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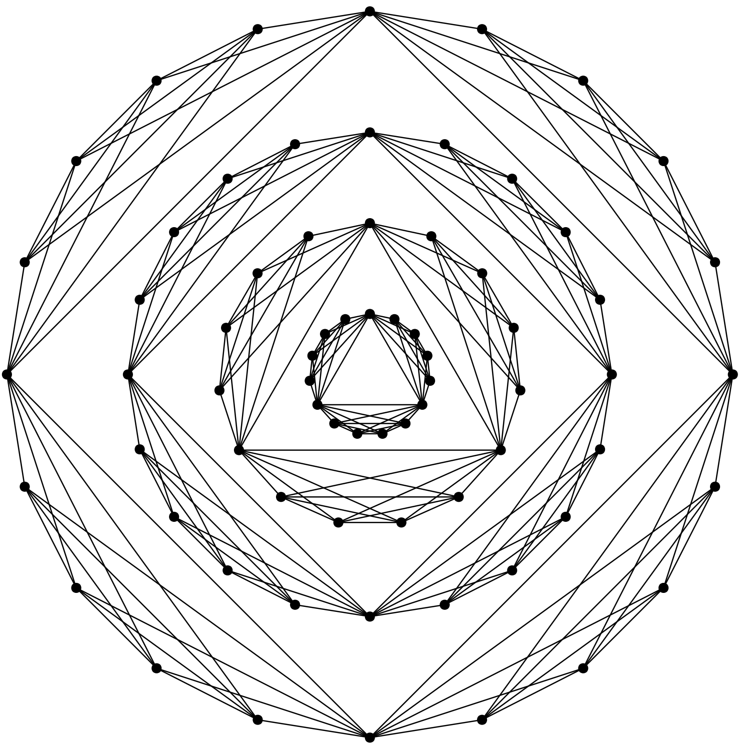

The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The Revision Weekend is a residential Friday to Sunday event providing revision and examination preparation for both undergraduate and postgraduate students. For details, please go to the Society's website.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, please go to the Society's website.

Advice for authors We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to the Editor, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. For more information, go to m500.org.uk/magazine/ from where a LaTeX template may be downloaded.

Colin Davies

We are sorry to have to inform you that M500 Society member Colin Davies died on 15 July 2021. Colin was a valued contributor to M500 for a really long time, to the earliest years of the magazine and of the OU itself. He was a fine supplier and solver of puzzles, and he was also an interesting source of information in areas peripheral to or outside mathematics, particularly the timber trade, electrical matters, and Finland and the Finnish language. We offer our sympathy to his widow, Anne, his family and his friends.

Trapping the primes

Martin Hansen

1. A simple beginning

An infinite arithmetic progression may be prime free,

$$\text{AP}\{4,10\}\colon 4,\ 14,\ 24,\ 34,\ 44,\ 54,\ 64,\ 74,\ 84,\ 94, \ldots,$$

or have a single prime as initial term,

$$\text{AP}\{5,10\} : \mathbf{5},\ 15,\ 25,\ 35,\ 45,\ 55,\ 65,\ 75,\ 85,\ 95, \ldots,$$

or contain many primes,

$$\text{AP}\{9,10\}\colon 9,\ \mathbf{19},\ \mathbf{29},\ 39,\ 49,\ \mathbf{59},\ 69,\ \mathbf{79},\ \mathbf{89},\ 99, \ldots.$$

These examples are amongst those depicted in the ten-wide number grid below in which only the primes within the respective sequences are shown, composite terms being indicated with a dash.

AP{1, 10}	$y = 1 + 10n$	-	11	-	31	41	-	61	71	-	-	101	...
AP{2, 10}	$y = 2 + 10n$	2	-	-	-	-	-	-	-	-	-	-	...
AP{3, 10}	$y = 3 + 10n$	3	13	23	-	43	53	-	73	83	-	103	...
AP{4, 10}	$y = 4 + 10n$	-	-	-	-	-	-	-	-	-	-	-	...
AP{5, 10}	$y = 5 + 10n$	5	-	-	-	-	-	-	-	-	-	-	...
AP{6, 10}	$y = 6 + 10n$	-	-	-	-	-	-	-	-	-	-	-	...
AP{7, 10}	$y = 7 + 10n$	7	17	-	37	47	-	67	-	-	97	107	...
AP{8, 10}	$y = 8 + 10n$	-	-	-	-	-	-	-	-	-	-	-	...
AP{9, 10}	$y = 9 + 10n$	-	19	29	-	-	59	-	79	89	-	109	...

2. No primes at all

The observation that a particular infinite arithmetic progression contains no primes at all is easily proven. For example, to prove that no number in the infinite arithmetic progression AP{4, 10} is prime I could observe that all terms in AP{4, 10} are of the form,

$$y \;=\; 4 + 10n, \qquad \text{for } n = 0,\ 1,\ 2,\ 3,\ \ldots,$$

and factorise to get

$$y \;=\; 2(2 + 5n),$$

which shows that these numbers are always divisible by 2. As the first term, 4, is also not prime, no term in AP{4, 10} is prime.

3. Where lie the primes?

The ten-wide number grid shows that, with the exception of 2 and 5, all primes must be in one of AP$\{1, 10\}$, AP$\{3, 10\}$, AP$\{7, 10\}$ or AP$\{9, 10\}$. Thus when a prime greater than 5 is divided by 10 the remainder must be one of 1, 3, 7 or 9. Furthermore, when a positive integer other than 2 and 5 is divided by 10, if the remainder is 0, 2, 4, 5, 6 or 8, then that number cannot be prime, a fact known to every school pupil in the land.

Interestingly, other than 2 and 5, of the ten available rows the primes are trapped in 4 of them. Although the rows are of infinite extent, in terms of knowing where the primes lie, there is a sense in which they are trapped in 40% of the available space as those rows head off to infinity. This argument can be made more rigorous by taking a limit as columns are added. A natural development is to wonder if we can do better than the 40% of the ten-wide number grid. We can. First, however, a small digression.

4. Can an all prime arithmetic progression exist?

At this point it's worth noticing that none of the sequences above were composed entirely of primes. It's tempting to jump to the conclusion that no sequence of the form $y = a + 10n$ can be composed entirely of primes. The conclusion would be correct, but pinning it down needs care. By way of highlighting the need for caution, consider AP$\{13, 10\}$,

$$\begin{aligned} y &= 13 + 10n \\ &= 3 + 10 + 10n \\ &= 3 + 10(n+1) \qquad \text{for} \quad n = 0,\ 1,\ 2,\ 3,\ \ldots . \end{aligned}$$

Thus AP$\{13, 10\}$ is the sequence AP$\{3, 10\}$ with the first term removed. So, the possibility exists that by throwing away sufficient initial terms a sequence remains composed entirely of primes. The search for an all prime arithmetic progression is degenerating into a chase after the infinite; in any given infinite arithmetic progression that contains primes, every time we encounter a composite number we can throw away the sequence up to that term, and hope that what comes next is an all prime arithmetic progression.

A strategy to kill the idea of the all prime infinite arithmetic progression is to show that all infinite arithmetic progressions contain an infinite sequence of composite numbers.

Consider the generalised infinite arithmetic progression

$$y = a + dn.$$

As n increments through all non-negative integers it will take on the values given by the infinite 'kill' sequence

$$k(m) \;=\; a + m + md$$

where m is a positive integer.

The corresponding terms in the arithmetic progression are of the form

$$\begin{aligned} y &= a + d(a + m + md) \\ &= a + ad + md + md^2 \\ &= (a + md)(1 + d), \end{aligned}$$

which is composite.

5. An improved prime trap

Returning to the main thrust of the article, having dealt exhaustively with the ten-wide prime trap, now consider one that is six-wide. Shown below, it's clear that, with the exception of 2 and 3, all primes lie in AP$\{1, 6\}$ or AP$\{5, 6\}$. Two out of the six rows gives this prime trap a rating $33\,{}^1\!/\!{}_3\,\%$.

AP$\{1,6\}$	$y = 1 + 6n$	-	7	13	19	-	31	37	43	-	-	61	...
AP$\{2,6\}$	$y = 2 + 6n$	2	-	-	-	-	-	-	-	-	-	-	...
AP$\{3,6\}$	$y = 3 + 6n$	3	-	-	-	-	-	-	-	-	-	-	...
AP$\{4,6\}$	$y = 4 + 6n$	-	-	-	-	-	-	-	-	-	-	-	...
AP$\{5,6\}$	$y = 5 + 6n$	5	11	17	23	29	-	41	47	53	59	-	...

6. A trap of prime-width

Constructing a trap that has a width which is a prime number is a disappointing endeavour. By way of illustration, in the seven-wide trap shown, once past the number 7, only the row that contained the 7 is prime free. In general, a p-wide trap will have primes in all rows except AP$\{0, p\}$, once past p.

AP$\{1,7\}$	$y = 1 + 7n$	-	-	-	-	29	-	43	-	-	-	71	...
AP$\{2,7\}$	$y = 2 + 7n$	2	-	-	23	-	37	-	-	-	-	-	...
AP$\{3,7\}$	$y = 3 + 7n$	3	-	17	-	31	-	-	-	59	-	73	...
AP$\{4,7\}$	$y = 4 + 7n$	-	11	-	-	-	-	-	53	-	67	-	...
AP$\{5,7\}$	$y = 5 + 7n$	5	-	19	-	-	-	47	-	61	-	-	...
AP$\{6,7\}$	$y = 6 + 7n$	-	13	-	-	-	41	-	-	-	-	-	...

7. Coprime is the key

In 1837 Peter Dirichlet [1805–1859] proved that for any two positive coprime integers, a and d, there are infinitely many primes of the form $y = a + dn$. An analysis of Dirichlet's intricate proof is one of the highlights of Tom Apostol's classic book *Introduction to Analytic Number Theory* [1]. The result confirms what the ten-, six- and seven-wide traps are suggesting: the rows were the primes are trapped are those coprime to the width of the trap. To be clear, the numbers coprime to ten are 1, 3, 7 and 9 corresponding to the observation that the ten-wide trap placed the primes in $\mathrm{AP}\{1, 10\}$, $\mathrm{AP}\{3, 10\}$, $\mathrm{AP}\{7, 10\}$ and $\mathrm{AP}\{9, 10\}$.

Dirichlet's result also explains why a p-wide number grid was such a poor trap of primes. For any given prime p, all positive integers less than p are coprime to p. So the primes can and do appear in all rows except that described by $\mathrm{AP}\{0, p\}$, which is equivalent to $\mathrm{AP}\{p, p\}$. As no prime is coprime with itself, this is the only prime free row, once past p.

8. Euler's totient function

In the developing quest to find which width of trap is most effective in relatively restricting the primes, it is the number of coprime integers to a given width that is of interest. This is given by Euler's totient function, $\phi(n)$, also often called Euler's phi function. There is a delightfully simple formula for $\phi(n)$, when n is a prime power, p^m,

$$\phi\left(p^m\right) \;=\; p^{m-1}(p-1).$$

(For a proof see, for example, [2].)

So, when $n = 7$,

$$\phi\left(7^1\right) \;=\; 7^{1-1} \times (7-1) \;=\; 6,$$

which matches the observation that in the seven-wide number grid, the primes are to be found in six of the seven rows, bar the exception of the prime 7.

For two coprime integers, p and q, Euler's totient function is multiplicative.

$$\phi(pq) \;=\; \phi(p) \times \phi(q) \qquad \text{for coprime } p \text{ and } q.$$

(For a proof, again see, for example, [2].)

So, when $n = 10$,

$$\phi(10) \;=\; \phi(2)\phi(5) \;=\; 4,$$

which matches the observation that in the ten-wide trap, the primes are to be found in four of the ten rows.

Here is a table of values of $\phi(n)$ for $1 \le n \le 39$.

+	0	1	2	3	4	5	6	7	8	9
0	-	1	1	2	2	4	2	6	4	6
10	4	10	4	12	6	8	8	16	6	18
20	8	12	10	22	8	20	12	18	12	28
30	8	30	16	20	16	24	12	36	18	24

A much bigger table for $0 < n < 2460$ is available at [3]. A good trap will have a value of $\phi(n)$ in the table that's low in relation to those around it. The entry $\phi(6) = 2$ was good because it seems to be the last occurrence of a value of 2 in the table; a comment based on the observation that the values in the table, on average, seem to be increasing. Assuming no more occurrences of 4 or 6 are to be found in an extended table, we have the following 'best so far' results;

$$\frac{\phi(6)}{6} = \frac{\phi(12)}{12} = \frac{\phi(18)}{18} = \frac{1}{3} \qquad \left(33\,^1\!/_3\,\%\right).$$

Motivation to keep going is provided by the entry for $\phi(30)$,

$$\frac{\phi(30)}{30} = \frac{4}{15} \qquad \left(26\,^2\!/_3\,\%\right).$$

In a thirty-wide table all the primes greater than 7 will be trapped into eight of the thirty available rows. It's the best result yet but comes with a caveat; how can we be sure that there's not another n for which $\phi(n)$ takes the value of 8 further on, beyond where our table (however big) stopped?

9. Establishing boundaries

Reassurance is sought that the equation

$$n = \phi^{-1}(s) \qquad \text{when } s = 8$$

has the finite set of solutions that are visible in our table of values of $\phi(n)$. That is, n is precisely the set of solutions

$$n = \{15, 16, 20, 24, 30\}.$$

A lovely result, originally due to Gupta [4], more accessible in Coleman [5], places bounds on the value of n for a given s that's known to be in the table of values for $\phi(n)$.

Theorem 1: Given that s is an even number, and p is prime, define $A(s)$ to be

$$A(s) = s \prod_{p-1|s} \frac{p}{p-1}.$$

If $n \in \phi^{-1}(s)$, then $s < n \leq A(s)$.

Proof: If $\phi(n) = s$ then, from the definition of $\phi(n)$ being the number of positive integers less than n that are coprime to n, it's clear that $s < n$. On the other hand, if n is decomposed into its product of primes,

$$n = p_0^{k_0} p_1^{k_1} \ldots p_r^{k_r}$$

for the necessary primes, p_0, p_1, …, p_r and positive integers, k_0, k_1, …, k_r, then, because Euler's totient function is multiplicative,

$$\begin{aligned}
s &= \phi(n) \\
&= \phi\left(p_0^{k_0} p_1^{k_1} \ldots p_r^{k_r}\right) \\
&= \phi\left(p_0^{k_0}\right) \phi\left(p_1^{k_1}\right) \ldots \phi\left(p_r^{k_r}\right) \\
&= p_0^{k_0-1}(p_0-1) p_1^{k_1-1}(p_1-1) \ldots p_r^{k_r-1}(p_r-1) \\
&= p_0^{k_0}\left(\frac{p_0-1}{p_0}\right) p_1^{k_1}\left(\frac{p_1-1}{p_1}\right) \ldots p_r^{k_r}\left(\frac{p_r-1}{p_r}\right) \\
&= n \prod_{i=0}^{r} \frac{p_i-1}{p_i} \\
\Rightarrow n &= s \prod_{i=0}^{r} \frac{p_i}{p_i-1}.
\end{aligned}$$

A consequence of the formula $\phi(p^m) = p^{m-1}(p-1)$ is that if p divides n then $p-1$ must divide $\phi(n)$ and it follows that, for each p, $p_i - 1$ divides s. Hence we have $n \leq A(s)$. □

From Theorem 1, bounds of the solutions to the equation

$$n = \phi^{-1}(8)$$

are found by adding one to each of 1, 2, 4 and 8, the divisors of 8, to obtain the prime numbers 2, 3 and 5, and the composite 9 which is thrown away. Thus

$$A(8) = 8 \times \frac{2}{1} \times \frac{3}{2} \times \frac{5}{4} = 30,$$

therefore

$$8 \; < \; \phi^{-1}(8) \; \leq \; 30,$$

and the desired reassurance has been obtained.

Theorem 1 has 'holes in it' in the sense that many integers, s, are not in $\phi^{-1}(n)$. For example, a consequence of the formula $\phi\left(p^m\right) = p^{m-1}(p-1)$ is that if either p is odd or $p = 2$ and $m > 1$ then $p^{m-1}(p-1)$ is even. Hence, for $n \geq 3$, $\phi(n)$ cannot be odd. Some even numbers also are missing, the first example being 14 for which I'll provide a proof in Proposition 1. Other low value examples are 26, 34, 38, 50 and 62, [6].

Proposition 1: $\phi(n) = 14$ has no solutions.

Proof: If a prime p is a divisor of n then $p - 1$ is necessarily a divisor of $\phi(n)$. The primes p for which $p - 1$ is a divisor of 14 are 2 and 3.

Case 1: Suppose that $n = 2^r$; then

$$\phi\left(2^r\right) \; = \; 2^{r-1} \neq 14.$$

Case 2: Suppose that $n = 3^s$; then

$$\phi\left(3^s\right) \; = \; 3^{s-1} \times 2 \neq 14.$$

Case 3: Suppose that $n = 2^r 3^s$; then

$$\phi\left(2^r 3^s\right) \; = \; \phi\left(2^r\right) \times \phi\left(3^s\right) \; = \; 2^{r-1} \times 3^{s-1} \times 2 \; = \; 2^r 3^{s-1} \neq 14.$$

Thus $\phi(n) = 14$ has no solutions. There is no width of number grid that can trap the primes in 14 of the available rows. $\square$

10. How good a trap can be built?

Some stabs in the dark reveal that better traps than those for $n = 6$ and $n = 30$ exist. For example, with $n = 420$,

$$\frac{\phi(420)}{420} \times 100 \; = \; 22^{6}/_{7}\,\%.$$

The curious will wish to know how low this percentage can go. In a natural manner the requirement has arisen to understand the function

$$f(n) \; = \; \frac{\phi(n)}{n}.$$

The percentage rating of a trap is then given by simply multiplying $f(n)$ by 100. Partial illumination is provided by Theorem 2. I'll prove this carefully and in detail because it's a thought-provoking result, and interested readers may wish to revisit the steps to explore easily obtained improvements either in general, or in specific cases, such as when n is divisible by 4 or when n has some other property yet to be identified as of importance.

Theorem 2:

$$\frac{1}{2\sqrt{n}} \;\le\; \frac{\phi(n)}{n} \;\le\; 1 \qquad \text{for } n \ge 1.$$

Proof: Without loss of generality, by the fundamental theorem of arithmetic, let the positive integer n be written as a product of primes of the form

$$n \;=\; 2^{k_0} p_1^{k_1} \ldots p_r^{k_r}$$

for necessary odd primes, $p_1, \ldots, p_r$ and positive integers, $k_1, \ldots, k_r$; k_0 is a non-negative integer. That is, k_0 can be zero.

By the multiplicative property of Euler's totient function,

$$\begin{aligned}
\frac{\phi(n)}{n} &= \frac{\phi\left(2^{k_0}\right)\phi\left(p_1^{k_1}\right)\phi\left(p_2^{k_2}\right)\ldots\phi\left(p_r^{k_r}\right)}{2^{k_0}p_1^{k_1}p_2^{k_2}\ldots p_r^{k_r}} \\
&= \left(\frac{2^{k_0-1}}{2^{k_0}}\right)\left(\frac{p_1^{k_1-1}(p_1-1)}{p_1^{k_1}}\right)\ldots\left(\frac{p_r^{k_r-1}(p_r-1)}{p_r^{k_r}}\right). \qquad (1)
\end{aligned}$$

For an upper bound we can further write (1) as

$$\begin{aligned}
\frac{\phi(n)}{n} &= \left(\frac{2-1}{2}\right)\left(\frac{p_1-1}{p_1}\right)\ldots\left(\frac{p_r-1}{p_r}\right) \\
&= \left(1-\frac{1}{2}\right)\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_r}\right)
\end{aligned}$$

and use the inequality that for any prime p,

$$1-\frac{1}{p} \;\le\; 1$$

to obtain the stated upper bound, also noticing that when n is even this can effortlessly be improved to give

$$\frac{\phi(n)}{n} \;\le\; \frac{1}{2} \qquad \text{for even } n.$$

For a lower bound we can use in (1) the fact that for any odd prime, p, $p-1 > \sqrt{p}$. Thus (1) becomes

$$\begin{aligned}\frac{\phi(n)}{n} &\geq \left(\frac{1}{2}\right)\left(\frac{p_1^{k_1-1}\sqrt{p_1}}{p_1^{k_1}}\right)\cdots\left(\frac{p_r^{k_r-1}\sqrt{p_r}}{p_r^{k_r}}\right)\\ &\geq \left(\frac{1}{2}\right)\left(\frac{p_1^{k_1-0.5}}{p_1^{k_1}}\right)\cdots\left(\frac{p_r^{k_r-0.5}}{p_r^{k_r}}\right).\end{aligned}$$

Now for $k \geq 1$, $k - 0.5 \geq 0.5k$,

$$\begin{aligned}\frac{\phi(n)}{n} &\geq \left(\frac{1}{2}\right)\left(\frac{p_1^{0.5k_1}}{p_1^{k_1}}\right)\cdots\left(\frac{p_r^{0.5k_r}}{p_r^{k_r}}\right)\\ &\geq \left(\frac{1}{2}\right)\left(\frac{2^{0.5k_0}}{2^{0.5k_0}}\right)\left(\frac{1}{p_1^{0.5k_1}}\right)\cdots\left(\frac{1}{p_r^{0.5k_r}}\right)\\ &\geq \left(\frac{1}{2}\right)\left(\frac{2^{0.5k_0}}{\sqrt{n}}\right).\end{aligned}$$

Finally, observing that for any non-negative integer value of k_0, $2^{0.5k_0} \geq 1$ yields the claimed lower bound. □

The upper bound of Theorem 2 is not a surprise because of the fact established by Euclid that there are an infinite number of primes. Each time a prime is encountered $f(n)$ is almost on the upper bound. In other words,

$$f(p) \;=\; \frac{\phi(p)}{p} \;=\; \frac{p-1}{p} \;=\; 1 - \frac{1}{p} \;\to\; 1 \quad \text{(i.e. 100\%) as } p \to \infty.$$

The lower bound of Theorem 2 is tantalising for it leaves open, but unresolved, the possibility that as n becomes larger there are widths of number grid to be found in which the primes become trapped in an ever decreasing percentage of the rows of the n-wide table. Of course, just because Theorem 2 established a lower bound, there's no guarantee that values of $\phi(n)$ come close to it. This worry is exacerbated by the fact the lower bound values seem well below the 'best in their neighbourhood' results previously obtained. That is,

$$\text{for } n = 6, \quad \frac{\phi(6)}{6} = 0.333, \qquad \text{but} \quad \frac{1}{2\sqrt{6}} = 0.204;$$

$$\text{for } n = 30, \quad \frac{\phi(30)}{30} = 0.267, \qquad \text{but} \quad \frac{1}{2\sqrt{30}} = 0.091.$$

As is so often experienced by those working with prime numbers, seeming progress can turn out to be an illusion.

11. Primorials

The search for a good prime trap alighted upon the fact that the number grids with widths associated with $\phi(6)$ and $\phi(30)$ were more restrictive than neighbouring values. Since 1987, when Harvey Dubner [7] invented the expression, 6 and 30 have become known as early consecutive terms of the sequence of 'primorials'. For the w^{th} prime number, $p_w\#$ is defined as the product of the first w primes.

$p_w\#$	product	value
$p_1\#$	2	2
$p_2\#$	2×3	6
$p_3\#$	$2 \times 3 \times 5$	30
$p_4\#$	$2 \times 3 \times 5 \times 7$	210
$p_5\#$	$2 \times 3 \times 5 \times 7 \times 11$	2310

What is sought is a sequence within $f(n)$ that ignores most of the terms in $f(n)$ that are displaying so much variation, and which instead steadily decreases in value towards 0 as Theorem 2 suggested is possible. Primorials are the key as Theorem 3 and its proof will show.

Theorem 3: For each primorial $p_w\#$,

$$f(p_w\#) \;=\; \frac{\phi\,(p_w\#)}{p_w\#}$$

is smaller than itself for any lesser primorial.

Proof: Let an arbitrary primorial $p_w\#$ with $w \geq 2$ be decomposed into its product of primes, for any $w \geq 2$, $p_w\# = p_0 p_1 \ldots p_w$. Then

$$\begin{aligned}
\frac{\phi\,(p_w\#)}{p_w\#} &= \frac{\phi\,(p_0 p_1 \ldots p_w)}{p_0 p_1 \ldots p_w} \\
&= \frac{\phi\,(p_0 p_1 \ldots p_{w-1})}{p_0 p_1 \ldots p_{w-1}} \times \frac{\phi\,(p_w)}{p_w} \\
&= \frac{\phi\,(p_{w-1}\#)}{p_{w-1}} \times \frac{(p_w - 1)}{p_w} \;=\; \frac{\phi\,(p_{w-1}\#)}{p_{w-1}} \times \left(1 - \frac{1}{p_w}\right);
\end{aligned}$$

therefore

$$\frac{\phi\,(p_w\#)}{p_w\#} < \frac{\phi\,(p_{w-1}\#)}{p_{w-1}} \qquad \text{for any } w \geq 2. \qquad \square$$

12. Conclusion

A simple idea has been pursued a long way with a pleasingly minimalist set of tools. Success has been achieved in trapping the primes into as small a proportion of the positive integers as desired. However, some quick numerical calculations show that the widths of the number grids required rapidly become vast. For example,

$$\frac{\phi(p_{10}\#)}{p_{10}\#} = \frac{1\,021\,870\,080}{6\,469\,693\,230} = 0.158.$$

This primorial-width number grid has a lot of rows in which the primes reside, even if the percentage rating of the trap is down to 15.8%.

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[2] H. Davenport, *The Higher Arithmetic*, Cambridge Univ. Press, 1999.

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[4] H. Gupta, Euler's Totient Function and its Inverse, *Indian Journal of Pure and Applied Mathematics* **12** (January 1981).

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[6] N. J. A. Sloane, *The On-Line Encyclopaedia of Integer Sequences*, A005277, `http://www.oeis.org/A005277`.

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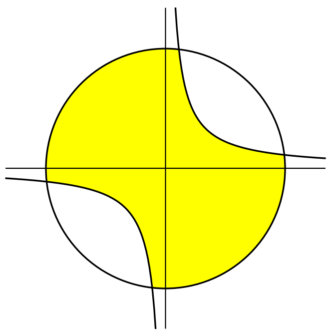
Problem 302.1 – A circle and a hyperbola

The circle has radius r and the hyperbola is $y = 1/x$. What's the area of the yellow (light grey) part?



I (TF) cannot recall ever seeing this in any textbook or other publication. Yet surely it must be amongst the most natural of the various shapes that you would want to consider as soon as you have learned how to calculate areas by integrating functions. If $r \leq \sqrt{2}$, the hyperbola can be ignored.

The necklace counting formula: addendum

Robin Whitty

In M500 Issue **285** I wrote down a formula for the number of necklaces with b_i beads of colour i, $i = 1, \ldots, t$. The formula was given in terms of the partially ordered set (poset) of divisors $d_1, \ldots, d_m$ of the greatest common divisor of the b_i, and of the Möbius function $\mu(x, y)$ for this poset:

$$N(b_1, b_2, \ldots, b_t) \;=\; \frac{1}{n}\sum_{i=1}^{m}\sum_{j=i}^{m} d_i \binom{n/d_j}{b_1/d_j, \ldots, b_t/d_j} \mu(d_i, d_j), \qquad (1)$$

where $n = b_1 + \ldots + b_t$. The formula works if some of the b_i are zero, using the fact that $\gcd(x, 0) = x$. The multinomial coefficient $\binom{x}{y_1, \ldots, y_t}$ is evaluated as $x!/(y_1! \cdots y_t!)$ and counts the number of non-circular permutations of x objects of which y_1 are colour 1, y_2 are colour 2, etc. The usual binomial coefficient $\binom{x}{y}$ is a short way of writing $\binom{x}{y, x-y}$. Recall that if we sum these binomial coefficients over all choices of y the result is 2^x and this generalises to monomial coefficients:

$$\sum_{y_1+\ldots+y_t=x} \binom{x}{y_1, \ldots, y_t} \;=\; t^x. \qquad (2)$$

By 'necklace' we mean a circular permutation of objects belonging to distinguished classes (colours) taking into consideration rotational symmetry. Formula (1) was derived from a 1956 *Mathematical Gazette* paper in which the number of such permutations was found as the solution to a set of simultaneous equations. I found the double sum above to be preferable to saying "the solution is now found by inverting the equation matrix". However, I didn't go far enough: a single summation was just around the corner!

I also suggested that it was textbook stuff to specify a formula for the number of necklaces with the number of colours, t, specified but with no restriction on their distribution. This is true; the formula is the following:

$$N(n, t) \;=\; \frac{1}{n}\sum_{d \le n} \varphi(d) t^{n/d}, \qquad (3)$$

using poset notation $d \le n$ to mean d divides into n. The function φ is Euler's totient function whose value at a positive integer x is the number of positive integers less than x and coprime to x. For example,

$$\begin{aligned} N(6,3) &= \frac{1}{6}\left(\varphi(1)\times 3^6 + \varphi(2)\times 3^3 + \varphi(3)\times 3^2 + \varphi(6)\times 3^1\right) \\ &= \frac{1}{6}(1\times 729 + 1\times 27 + 2\times 9 + 2\times 3) \;=\; 130. \end{aligned}$$

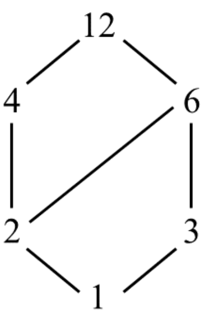
There are many ways to determine the values of $\varphi(x)$. We are going to find convenient the following recursive definition, due to Gauss:

$$\begin{aligned} \varphi(1) &= 1, \\ \varphi(y) &= y - \sum_{x<y}\varphi(x),\;\; y>1, \end{aligned} \tag{4}$$

where we are still using poset notation and '$<$' means 'strictly divides'.

Formula (3) is sometimes misattributed to Captain Percy Alexander MacMahon who indeed wrote about it in 1892 but acknowledged its prior discovery by another soldier, Monsieur le Colonel Charles Paul Narcisse Moreau. Moreau had solved our counting problem in 1872 and was, as far as I know, the first to do so. Formula (3) is a special case of his solution and I would now like to explain how we get to it from formula (1).

Let me recall what, in my original contribution, I said about the Möbius function. We are concerned with the version for the poset of divisors, as illustrated below left. Somewhat informally the value of the Möbius function $\mu(x,y)$ for two elements x and y of this poset is $\sum(-1)^{l(c)}$ where the sum is over all upward 'chains' c from x to y and $l(c)$ is the number of edges in the chain. The diagram of the poset only shows 'immediate' division but our summation must also include all implied edges, such as the edge from 3 to 12.



	1	2	3	4	6	12
1	1	−1	−1	0	1	0
2		1	0	−1	−1	1
3			1	0	−1	0
4				1	0	−1
6					1	−1
12						1

The table shows the values of $\mu(x,y)$ for the poset on the left. For example, $\mu(3,12)=0$ because there are two chains from 3 to 12: the two-edge chain $3-6-12$ which contributes $(-1)^2$ to the calculation and the one-edge chain $3-12$, not included in the diagram, which contributes $(-1)^1$.

I will rewrite formula (1) in a more poset-friendly form:

$$N(b_1, b_2, \ldots, b_t) \;=\; \frac{1}{n} \sum_{e \le M} \left\{ \binom{n/e}{b_1/e, \ldots, b_t/e} \sum_{d \le e} d\,\mu(d, e) \right\}, \qquad (5)$$

where $M = \gcd(b_1, \ldots, b_t)$ and I no longer need to list the divisors of M explicitly because the poset notation $e \le M$ takes care of that. The order of summation has changed from (1) but that is just a matter of counting by column instead of by row. The important thing is to isolate the sum $\sum_{d \le e} d\,\mu(d, e)$, because such a sum is amenable to Möbius inversion, a contribution to number theory by August Ferdinand Möbius in the 1830s, transferred to posets by group theorists in the 1930s as follows:

$$g(y) \;=\; \sum_{x \le y} f(x), \text{ for all } y, \text{ if and only if } \; f(y) \;=\; \sum_{x \le y} g(x)\mu(x, y), \text{ for all } y.$$

Define $f(e) = \sum_{d \le e} d\,\mu(d, e)$, for all $e \le M$. Then $e = \sum_{d \le e} f(d)$, or $f(e) = e - \sum_{d < e} f(e)$. And now from Gauss's formula (4):

$$N(b_1, b_2, \ldots, b_t) \;=\; \frac{1}{n} \sum_{e \le M} \binom{n/e}{b_1/e, \ldots, b_t/e} \varphi(e). \qquad (6)$$

This is what Colonel Moreau wrote down almost 150 years ago, and what I should have written down, two and half years ago, for M500. Suppose we now sum over all possible distributions of the n beads among the t colours. We get

$$N(n, t) \;=\; \frac{1}{n} \sum_{b_1 + \ldots + b_t = n} \left(\sum_{e \le \gcd(b_1, \ldots, b_m)} \binom{n/e}{b_1/e, \ldots, b_t/e} \varphi(e) \right).$$

This double sum groups the monomial terms according partitions of n; we will get the same result if we group according to divisors of n:

$$\begin{aligned} N(n, t) \;&=\; \frac{1}{n} \sum_{e \le n} \left(\sum_{(b'_1 + \ldots + b'_t) e = n} \binom{n/e}{b'_1, \ldots, b'_t} \varphi(e) \right) \\ &=\; \frac{1}{n} \sum_{e \le n} \left(\varphi(e) \sum_{b'_1 + \ldots + b'_t = n/e} \binom{n/e}{b'_1, \ldots, b'_t} \right), \end{aligned}$$

and we recover Moreau's formula (3) by applying identity (2).

Solution 295.7 – Divisibility

> For positive integer n, show that $n^n + (n+1)^{n-1} - 1$ is divisible by $n(n+1)$ if and only if n is even.

Peter Fletcher

Let

$$n^n + (n+1)^{n-1} - 1 \;=\; f(n).$$

Then $f(1) = 1^1 + 2^0 - 1 = 1$, which is not divisible by $1(1+1) = 2$.

For $n > 1$, if n is even,

$$n^n - 1 = (n+1)(n^{n-1} - n^{n-2} + \cdots + n - 1)$$

and if n is odd,

$$n^n - 1 = (n-1)(n^{n-1} + n^{n-2} + \cdots + n + 1).$$

These may be confirmed by trying a few even and odd integers.

We also have, for $n > 1$,

$$(n+1)^{n-1} = (n+1)\biggl(n^{n-2} + (n-2)n^{n-3} + \cdots + \frac{(n-2)(n-3)}{2}n^2 + (n-2)n + 1\biggr).$$

Clearly $f(n)$ is divisible by $(n+1)$ only if n is even, in which case

$$f(n) = (n+1)\biggl((n^{n-1} - n^{n-2} + \cdots + n - 1) + \Bigl(n^{n-2} + (n-2)n^{n-3} + \cdots + \frac{(n-2)(n-3)}{2}n^2 + (n-2)n + 1\Bigr)\biggr).$$

The -1 and $+1$ cancel to leave sums of powers of n greater than 0, which are hence divisible by n.

Therefore $f(n)$ is divisible by $n(n+1)$ only if n is even.

Problem 302.2 – Divisibility

Tony Forbes

Given integers a and b, for which positive integers n is $(n+a)(n+b)$ divisible by n^2?

Solution 296.3 – Elliptic curve

Let a be a positive real number. Then the elliptic curve $y^2 = x(x^2 - a^2)$ has two components, an unbounded curve that passes through $(a, 0)$ and a closed 'bubble' that passes through $(0, 0)$ and $(-a, 0)$. What area does the bubble enclose?

Ted Gore

We have

$$y^2 = x(x^2 - a^2) = a^2 z(1+b),$$

where $z = -x$ and $b = -z^2/a^2$, so that

$$\begin{aligned} y &= az^{1/2}(1+b)^{1/2} \\ &= az^{1/2}\left(1 + \frac{b}{2} + \frac{1/2(1/2-1)b^2}{2!} + \frac{1/2(1/2-1)(1/2-2)b^3}{3!} + \dots\right). \end{aligned}$$

Substituting back for b we get

$$\begin{aligned} \int_{-a}^{0} y dx &= \int_{0}^{a} y dz \\ &= a\left[\frac{z^{3/2}}{3/2} - \frac{1/2z^{7/2}}{7/2a^2} + \frac{1/2(1/2-1)z^{11/2}}{11/2a^4 2!}\right. \\ &\qquad \left. - \frac{1/2(1/2-1)(1/2-2)z^{15/2}}{15/2a^6 3!} + \dots\right]_0^a \\ &= a^{5/2}\left(\frac{1}{3/2} - \frac{1/2}{7/2} + \frac{1/2(1/2-1)}{11/2 \cdot 2!} - \frac{1/2(1/2-1)(1/2-2)}{15/2 \cdot 3!} + \dots\right) \\ &= a^{5/2}\, 0.4792560938942 \end{aligned}$$

(calculating 1,000,000 terms using double precision arithmetic). This gives the area of the bubble above the x-axis. It needs to be doubled for the total area.

a	1	2	3	4	5
area	0.959	5.422	14.942	30.672	53.582

If you want the exact answer, the constant is $\dfrac{\sqrt{\pi}}{5}\,\dfrac{\Gamma(3/4)}{\Gamma(5/4)}$, but I do not know where it comes from. — TF

Solution 296.4 – Cubic curve

Let a be a positive real number. Then the cubic curve $y^2 = x(x-a)^2/(3a)$ has a loop that passes through $(0,0)$ and $(a,0)$. What is its length and what area does it enclose?

Curiously, the denominator $3a$ in the definition of the curve has some significance. Remove it, and the loop length of the curve $y^2 = x(x-a)^2$ becomes much more difficult to compute. You are welcome to try!

Ted Gore

We have $y^2 = x(x-a)^2/(3a)$. So that

$$y = \frac{x^{3/2} - ax^{1/2}}{\sqrt{3a}}$$

and

$$\int_0^a y dx = \frac{2}{\sqrt{3a}} \left[\frac{3x^{5/2} - 5ax^{3/2}}{15} \right]_0^a = \frac{-4a^2}{15\sqrt{3}}.$$

This represents the area below the x-axis between 0 and a. We double the absolute value for the enclosed area.

The length of the curve is

$$s = \int_0^a \sqrt{1 + (dy/dx)^2} dx.$$

But

$$\frac{dy}{dx} = \frac{3x^{1/2} - ax^{-1/2}}{\sqrt{12a}}, \qquad \left(\frac{dy}{dx}\right)^2 = \frac{9x + a^2x^{-1} - 6a}{12a}.$$

The $3a$ in the denominator makes possible the next step.

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{9x + a^2x^{-1} + 6a}{12a}.$$

So that

$$s = \int_0^a \frac{3x^{1/2} + ax^{-1/2}}{\sqrt{12a}} dx = \left[2x^{3/2} + 2ax^{1/2}\right]_0^a = \frac{2a}{\sqrt{3}}.$$

This is the curve length above the x-axis between 0 and a. We double it for the total length of the loop.

a	1	2	3	4	5
area	0.308	1.232	2.771	4.927	7.698
length	1.155	2.309	3.464	4.619	5.774

Solution 292.1 – Angle trisection

Let

$$D = 10\cos(\theta/2)(6-\cos\theta)\sqrt{29-4\cos\theta}$$

and suppose $-120^\circ \le \theta \le 120^\circ$. Show that

$$\tan\frac{\theta}{3} \approx \frac{\sin\theta}{6-\cos\theta}\cdot\frac{1608-676\cos\theta+68\cos^2\theta-D}{168+100\cos\theta-68\cos^2\theta+D}.$$

Peter Fletcher

Let the RHS of the expression in the question be z.

We begin by noting that

$$1608-676\cos(\theta)+68\cos^2(\theta) = \Bigl(6-\cos(\theta)\Bigr)\Bigl(268-68\cos(\theta)\Bigr).$$

The factor $\Bigl(6-\cos(\theta)\Bigr)$ cancels, so

$$z = \frac{\sin(\theta)\Bigl(268-68\cos(\theta)-10\cos\left(\theta/2\right)\sqrt{29-4\cos(\theta)}\Bigr)}{168+100\cos(\theta)-68\cos^2(\theta)+10\cos\left(\theta/2\right)\Bigl(6-\cos(\theta)\Bigr)\sqrt{29-4\cos(\theta)}}$$

Now we let $\theta = 3\alpha$, expand everything, factorise and divide the numerator by $\sin(\alpha)$ and the denominator by $\cos(\alpha)$. This allows us to write

$$z = \frac{\sin(\alpha)}{\cos(\alpha)}\cdot\frac{n}{d} = \tan\left(\frac{\theta}{3}\right)\frac{n}{d}$$

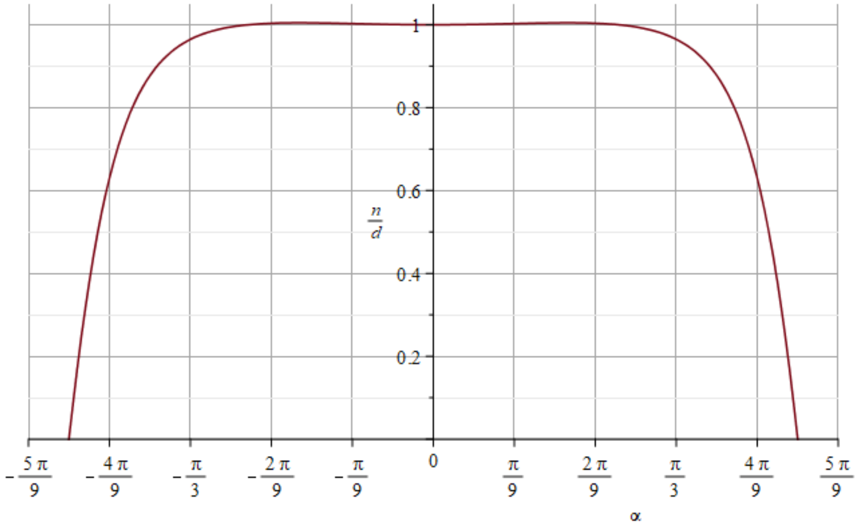
where

$$\begin{aligned} n = {} & 2\Bigl(4\cos^2(\alpha)-1\Bigr)\Bigl(136\cos^3(\alpha)-102\cos(\alpha)-134 \\ & + 5\cos\left(\frac{3\alpha}{2}\right)\sqrt{29+12\cos(\alpha)-16\cos^3(\alpha)}\Bigr)\cos(\alpha) \end{aligned}$$

and

$$\begin{aligned} d = {} & -4\Bigl(\cos(\alpha)+1\Bigr)\Bigl(68\cos^3(\alpha)-51\cos(\alpha)-42\Bigr)\Bigl(2\cos(\alpha)-1\Bigr)^2 \\ & -10\Bigl(4\cos^3(\alpha)-3\cos(\alpha)-6\Bigr)\cos\left(\frac{3\alpha}{2}\right)\sqrt{29+12\cos(\alpha)-16\cos^3(\alpha)}. \end{aligned}$$

If we now plot n/d against α, we can see that n/d is very close to 1 for $-2\pi/9 \le \alpha \le 2\pi/9$, which is the same as $-2\pi/3 \le \theta \le 2\pi/3$ or $-120^\circ \le \theta \le 120^\circ$. Therefore for θ in this range, $\tan(\theta/3)$ is very close to z, the expression in the question.



Plot of n/d against α.

Problem 302.3 – Cycles

Let p be a permutation of the elements of a set S. Let t be a transposition of two elements of S. Show that when pt and p are each written as products of mutually disjoint cycles, the number of cycles in pt is either one more or one less than the number of cycles in p.

Problem 302.4 – Ball-point energy

Tony Forbes

What is the significance of the energy given by the formula $mg(h_2 - h_1)$?

Here, g is the acceleration due to gravity and the other parameters arise as follows. Consider a bog-standard retractable ball-point pen of the type where you change its state by pressing a spring-loaded cylindrical button set into the upper end of the body. The process is usually accompanied by two plainly audible clicks. The pen's mass is m. The parameters h_1 and h_2 are the minimum heights from which the pen must be dropped on to a hard surface, upper end downwards, for its state to change, either from retracted to open (i.e. ready to write with), h_2, or from open to retracted, h_1. Simple experiments should confirm that $h_2 > h_1$. We would be interested in actual values if you are willing to experiment.

Solution 266.2 – Snooker without friction

Is it sensible to play snooker on a frictionless table? If it is, devise a strategy for winning a frame in a finite amount of time.

Kim Forbes and Tony Forbes

Here we describe a possible strategy, which should almost certainly work. Remarkably, by some kind of bizarre coincidence, the game is actually unbiased. First, we remind the reader of some reasonable interpretations of the rules of snooker.

(i) The cue ball must appear to move after being struck by the cue.

(ii) Reds that end up in pockets legitimately or by foul play are not returned to the table.

(iii) A shot is completed when every ball on the table has come to rest.

(iv) A ball on the table has come to rest if, relative to the table, its speed is less than $10^{-10}\,\mathrm{m\,s^{-1}}$ and its spin angular velocity has absolute value less than $10^{-10}\,\mathrm{s^{-1}}$.

We believe (iv) applies in real matches. It also permits frictionless games to be played on tables that are not truly horizontal.

As in the official Rules, [`https://wpbsa.com/wp-content/uploads/WPBSA-Official-Rules-of-the-Games-of-Snooker-and-Billiards-2020.pdf`], we use the verb 'pocket' when a ball ends up in a pocket as a result of a foul shot. The verb 'pot' is reserved for legitimate shots. Let the players be P and Q. Perfect play proceeds thus.

(1) Player P pockets all 22 balls with a single foul shot. This looks like a strange way to start a frame, but there doesn't seem to be any reasonable alternative. There is no way for P to pot a red. Nor is it possible to make a legitimate no-score shot. Once significant energy has been introduced into the system perpetual motion takes over and all the players can do is wait until each ball finds a pocket to drop into. Of course, it is theoretically possible for a ball to go into an orbit that never ends in a pocket. However, that has probability 0 of happening and therefore its possibility can be ignored.

A red-avoidance option, such as pocketing the cue ball for a penalty of 4, will not be good. For then Q would clear the table and consequently be in the same position as if P had done so but with the advantage of a 4-point lead. In fact, to have any chance of winning the frame P should clear the table now and hope future breaks provide opportunities to recoup the lost points. The current score is **0–7**.

(2) Yellow, green, brown, blue, pink and black are respotted. With careful placement of the cue ball and careful aim, Q pots the yellow and the cue ball comes to rest at the point where it struck the yellow and without disturbing any other ball. If the aim is perfect, by conservation of momentum there is a simple exchange of velocities: $(\mathbf{v}, 0) \rightarrow (0, \mathbf{v})$. With our rule (iv) it is permissible for aiming accuracy to be merely very good rather than perfect, and it won't matter if the collision results in an undetectable residual velocity for the cue ball.

Unfortunately Q does not seem to be in a position for potting the green and bringing the cue ball to rest. The problem is that the location of the cue ball is now fixed. We do not have the continuous range of placements available in the D. For a perfect collision that truly halts the cue ball, the number of possible orbits is probably finite but certainly countable. They correspond to the various ways one can bounce the cue ball off the sides of the table before it strikes the green. The number increases to uncountably infinite—but only slightly—by rule (iv), which allows some additional flexibility. Around each perfect orbit there is a tiny range of valid imperfect ones, where the cue ball merely comes to rest after the collision. However, in any orbit, perfect or otherwise, the green would have to avoid collisions with the other balls on the table. It is not clear that a suitable orbit can be found. A reasonable strategy for Q is to finish the break by pocketing the cue ball. Although the deliberate giving away of four points would be an unusual tactic in normal games, on a frictionless table this play is actually quite sensible. Any other option will almost certainly incur a penalty of 7 by clearing the table. **4–9**.

(3) With careful placement of the cue ball and careful aim, P pots the green and the cue ball comes to rest at the collision point and without disturbing any other ball. Unfortunately P probably cannot pot the brown, and finishes the break by pocketing the cue ball. **7–13**.

(4) Similarly, Q pots the brown and then pockets the cue ball. **11–17**.

(5) Player P pots the blue and then pockets the cue ball. To make this shot, P can place the cue ball at a precisely defined position somewhere in the vicinity of a point 0.128196 m west of the yellow spot, and aim directly at the blue. The blue rebounds off the north side of the table into the south-west pocket. **16–21**.

(6) Player Q pots the pink and then pockets the cue ball. A suitable location for the cue ball is approximately 0.0884284 m west of the yellow spot. The pink rebounds from the north and south sides to go into the north-west pocket. **20–27**.

(7) Player P pots the black. An orbit can be found such that after the collision the black ball narrowly avoids the cue ball on rebounding from the north side of the table and ends up in the south-west pocket after a two further rebounds. A suitable position for the cue ball is approximately 0.100698 m west of the yellow spot. You can verify this by drawing a scale model of a snooker table like the one on the next page. **27–27**.

(8) The scores are equal! The black is respotted, a coin is tossed, the winner pots the black.

Surprisingly, the game is fair. In the ideal case, where both players are sufficiently skilful, it is the coin toss at the end that decides the winner of the frame. In the real world, however, frictionless snooker is a game of skill because less than perfect players might make mistakes. And whenever there is a mistake it will almost surely clear the table.

As we have seen, frictionless snooker is a very sensible option for an interesting competitive sport and it clearly deserves greater recognition than it currently enjoys. We believe that the main reason for its relative unpopularity is the unavailability of suitable tables. However, we can easily get around this difficulty by simulating frictionlessness on a real table. All we need is a small amendment to the way in which snooker is normally played. We allow only three types of shot.

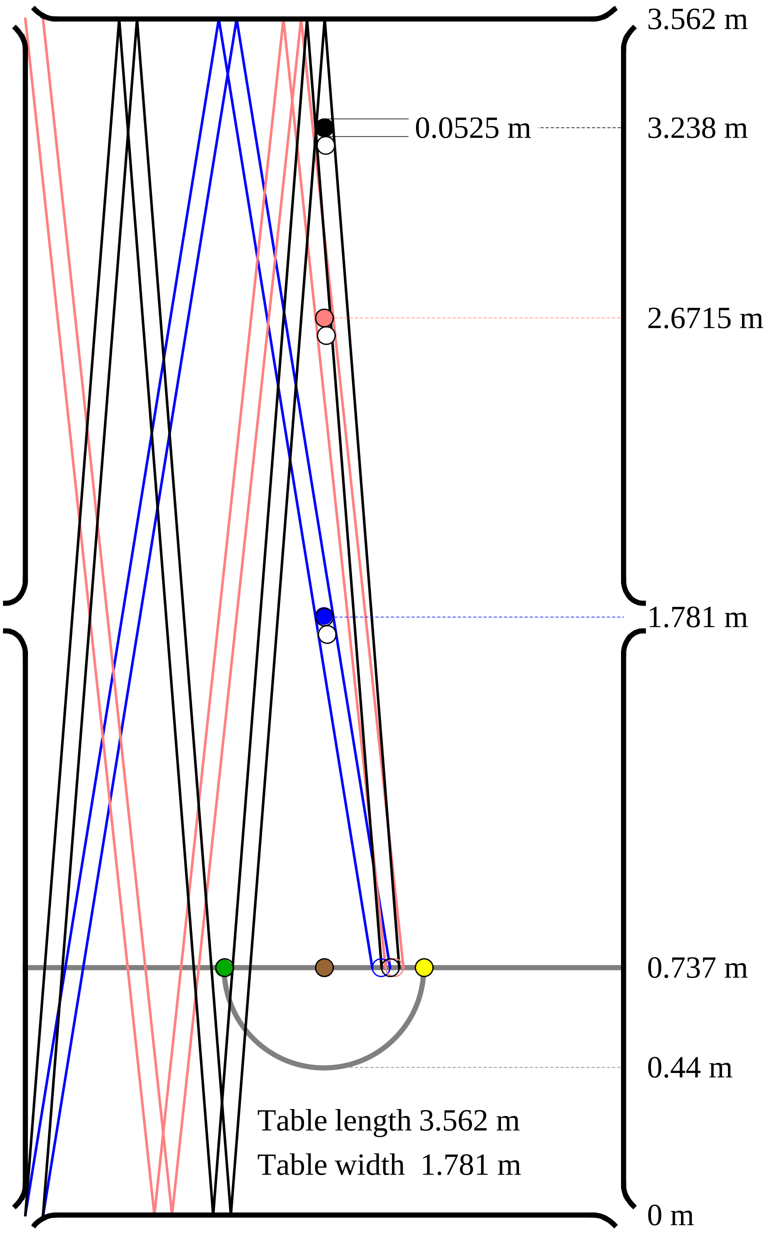
(A) Table clearance. This is conveniently achieved by just informing the referee of your intention. The referee removes everything from the table, returns the colours in play to their correct spots, awards you a penalty of 7, and hands the cue ball to the other player, who is then free to place it anywhere in the D.

(B) Pocket the cue ball. You get a penalty of 4 if the shot is 'successful'. Otherwise it is treated as a table clearance, type (A).

(C) Pot a colour in such a manner that the cue ball comes to rest at the collision point. Proceed as in normal snooker if the shot is successful. Otherwise—in particular, if the cue ball fails to come to rest at the site of the collision—it's a table clearance.

For simplicity, any deviation from these three types, such as what would be a no-score shot in normal snooker, is to be treated as a table clearance.

The cue ball placements described in (5)–(7), above, should still apply, at least approximately. If the shots are performed correctly, the orbits are the same as in the frictionless case except that you might need a more powerful cue stroke, especially for the black, to overcome friction.


3.562 m
0.0525 m
3.238 m
2.6715 m
1.781 m
0.737 m
0.44 m
Table length 3.562 m
Table width 1.781 m
0 m

Tiling a torus – suggested solution

Tommy Moorhouse

Consider the unit square with corners $(0,0), (0,1), (1,0)$ and $(1,1)$. We identify the opposite sides in pairs to make a torus T. A (closed) line on this torus is the image of a line from $(0,0)$ to (m,n) where m and n are relatively prime positive integers. We will call this image line $L(m,n)$. Recall that in my article, 'Tiling a torus', M500 **300**, we finished with a problem. Two distinct lines on the torus give rise to a tiling of the torus. How many tiles are there for two distinct lines $L(m,n)$ and $L(\hat{m},\hat{n})$? The answer is related to the number of times the lines intersect on the torus.

In the plane two distinct lines L_1 and L_2 form two sides of a parallelogram $P(L_1, L_2)$. This parallelogram has integer area A which is easy to calculate. We take L_1 to be given by (m,n) and L_2 to be given by $(\hat{m},\hat{n})$, say. You can use a linear map from T to $T(P)$ or a direct calculation but there is a pleasing, simple geometric construction involving sliding triangles. The conclusion is that $A = m\hat{n} - \hat{m}n$. Note that A can be positive or negative, but this is not important for this problem.

We identify opposite sides of P in pairs so that P becomes a torus, denoted $T(P)$ simply to emphasise the change of viewpoint. It is simple to check that every point of the unit torus corresponds to exactly A points in the torus $T(P)$. We will say that a point of T 'lifts' to A points of $T(P)$.

We will denote the image line of L_1 in the unit torus T by t_1 and so on. Since every point in T is covered by A points of $T(P)$ the line t_1 on T lifts to a set S_1 of A distinct lines on $T(P)$ each parallel to L_1. Similarly t_2 lifts to a set S_2 of A lines on $T(P)$, each parallel to L_2. On the 'dual' torus $T(P)$ these distinct lines are circles around the torus. Each line in S_1 meets the set of lines S_2 exactly A times in $T(P)$, so in $T(P)$ there are exactly A^2 intersection points. These cover points of T in sets of size A so we have found that $I(t_1, t_2) = A^2/A = A$.

The lines t_1 and t_2 cut T into identical tiles similar (in the geometric sense) to P. We can count the tiles in terms of $I(t_1, t_2)$, since every tile has a unique 'bottom left' corner, and we conclude that the number of tiles is $|m\hat{n} - \hat{m}n|$.

Reference

Barton Zwiebach, *A First Course in String Theory*, Cambridge University Press, Chapter 21 outlines the derivation of the intersection number for a particular application.

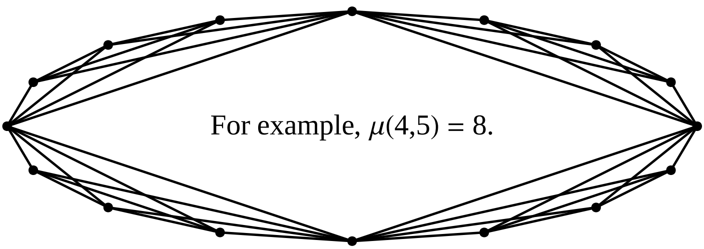
Problem 302.5 – Eigenvalues

Tony Forbes

(i) Find a closed formula for the function $\mu(m, n)$ defined for integers $m, n \geq 3$ by

$$\mu(m, 3) \; = \; 0, \qquad \mu(m, n + 1) \; = \; \mu(m, n) + m, \quad n \geq 3.$$

(ii) Let $m, n \geq 3$ be integers. Take m copies of the complete graph K_n and join them together to form a cycle where two adjacent K_n graphs have precisely one vertex in common. Show that the multiplicity of eigenvalue -1 of the graph's adjacency matrix is $\mu(m, n)$. Or find a counter-example.



Problem 302.6 – Numbers avoided by phi

As usual, let $\phi(n)$ be the number of integers k such that $1 \leq k \leq n$ and $\gcd(k, n) = 1$. Show that for every positive even integer m there is a prime p such that $\phi(n) \neq pm$ for any n.

In case it helps, here are the first few even numbers that $\phi(n)$ avoids: 14, 26, 34, 38, 50, 62, 68, 74, 76, 86, 90, 94, 98, 114, 118, 122, 124, 134, 142, 146, 152, 154, 158, 170, 174, 182, 186, 188, 194, 202, 206, 214, 218, 230, 234, 236, 242, 244, 246, 248, 254, 258, 266, 274, 278, 284, 286, 290, 298, 302, 304, 308, 314, 318, 322, 326, 334, 338, 340, 350, 354, 362, 364, 370, 374, 376, 386, 390, 394, 398, 402, 404, 406, 410, 412, 414, 422, 426, 428, 434, 436, 446, 450, 454, 458, 470, 472, 474, 482, 484, 488, 494, 496, 510, 514, 516, 518, 526, 530, 532, 534, 538, 542, 548, 550, 554, 558, 566.

Thus for $m = 2$, we can let $p = 7$ and indeed $\phi(n) \neq 14$ for any n. Of course, we already knew that—see Proposition 1 on page 7. Thereafter we have the following (m, p) pairs, and you will probably notice that multiples of 12 seem to give trouble.

m	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
p	17	19	19	5	43	7	19	5	17	7	167	7	11	3	19	2	67	2	17	17	7	5	211	7

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Problem 302.7 – Raffle

You and 99 other people buy raffle tickets, one each. There is only one prize and it is a rule that only persons present where the draw takes place are eligible to win it. At the time of the draw only 10 people turn up. So the organizer discards 90 randomly chosen tickets. Then tickets are drawn at random from the remaining 10 until one of the persons present wins the prize. What's your probability of getting the prize?

Front cover A graph where the eigenvalue -1 occurs with multiplicity 42 in its adjacency matrix. See problem 302.5.