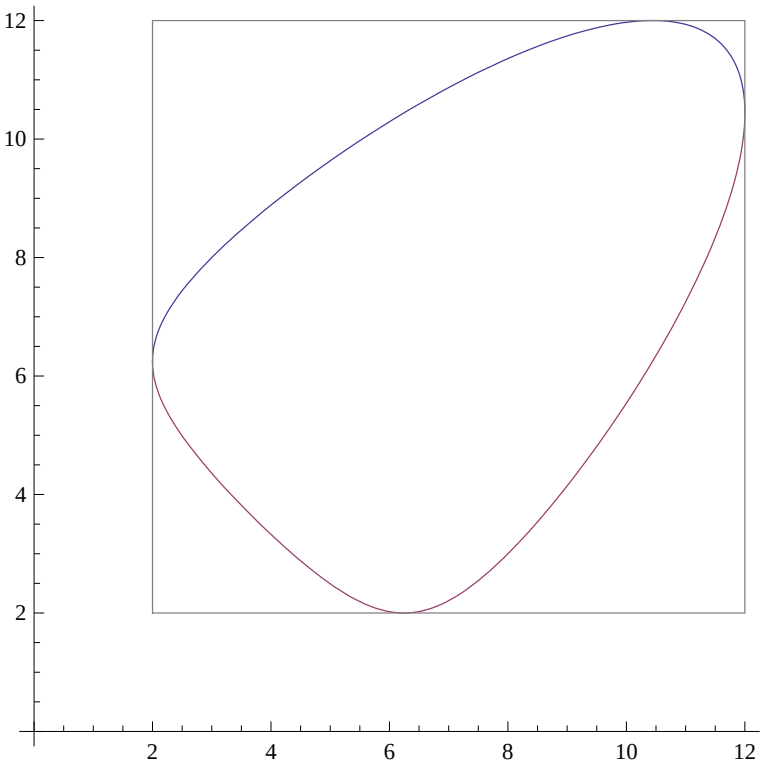
The problem has a straightforward solution. Consider the quadrilateral $PRST$. By Euler's generalization of Ptolemy's theorem concerning the diagonals of a quadrilateral, we have $cd \leq ad + bd$ with equality when the quadrilateral is cyclic. Hence the maximum attained by c is $a + b$.

However, we think it is interesting to obtain a formula for c. Assume $b \geq a \geq 0$ and let $\theta = \angle PRT$. Using the cosine rule on triangle SRP,

$$c^2 \;=\; a^2 + d^2 - 2ad\cos(60^\circ + \theta) \;=\; a^2 + d^2 - ad(\cos\theta - \sqrt{3}\sin\theta).$$

From the cosine rule on triangle PRT, $\cos\theta = (a^2 + d^2 - b^2)/(2ad)$. Hence

$$\begin{aligned} c &= \sqrt{a^2 + d^2 - ad\,\frac{a^2 + d^2 - b^2}{2ad} \pm \sqrt{3}ad\sqrt{1 - \left(\frac{a^2 + d^2 - b^2}{2ad}\right)^2}} \\ &= \sqrt{\frac{a^2 + b^2 + d^2}{2} \pm \frac{\sqrt{3}}{2}\sqrt{4a^2d^2 - (a^2 + d^2 - b^2)^2}} \\ &= \sqrt{\frac{1}{2}\left(a^2 + b^2 + d^2 \pm \sqrt{3((b+a)^2 - d^2)(d^2 - (b-a)^2)}\right)}. \end{aligned} \tag{1}$$



On the previous page we have plotted c against d for $a = 5$, $b = 7$ and both options of the sign of the inner square root. There are interesting features.

The graph is a closed curve that looks as if it fits snugly into a square of side $2a$.

The graph appears to be symmetric about the line $c = d$. Hence there should be a symmetric relation between c and d. To confirm this, square (1) and rearrange to isolate the inner square root,

$$2c^2 - (a^2 + b^2 + d^2) \;=\; \pm\sqrt{3((b+a)^2 - d^2)(d^2 - (b-a)^2)}.$$

Now square both sides to obtain

$$(2c^2 - (a^2 + b^2 + d^2))^2 \;=\; 3((b+a)^2 - d^2)(d^2 - (b-a)^2),$$

which simplifies to

$$a^4 + b^4 + c^4 + d^4 \;=\; a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2. \qquad (2)$$

To get back to the problem, note that the inequality

$$b - a \;\le\; d \;\le\; b + a \qquad (3)$$

can be deduced directly from the original diagram; the limits are reached when P, R and T are collinear. Since (2) is symmetric in c and d, inequality (3) must hold for c in place of d. Thus $b - a \le c \le b + a$.

Pi Day

Jeremy Humphries

As I expect you know, Pi Day is usually celebrated on 14 March. A friend of mine, Stuart Hillman, has told me that if we went for 22/7, instead of 3.14, we could celebrate it on 22 July instead.

I mused, and found that if we went for a Pi Minute, rather than a Pi Day, we could celebrate it on St George's Day (and Shakespeare's birthday). The best rational approximation to π using fewer than five digits in the numerator and denominator is 355/113. The 113th day of the year is 23 April (provided the year is not leap). Mind you, we'd have to get up early, because the 355th minute starts at 05:54.

Solution 294.4 – Mod 9

Two three-digit numbers, a and b, are chosen at random. What's the probability of them being anagrams of each other given that $a \equiv b \pmod 9$?

Ted Gore

Let $a = 100x + 10y + z$ and $b = 100y + 10x + z$ so that a and b are anagrams. Clearly $b \equiv b \pmod 9$, but

$$b \;=\; a - 100x - 10y + 100y + 10x \;=\; a + 9(10y - 10x) \;\Rightarrow\; a \;\equiv\; b \pmod 9.$$

A similar result is obtained whichever two digits we switch and we can compose switches so that all anagrams remain congruent mod 9.

Let m be the number of trials with pairs of random numbers. Let p be the number of congruent pairs, $p = m/9$. Let q be the number of anagrams.

Working with a batch of 1000 pairs we have the following.

Numbers consisting of three different digits. There are $10 \cdot 9 \cdot 8 = 720$ of these and there are $3 \cdot 2 \cdot 1 = 6$ possible anagrams for each number.

Numbers consisting of two of one digit and one of another. There are 270 of these and there are three possible anagrams for each number.

Numbers consisting of three of one digit. There are 10 of these and there is one possible anagram for each number.

So the expected number of anagrams in the 1000 pairs is

$$720 \cdot \frac{6}{1000} + 270 \cdot \frac{3}{1000} + 10 \cdot \frac{1}{1000} \;=\; 5.14.$$

The expected number of anagrams in m trials is $5.14m/1000$.

The probability that a pair are anagrams is $q/m = 0.00514$.

The probability that a pair are anagrams given that they are congruent modulo 9 is

$$\frac{q}{p} \;=\; 0.04626.$$

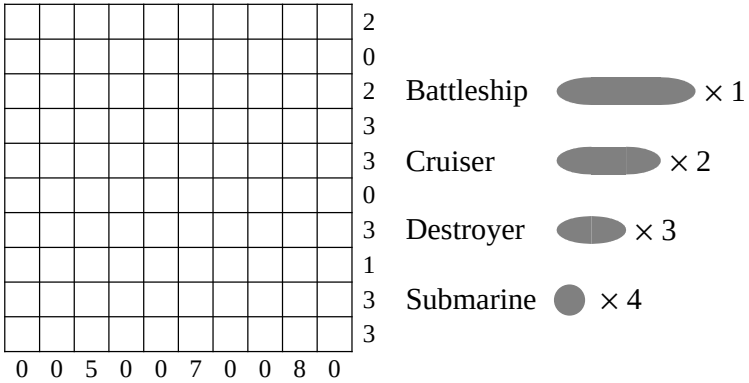
A computer simulation gave the following result:

$$m \;=\; 10^8, \quad p \;=\; 11111417, \quad q \;=\; 514103, \quad \frac{q}{p} \;=\; 0.04627.$$

Problem 300.2 – Warships

Tony Forbes

Here is a puzzle of a type that has been around for decades in one form or another and which you can probably find an instance of on the 'coffee-break' page of your local newspaper. Usually, but not in this particular case, the grid will have clues in the form of specific squares indicating the presence or absence of fragments of ships.



The fleet consists of a battleship (occupying four adjacent squares), two cruisers (three squares), three destroyers (two squares) and four submarines (one square). Surface ships are orientated only north–south or east–west. Two ships cannot overlap or occupy adjacent squares, even diagonally. The numbers on the right or at the bottom of the grid indicate how many parts of ships are present in that row or column respectively. Observe that nine of these numbers are zeros. Your task is to locate the fleet.

Well, that wasn't too difficult, was it? Now for the problem. Devise a similar puzzle (on a blank 10×10 grid, as above, and with the same fleet) but with fewer than nine zeros; the fewer the better.

I have repeatedly and often wondered what sort of detection mechanism is at work. A human observation crew on board an aircraft would surely radio the locations of entire vessels. I can only speculate that what we have here is a drone equipped with fairly crude electronics. It scans the sea north–south and east–west. At the end of each pass it is capable of reporting only the total amount of iron that it has flown over.

Solution 297.2 – Two trigonometric integrals

Richard Gould

Problem

Evaluate

$$\int \frac{\sin^7 x}{\cos^5 x}\,dx \quad \text{and} \quad \int \frac{\sin^5 x}{\cos^7 x}\,dx.$$

The author (TF) is of the opinion that exactly one of these is not easy.

Solution

Using integration by parts we have, for $n > 3$,

$$\begin{aligned} I_n &= \int \cos^{-(n-2)} x \, \sin^n x \, dx \\ &= \frac{1}{n-3} \cos^{-(n-3)} x \, \sin^{n-1} x - \frac{n-1}{n-3} \int \cos^{-(n-4)} x \, \sin^{n-2} x \, dx \\ &= \frac{1}{n-3} \frac{\sin^{n-1} x}{\cos^{n-3} x} - \frac{n-1}{n-3} I_{n-2}. \end{aligned}$$

Also

$$\begin{aligned} I_3 &= \int \frac{\sin^3 x}{\cos x}\,dx = \int \frac{1-\cos^2 x}{\cos x} \sin x \, dx \\ &= -\ln \cos x + \frac{1}{2}\cos^2 x. \end{aligned}$$

So

$$\begin{aligned} I_7 &= \frac{1}{4}\frac{\sin^6 x}{\cos^4 x} - I_5 = \frac{1}{4}\frac{\sin^6 x}{\cos^4 x} - \frac{3}{4}\frac{\sin^4 x}{\cos^2 x} + 3I_3 \\ &= \frac{1}{4}\frac{\sin^6 x}{\cos^4 x} - \frac{3}{4}\frac{\sin^4 x}{\cos^2 x} - 3\ln \cos x + \frac{3}{2}\cos^2 x + c, \end{aligned}$$

where c is an arbitrary constant. Note that, with a change of arbitrary constant, the penultimate term may be written as $-\frac{3}{2}\sin^2 x$. Computer algebra systems may produce this variant.

Now for the easy one:

$$\begin{aligned} \int \frac{\sin^5 x}{\cos^7 x}\,dx &= \int \tan^5 x \, \sec^2 x \, dx \\ &= \frac{1}{6}\tan^6 x + c, \end{aligned}$$

where c is an arbitrary constant.

Bruce Roth

The second integral can be done by inspection as follows:

$$\begin{aligned}\int \frac{\sin^5 x}{\cos^7 x}\,\mathrm{d}x &\equiv \int \frac{\sin^5 x}{\cos^5 x}\,\frac{1}{\cos^2 x}\,\mathrm{d}x \\ &= \int \tan^5 x\,\sec^2 x\,\mathrm{d}x \;=\; \frac{1}{6}\,\tan^6 x + \text{const.}\end{aligned}$$

Generalizing it gives

$$\begin{aligned}\mathrm{I}_n &= \int \frac{\sin^{n-2} x}{\cos^n x}\,\mathrm{d}x \qquad \text{for } n \geq 3 \\ &= \int \frac{\sin^{n-2} x}{\cos^{n-2} x}\,\frac{1}{\cos^2 x}\,\mathrm{d}x \\ &= \int \tan^{n-2} x\,\sec^2 x\,\mathrm{d}x; \\ \mathrm{I}_n &= \frac{1}{n-1}\,\tan^{n-1} x + \text{const.} \qquad (1)\end{aligned}$$

Using the same idea we can generate a recurrence relation for powers of $\tan x$ as follows:

$$\begin{aligned}\mathrm{T}_n &= \int \tan^n x\,\mathrm{d}x \qquad \text{for } n \geq 3 \\ &= \int \frac{\sin^n x}{\cos^n x}\,\mathrm{d}x \\ &= \int \frac{\sin^{n-2} x}{\cos^n x}\,\sin^2 x\,\mathrm{d}x \\ &= \int \left(\frac{\sin^{n-2} x}{\cos^n x}\right)(1-\cos^2 x)\,\mathrm{d}x \\ &= \int \left(\frac{\sin^{n-2} x}{\cos^n x}\right) - \left(\frac{\sin^{n-2} x}{\cos^n x}\right)(\cos^2 x)\,\mathrm{d}x \\ &= \int \left(\frac{\sin^{n-2} x}{\cos^n x}\right)\mathrm{d}x - \int \left(\frac{\sin^{n-2} x}{\cos^n x}\right)(\cos^2 x)\,\mathrm{d}x \\ &= \mathrm{I}_n - \int \left(\frac{\sin^{n-2} x}{\cos^{n-2} x}\right)\mathrm{d}x; \\ \mathrm{T}_n &= \frac{1}{n-1}\,\tan^{n-1} x - \mathrm{T}_{(n-2)}. \qquad (2)\end{aligned}$$

We can now tackle the first integral,

$$\begin{aligned}
\int \frac{\sin^7 x}{\cos^5 x}\,\mathrm{d}x &\equiv \int \frac{\sin^3 x}{\cos^5 x}\,\sin^4 x\,\mathrm{d}x \\
&= \int \frac{\sin^3 x}{\cos^5 x}\,(\sin^2 x)^2\,\mathrm{d}x \\
&= \int \left(\frac{\sin^3 x}{\cos^5 x}\right)\,(1-\cos^2 x)^2\,\mathrm{d}x \\
&= \int \left(\frac{\sin^3 x}{\cos^5 x}\right)\,(1-2\cos^2 x+\cos^4 x)\,\mathrm{d}x \\
&= \int \left(\frac{\sin^3 x}{\cos^5 x}\right) - 2\left(\frac{\sin^3 x}{\cos^5 x}\right)(\cos^2 x) + \left(\frac{\sin^3 x}{\cos^5 x}\right)(\cos^4 x)\,\mathrm{d}x \\
&= \int \frac{\sin^3 x}{\cos^5 x}\,\mathrm{d}x - 2\int \frac{\sin^3 x}{\cos^3 x}\,\mathrm{d}x + \int \frac{\sin^3 x}{\cos x}\,\mathrm{d}x.
\end{aligned}$$

Making use of our two generalizations ((1) and (2)) we can write this as

$$\begin{aligned}
\int \frac{\sin^7 x}{\cos^5 x}\,\mathrm{d}x &\equiv \mathrm{I}_5 - 2\mathrm{T}_3 + \int \frac{\sin^3 x}{\cos x}\,\mathrm{d}x \qquad (3) \\
&= \frac{1}{4}\,\tan^4 x - 2\left(\frac{1}{2}\,\tan^2 x - \mathrm{T}_1\right) + \int \frac{\sin^3 x}{\cos x}\,\mathrm{d}x \\
&= \frac{1}{4}\,\tan^4 x - \tan^2 x + 2\mathrm{T}_1 + \int \frac{\sin^3 x}{\cos x}\,\mathrm{d}x.
\end{aligned}$$

Applying the original idea one more time gives

$$\begin{aligned}
\int \frac{\sin^7 x}{\cos^5 x}\,\mathrm{d}x &\equiv \frac{1}{4}\,\tan^4 x - \tan^2 x + 2\int \tan x\,\mathrm{d}x + \int \left(\frac{\sin x}{\cos x}\right)\,(\sin^2 x)\,\mathrm{d}x \\
&= \frac{1}{4}\,\tan^4 x - \tan^2 x + 2\ln\sec x + \int \left(\frac{\sin x}{\cos x}\right)\,(1-\cos^2 x)\,\mathrm{d}x \\
&= \frac{1}{4}\,\tan^4 x - \tan^2 x + 2\ln\sec x + \int \tan x\,\mathrm{d}x - \int \sin x\,\cos x\,\mathrm{d}x \\
&= \frac{1}{4}\,\tan^4 x - \tan^2 x + 2\ln\sec x + \ln\sec x - \int \sin x\,\cos x\,\mathrm{d}x \\
&= \frac{1}{4}\,\tan^4 x - \tan^2 x + 3\ln\sec x + \frac{1}{2}\cos^2 x + \text{const.}
\end{aligned}$$

Tommy Moorhouse

The second integral is a straightforward one, since

$$\frac{d}{dx}\tan x \;=\; \frac{1}{\cos^2 x}.$$

We see at once that

$$\int \frac{\sin^5 x}{\cos^7 x}dx \;=\; \frac{1}{6}\tan^6 x + C.$$

We can use the same argument to evaluate the first integral. First we use $\sin^2 x = 1 - \cos^2 x$ to rewrite the integral

$$\begin{aligned}
\int \frac{\sin^7 x}{\cos^5 x}dx &= \int \frac{\sin^5 x}{\cos^5 x}dx - \int \frac{\sin^5 x}{\cos^3 x}dx \\
&= \int \left(\frac{\sin^3 x}{\cos^5 x} - 2\frac{\sin^3 x}{\cos^3 x} + \frac{\sin^3 x}{\cos x}\right)dx \\
&= \frac{1}{4}\tan^4 x - 2\int \frac{\sin x}{\cos^3 x}dx + 3\int \frac{\sin x}{\cos x}dx - \int \sin x\,\cos x\,dx \\
&= \frac{1}{4}\tan^4 x - \tan^2 x - 3\log\cos x + \frac{1}{2}\cos^2 x + C.
\end{aligned}$$

In each case C is an arbitrary constant.

Problem 300.3 – Bomb

A bomb dropped from an aircraft explodes above the ground into $n \geq 2$ fragments, B_1, B_2, …, B_n, where B_i has mass m_i, $i = 1, 2, \ldots$. Show that the total kinetic energy of the fragments increases by

$$\frac{1}{2\,(m_1 + m_2 + \cdots + m_n)} \sum_{1 \leq i < j \leq n} m_i\, m_j\, v_{i,j}^2,$$

where $v_{i,j}$ is the speed of B_j relative to B_i immediately after the explosion.

For example, let $n = 2$ and suppose that the bomb has velocity $\mathbf{u}$ at the time of the explosion and that B_i has velocity $\mathbf{u}_i$, $i = 1, 2$, just after. By momentum conservation, $(m_1 + m_2)\mathbf{u} = m_1\mathbf{u}_1 + m_2\mathbf{u}_2$. So the increase in kinetic energy is

$$\frac{m_1|\mathbf{u}_1|^2}{2} + \frac{m_2|\mathbf{u}_2|^2}{2} - \frac{(m_1+m_2)|\mathbf{u}|^2}{2} \;=\; \frac{m_1 m_2|\mathbf{u}_1 - \mathbf{u}_2|^2}{2(m_1+m_2)}.$$

Problem 300.4 – Integral

Tony Forbes

Show that

$$\int_0^{\pi/2} \left((\sin x)^{4/5} + (\cos x)^{4/5}\right)^5 \, dx \;=\; \left(\frac{53\sqrt{5}}{40} - \frac{2}{5}\right)\pi.$$

According to Mathematica, $\int_0^{\pi/2} \left((\sin x)^{(n-1)/n} + (\cos x)^{(n-1)/n}\right)^n dx$ probably has an elementary and not too complicated evaluation for positive integer n only when $n = 1$ (trivial), $n = 3$ (Problem 299.2), $n = 5$ and possibly $n = 7$.

Problem 300.5 – Sum

Tony Forbes

Show that

$$\frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{6\cdot 7\cdot 8\cdot 9} + \frac{1}{11\cdot 12\cdot 13\cdot 14} + \cdots$$
$$= \frac{\sqrt{5}\pi}{150}\left(\sqrt{5+2\sqrt{5}} - 3\sqrt{5-2\sqrt{5}}\right).$$

Compare with Problem 398.4, $1/(1\cdot 2\cdot 3) + 1/(5\cdot 6\cdot 7) + \cdots = (\log 2)/4$.

Problem 300.6 – Symmetrizing a matrix

Tony Forbes

A matrix M has the following properties: M is square, each entry of M is either 0 or 1, and for some integer n, M has exactly n 1s in each row and in each column. Can M be made symmetric by permuting the rows and columns?

For example,

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(swap columns 1, 3; swap columns 2, 4; swap rows 2, 5).

How many people understand what r means?

Colin Aldridge

How many people understand what r means? The best answer will receive an M500 mug and all other answers to this question have a $1/n$ chance of winning an M500 mug, where n is 4.669, for 'obvious reasons'.

1. Susceptibility and Immunity

The r coefficient has suddenly come to the consciousness of the entire population and may have raised the profile of mathematics and statistics in the UK. Will it last and what does this mean for the M500 Society? Probably nothing but ....

The initial reproduction rate is r_0, the number of people an infected person will infect. So an r_0 of 2 means the number of infected people will double in a fully susceptible population.

The rate r_t at a point in time t will depend on the proportion of the population which is susceptible. Herd immunity for diseases with an r above 1 is achieved when the proportion of the population susceptible to the disease is $1/r_0$. The infection rate at time t is

$$N(t_{n+1}) \;=\; r_t N(t_n).$$

Of course, you will need to know what the time interval t_n to t_{n+1} is in order to calculate the rate of increase per day, and knowing this turns out to be problematic.

The initial view for COVID was that this was about five days but because of uncertainty most statisticians and epidemiologists now talk about rate of change per day. As a rule of thumb an r of 0.8 and a time to transmission of five days equates to a halving of cases every fortnight, which is where the UK is as of January and February 2021. For the Kent variant, with an r of around 4 to 5, we need the susceptible population to be around 20 to 25 per cent of the total.

It is interesting to note that the current lockdown is less severe than the March 2020 one, but the rate of decline in cases, deaths, etc. is at least as fast now as it was in spring 2020. So what is the population susceptibility as at 15th February 2021? My guess is around half.

2. How many people understand r?

Some proportion of the country knows the difference between an r of greater or less than 1, but I know of no study which estimates the r for the understanding of r. This would be affected by

a. a definition of what it means to understand r,

b. the starting number of people who understand r,

c. the number of the uneducated who then understand r in a given time interval, and perhaps a modified form of the logistics equation could represent this.

Clearly, we have

$$K_{t+1} = rK_t(1 - M),$$

the logistics equation, where K_t is the percentage of people that understand r at time t and M is the percentage of people who are incapable of understanding it.

This equation makes the assumptions that only people who understand r can explain it to others, which is probably true, and that no one forgets who has at one time understood it. This is undoubtedly false but we can adjust for this by a similar equation

$$F_{t+1} = pK_t,$$

where F is the percentage of people forgetting what r means amongst the knowledgable at time t and p is the chance of a knowledgeable person forgetting. Our equation now becomes

$$K_{t+1} = rK_t(1 - M) - pK_t.$$

This is the 'herd ignorance equation'.

d. This, however, needs to be adjusted since the chances of a person forgetting what r means is related to how long they have remembered it for.

e. It also needs to be adjusted for people who die knowing what r meant.

f. And might it need to be adjusted for other things?

By the way, if you do not understand why your chances of winning a mug are 1/4.669 then this is entertainingly explained on YouTube at `https://www.youtube.com/watch?v=ovJcsL7vyrk`.

04.01

Problem 300.7 – Squares

Find all solutions in positive integers a, b, c, d of

$$a^2 + b^2 + ab = c^2, \qquad a^2 + b^2 - ab = d^2.$$

Front cover The projective plane of order 4. There are 21 points and 21 lines. Two points are joined by a unique line and two lines meet at a unique point. Exercise for reader: construct this thing from a sheet of plywood, 21 nails and a long piece of string.