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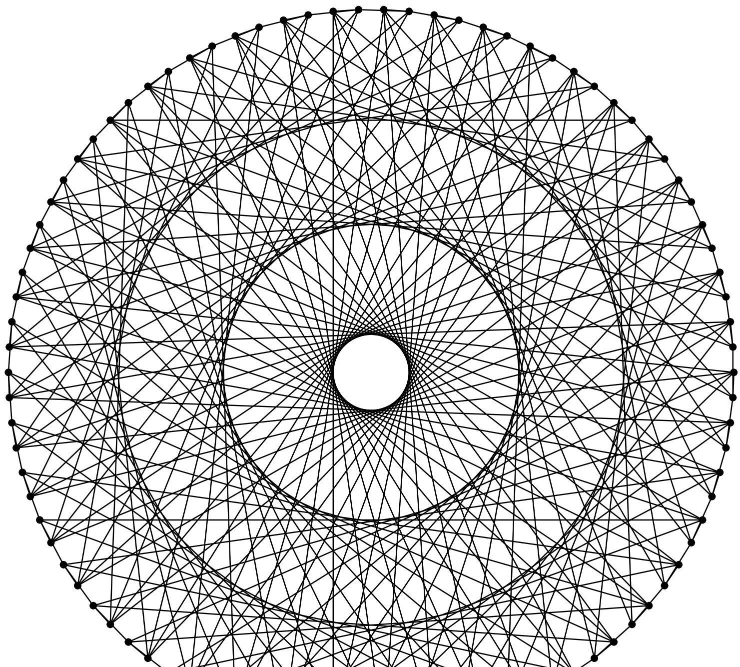
M500 304



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Graph thermodynamics

Tommy Moorhouse

Introduction

In this investigation we use simple ideas from thermodynamics to explore an energy function defined on connected graphs. We consider the mathematical construction of the partition function for a particular family of graphs.

Connected graphs, graph states and the energy function

A connected graph will be a set of points (vertices) connected by edges such that each edge connects exactly two distinct vertices and every vertex is connected to at least one other. The graph cannot be separated into two or more connected graphs except by 'cutting' an edge. Call the graph G.

We will assign to each vertex of G a colour and call the coloured graph a 'graph state' which we denote G_s. If an edge connects two vertices with the same colour then the energy associated with the edge is $+\epsilon$ in suitable units. Here we have set the zero of energy to make the calculations simple. If an edge connects two differently coloured vertices the energy associated with the edge is $-\epsilon$. The energy $E_s = E(G_s)$ of the graph state G_s is then the sum of the energies of its edges. (Aside: this assignment of energies favours graph states with adjacent vertices of different colours.)

For example, consider the simplest nontrivial connected graph, two vertices connected by an edge, which we call K_2. If the vertices have the same colour the state has energy $+\epsilon$, and there is one state of negative energy $-\epsilon$ having different coloured vertices. In general we will use only as many colours as strictly necessary for the graph in question, and in calculating edge energies we are only concerned whether adjacent vertices have the same colour or not.

The partition function

First we fix a graph G. We imagine that we have a collection ('ensemble') of graph states in thermal equilibrium at inverse temperature β. Don't worry too much about this: all it means is that at a certain temperature the copies of the graph will be distributed across all possible states, with a well defined probability of a graph having a certain energy. Given a graph G we form the partition function derived from the energies of each possible state,

$$Z_G(\beta) \;=\; \sum_{s\in\text{states}} e^{-\beta E_s}.$$

For example, the partition function of K_2 is then

$$Z_{K_2}(\beta) \;=\; (e^{\beta\epsilon} + e^{-\beta\epsilon}) \;=\; 2\cosh\beta\epsilon.$$

The probability that a graph in the large ensemble is in state of energy E_s is

$$P(E_s) \;=\; \frac{g(E_s)e^{-\beta E_s}}{Z_G(\beta)}.$$

Here $g(E_s)$ accounts for the fact that there may be more than one state with energy E_s in general: $g(E_s)$ counts them. You may like to show that close to absolute zero (when $\beta \to \infty$) the graphs are in the lowest energy states, and that in the limit of very high temperature ($\beta \to 0$) every graph state is equally probable.

Complete graphs

As a concrete example we consider the colour energy of a collection of graph states. A complete graph K_n has n vertices and every vertex is connected to every other, each by a single edge. We wish to consider the energies of states with one, two, and so on up to n colours. We can make use of an interesting property of complete graph states, namely that each colour can be separately carried by a complete subgraph. By this we mean that a complete graph state with, say, two colours is the product of two monochrome complete graph states, where the product is the smash product. The smash product of two graphs G and H, denoted $G \vee H$, is the graph including the same edges and vertices as the two components, but with every vertex of G joined by a single edge to each vertex of H (and every vertex of H joined by a single edge to each vertex of G) (see M500 **294** Problem 294.6).

We can decompose any complete graph state into smash products of its monochrome components, using superscripts to denote colours:

$$K_m \;=\; K_{k_1}^{A_1} \vee K_{k_2}^{A_2} \vee \cdots \vee K_{k_r}^{A_r}$$

where $\sum k_i \;=\; m$. This applies to any partition of m into integers $k_1, k_2, \cdots k_r$. You might like to prove this, possibly by induction. The next step is to find the colour energy of a complete graph state, say K_k, taking the state to be monochrome. Each edge contributes $+\epsilon$ to the energy and there are $k(k-1)/2$ edges. Taking the smash product of two complete graph states K_a^A and K_b^B, each graph being monochrome (colours indicated by superscripts), gives a bicoloured graph state. Each new edge contributes $-\epsilon$ to the total energy, and there are ab such new edges. This means that the energy of $K_a^A \vee K_b^B$ can be expressed as

$$E(K_a^A \vee K_b^B) \;=\; E(K_a) + E(K_b) - \epsilon ab.$$

We have dropped the colour superscripts on the right as they are not needed for monochrome graphs.

The general partition function

A state of a complete graph has been shown to be a smash product of monochrome complete graphs. Use this fact and the associative property of the smash product to deduce that the partition function for an ensemble of complete graph states K_n with at most n colours is

$$Z_{K_n}(\beta) \;=\; e^{\epsilon\beta n(n+1)/2} \sum_{\sum k = n} e^{-\{\epsilon\beta \sum k^2\}}.$$

The sum on the right is over the partitions of n. Hint: start from a smash product

$$K_{k_1}^{A_1} \vee K_{k_2}^{A_2} \vee \cdots \vee K_{k_r}^{A_r}$$

of complete graphs to calculate the energy of a general graph state, then use $\sum k_i = n$ to tidy up the algebra.

A thermodynamic property

A simple example of a thermodynamic property is the internal energy U of a system. This is defined as

$$U \;=\; -\frac{\partial \log Z_G(\beta)}{\partial \beta}.$$

You might like to plot the internal energy of a complete graph ensemble and interpret the result.

Other families

Other simple families of graphs are the linear and cyclic graphs. Are there simple formulae for the thermodynamic quantities we have examined in the case of complete graphs?

Graph thermodynamics: solution and alternative result

Tommy Moorhouse

Distinct graph states

In the problem set in the investigation into the thermodynamics of graphs and graph states (coloured graphs) we made two assumptions. First we use only as many colours as strictly necessary to create all the states of the graph in question (e.g. for K_n we use n colours). Secondly, in calculating edge energies we are only concerned whether or not adjacent vertices have the same colour. In the case of the complete graphs K_n this amounts to treating all states built from the same set of monochrome components as the same graph state, regardless of the colours of the monochrome components (a 'colourblind' approach). We will consider a different approach, leading to a different partition function, after proving the result for the given assumptions.

The 'colourblind' partition function

The partition function of K_n can be built using the expression for the energy of a general graph state, $K_{k_1} \vee K_{k_2} \vee \cdots \vee K_{k_r}$. Here there are r monochrome components, the k_i label the component sizes (which need not be distinct), and $\sum_{i=1}^{r} k_i = n$. In units of ϵ the energy of a single component, say K_k, is $k(k-1)/2$. Smashing this component with a second, say K_m produces km new edges, each of energy $-\epsilon$. Continuing to form the full smash product of components we find

$$\begin{aligned}
E(K_{k_1} \vee K_{k_2} \vee \cdots \vee K_{k_r}) &= \sum_{i=1}^{r} \left(\frac{1}{2} k_i(k_i - 1) \right) - \sum_{j \neq m} k_m k_j \\
&= \sum_{i=1}^{r} \frac{1}{2} k_i (k_i - \sum_{j \neq i} k_j - 1) \\
&= \sum_{i=1}^{r} \frac{1}{2} k_i (k_i - (n - k_i) - 1) \\
&= \sum_{i=1}^{r} k_i^2 - \frac{1}{2} n(n+1).
\end{aligned}$$

In the second line we have split $k_i k_j$ into two equal parts and used this to rewrite the sum over i. The next line follows from $\sum_{j \neq i} k_j = n - k_i$.

This leads, summing over partitions of n representing the different smash products that contribute, to the partition function

$$Z_{K_n}(\beta) \;=\; e^{\epsilon\beta n(n+1)/2} \sum_{\sum k=n} e^{-\{\epsilon\beta \sum k^2\}}$$

set out in the problem.

An alternative approach and a different result

An objection might be raised to the above argument that differently coloured graph states can have the same energy. In the case of complete graphs this just depends on the component monochrome complete graphs. We will consider the combinatorial factors that account for this degeneracy.

N colours and r components

If the only type of component in the smash product is K_1 then the graph state is the smash product of n K_1s, with all n colours used. At the other extreme the only monochrome component is K_n. We can write the general decomposition of K_n (dropping the colour labels on the monochrome factors) as

$$K_n \;=\; K_1^{m_1} \vee K_2^{m_2} \vee \cdots \vee K_p^{m_p} \vee \cdots \vee K_n^{m_n}.$$

Here m_i can be zero or a positive integer (K_m^0 represents the empty graph $\emptyset$, with $G \vee \emptyset = G$), and the powers are smash products. Observe that

$$\sum_{i=1}^{n} m_i \;=\; r.$$

First we consider how many graph states can be made using these components. For now let the number of available colours be N, so that the number of ways of choosing the r colours for our r-components is $\binom{N}{r}$. Some of the r-component graph states will have the same energy. In particular, if two or more components are of the same size then switching the colours of those components gives exactly the same graph state. We account for this by counting each state with the combinatorial factor

$$\binom{r}{m_1 m_2 \cdots m_n} \;=\; \frac{r!}{m_1! m_2! \cdots m_n!},$$

defining the multinomial coefficient.

The alternative partition function

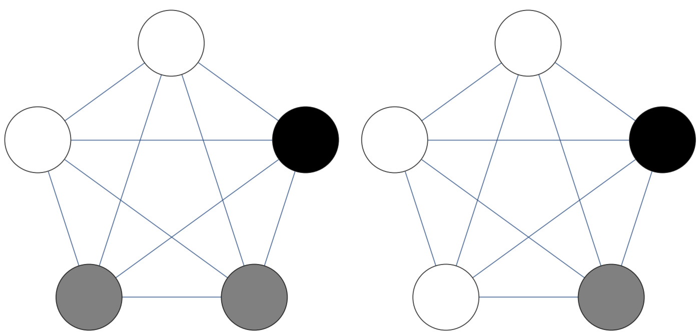
The alternative partition function, accounting for degeneracies and setting the number of colours to n, is then the sum over all graph states of K_n

$$Z_{K_n}(\beta) \;=\; e^{\beta\epsilon n(n+1)/2} \sum_{r=1}^{n} \binom{n}{r} \sum_{\sum m=r} \binom{r}{m_1 m_2 \cdots m_n} \sum_{\sum k m_k = n} e^{-\beta\epsilon \sum m_k k^2}.$$

Here the first sum is over the number of components in the graph state, the second is the combinatorial factor that accounts for degenerate states and the final sum is over partitions of n, so that $n = 3m_3 + 7m_7$ gives rise to a term $e^{-\beta\epsilon(9m_3+49m_7)}$ for example.

To see how this works out consider K_5. Suppose $r = 3$. Then we have $m_1 + m_2 + m_3 + m_4 + m_5 = 3$ and $m_1 + 2m_2 + 3m_3 + 4m_4 + 5m_5 = 5$. It is actually easier to work with $m_2 + 2m_3 + 3m_4 + 4m_5 = 2$, when we can see that there are two possibilities given that the m_k are zero or positive. First $m_1 = 2$, $m_3 = 1$, (other m_ks vanishing) denoting two components each with a single vertex and one with three vertices; and second $m_1 = 1, m_2 = 2$ denoting one component with a single vertex and two others each with two vertices. The corresponding graph states are shown below. The combinatorial factors are

$$\binom{5}{3}\binom{3}{2\;0\;1\;0\;0} \text{ and } \binom{5}{3}\binom{3}{1\;2\;0\;0\;0}.$$



(a) $n = 5, m_1 = 1, m_2 = 2$ (b) $n = 5, m_1 = 2, m_3 = 1$

Figure 1: Two K_5 graph states.

Accidental degeneracies

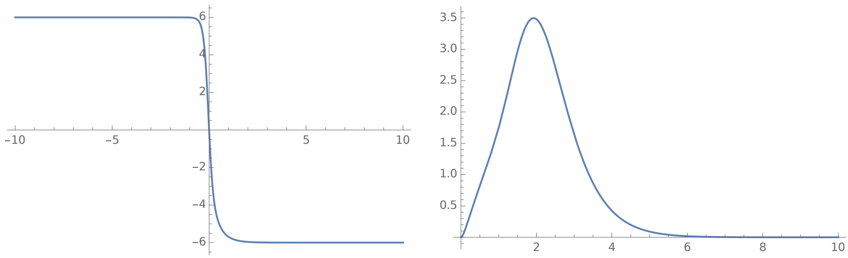
Although we have accounted for the ‘obvious’ degeneracies there remains the possibility that some of the sums $\sum m_k k^2$ where $m_1 + 2m_2 + \cdots + nm_n = n$ are equal for different values of the m_k subject to $\sum m_k = r$. We will call this an accidental degeneracy. You may wish to look into whether accidental degeneracies occur in this case. It appears to be a number theory question.

Internal energy

The internal energy U of an ensemble of graph states is defined as

$$U = -\frac{\partial \log Z_G(\beta)}{\partial \beta}.$$

The shape of the graph of $U(\beta)$ appears to be quite universal, shown in the figure.



(a) Internal energy $U(\beta)$ (b) Heat capacity vs temperature

Figure 2: Two thermodynamic properties for K_5 states, units arbitrary.

Postscript

In borrowing terms from thermodynamics we have developed a toy model, imagining that the graph states can exchange energy like molecules in a gas. It is not hard to plot the various thermodynamic properties starting from our partition functions, and it may be interesting to explore graph states in their own right. The two alternative models here might be envisaged to cover two distinct situations: in a physics context experiment would decide which was more appropriate to model a physical system. Here it is just for fun!

Solution 299.3 – Hair

> Assume: a haircut costs £H after which your hair will be h metres long; hair grows at d metres per second; you wash your hair n times per second; hair shampoo costs £S per kilogram; a hair wash requires w kilograms per metre (for example, if $w = 1$ and your hair length is 2 metres, you will need 2 kilograms of shampoo). How often should you visit the hairdresser to minimize the cost rate.

Ted Gore

Let k be the number of washes that give a minimum overall cost. The time between washes is $1/n$ seconds and in that time hair will grow d/n metres.

At the first wash, hair length will be $h + d/n$ metres; at the second $h + 2d/n$ and so on. After k washes, the cumulative length of hair washed will be

$$kh + \frac{d}{n}\sum_{i=1}^{k} i \;=\; kh + \frac{dk(k+1)}{2n}$$

and the shampoo cost will be

$$\left(kh + \frac{dk(k+1)}{2n}\right) Sw.$$

The overall cost is

$$H + \left(kh + \frac{dk(k+1)}{2n}\right) Sw.$$

The average cost per wash,

$$A \;=\; \frac{H}{k} + \left(h + \frac{d(k+1)}{2n}\right) Sw.$$

Differentiating, we get

$$\frac{dA}{dk} \;=\; -\frac{H}{k^2} + \frac{dSw}{2n}$$

and setting this equal to zero we have

$$k \;=\; \sqrt{\frac{2Hn}{dSw}}.$$

Taking, as an example, $k = \sqrt{2 \cdot 11 \cdot 5/(1 \cdot 3 \cdot 2)} = 4.2817$, we get $A = 47.7381$. We would like k to be an integer; $k = 4$ gives 47.75; $k = 5$ gives 47.8. Our solution is $k = 4$ and the overall cost is 191.

This answer assumes that the first wash takes place $1/n$ seconds after the haircut. An alternative would be to do the first wash immediately after the haircut.

Now, at the first wash, hair length will be h metres; at the second $h + d/n$; at the third $h + 2d/n$ and so on. The shampoo cost will be

$$\left(kh + \frac{dk(k-1)}{2n}\right) Sw$$

and the overall cost

$$H + \left(kh + \frac{dk(k-1)}{2n}\right) Sw.$$

We would still have $k = \sqrt{2Hn/(dSw)}$ at the minimum average cost but now for our example $k = 4$ gives 46.55; $k = 5$ gives 46.6. Our solution is still $k = 4$ but the overall cost is 186.2.

Problem 304.1 – Perfume

Tony Forbes

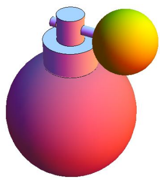
A cosmetics factory produces $n \geq 4$ types of perfume. They probably have fancy names but we will just call them P_1, P_2, ..., P_n. There are n ingredients, I_1, I_2, ..., I_n.

Perfume P_i sells for i pounds per kilogram and requires equal quantities of each ingredient except I_i, and there are only $n-i+3$ tonnes of I_i available, $i = 1, 2, \ldots, n$.

How much of each type of perfume must be produced to maximize income?

For a related problem, try deciphering this.

1/- 6d me



Answer on page 10.

Is it true that the star of *The Magnificent Seven*, a great fan of Liverpool FC, didn't use aftershave? Yes, Yul never wore cologne. —Jeremy Humphries

Solution 297.7 – Two queens

> Two queens are placed on an 8 metre square chessboard at random given that they must be separated from each other by at least 2 metres. What's the probability that they do not attack each other.

Ted Gore

We randomly assign a square to each queen. The number of possible pairs of squares is $64 \cdot 64 = 4096$. Certain pairs must be excluded because they are too close. If the second square of a pair is the same as or adjacent to the first then the pair must be excluded. If the first is a corner square then there are four pairs to exclude; for a non-corner edge square then six exclusions and for an internal square there are nine. The number of excluded pairs is

$$4 \cdot 4 + 24 \cdot 6 + 36 \cdot 9 \;=\; 484.$$

There are $4096 - 484 = 3612$ non-excluded pairs.

Now picture the board as an 8×8 grid. Each square lies in one row, one column and two diagonals. Consider the case where one queen is at row 3 column 2. There are 5 valid positions in the row that the second queen can take, 5 in the column and 5 in the two diagonals that intersect at row 3 column 2. A similar calculation can be carried out for the other 15 squares in the top left quadrant. There are 259 valid pairs of squares in the quadrant. Taking into account the symmetries of the board there are

$$4 \cdot 259 \;=\; 1036$$

valid pairs on the board. The probability of two queens attacking each other given that they are at least one square apart is $1036/3612 = 0.2868$ and of them not attacking is $92/129 = 0.7132$.

Problem 304.2 – Cubic

Let $f(x)$ be a cubic that has three distinct rational roots. Show that there exists $m \neq 0$ such that $f(x) - m$ also has three distinct rational roots. Thanks to *LMS Newsletter* **492** (Jan 2021) for suggesting this problem.

Answer to the picture thingy on page 9:

Bob Tanner sent me a bottle of perfume.

Bob, 1/-, shilling, £0.05, as in, for example, ten bob, ten shillings, 10/-, £0.50. Tanner, a pre-decimal six pence coin, 6d, £0.025. ...

Sudoku BUG

Ken Greatrix

Recently I have looked into the BUG + 1 formation in a sudoku puzzle and I note that the arrangement in the puzzle in M500 **207** on page 21 fits this pattern. This being that a BUG (bi-value universal grave) consists of remaining candidates in bi-values only, the +1 bit means there is one extra candidate. See, for example, `https://www.sudoku9981.com/sudoku-solving/bivalue-universal-grave.php`. As I look at this puzzle, reproduced below, I see that the extra candidate is in row 4, column 1; a 4, which is the correct solution to complete the puzzle. Also, each digit appears in pairs in each row, column and box except for digit 4 in row 4, column 1 and box 4, where there are three occurrences.

If my assumption is correct, then the most digits that can be placed is 74 leaving the remaining candidates as ...

AB	...	AB		
AB	...	ABC	...	BC
		BC	...	BC

..., where the puzzle can be completed by placing B in the middle cell of this formation. Sudoku fans would recognise this as a swordfish pattern. I think each cell in this should be in its own box.

3	9	5	2	1	6	7	4	8
8	4	7	9	3	5	1	6	2
1	6	2	8	4	7	9	3	5
	3		1	5	8	2	7	
	2	8	3	7	9		5	1
7	5	1	6	2	4	3	8	9
	7	3	5		2	8	1	
5	8		7		1		2	3
2	1	6	4	8	3	5	9	7

Answer to page 9, continued. ... "Where's the 'sent'?" I hear you ask. ...

Solution 301.1 – Power sum

Let b and k be integers greater than 1. Show that

$$S(b,k) \;=\; \sum_{i=0}^{\infty} \frac{1}{\displaystyle\sum_{j=0}^{k-1} b^{ki+j}} \;=\; \frac{(b-1)\,b^k}{(b^k-1)^2}.$$

For example,

$$S(2,2) \;=\; \frac{1}{1+2} + \frac{1}{4+8} + \frac{1}{16+32} + \ldots \;=\; \frac{4}{9},$$

$$S(10,3) \;=\; \frac{1}{1+10+100} + \frac{1}{1000+10000+100000} + \ldots \;=\; \frac{1000}{110889}.$$

Peter Fletcher

The denominator of each term in $S(b,k)$ is the sum of a geometric series with first term b^{ki} and common ratio b, so we can immediately write down

$$\sum_{j=0}^{k-1} b^{ki+j} \;=\; b^{ki}\left(\frac{b^k-1}{b-1}\right).$$

Putting this into $S(b,k)$, we get

$$S(b,k) \;=\; \sum_{i=0}^{\infty} \frac{b-1}{b^{ki}(b^k-1)} \;=\; \frac{b-1}{(b^k-1)} \sum_{i=0}^{\infty} \left(\frac{1}{b^k}\right)^i.$$

Since $b, k > 1$ the common ratio is less than 1, so

$$\sum_{i=0}^{\infty} \left(\frac{1}{b^k}\right)^i \;=\; \frac{1}{1-1/b^k} \;=\; \frac{b^k}{b^k-1}$$

and

$$S(b,k) \;=\; \frac{(b-1)b^k}{(b^k-1)^2}.$$

Problem 304.3 – Smallest quadratic non-residue

Let p be an odd prime. Let q be the smallest positive integer such that the Legendre symbol (q/p) is equal to -1. Show that q is prime.

Answer to page 9, continued. … It's in the bottle! (Scent.)

Solution 296.3 – Elliptic curve

Let a be a positive real number. Then the elliptic curve $y^2 = x(x^2-a^2)$ has two components, an unbounded curve that passes through $(a,0)$ and a closed 'bubble' that passes through $(0,0)$ and $(-a,0)$. What area does the bubble enclose?

Ted Gore

From my answer to this problem in M500 **302** we have

$$\begin{aligned} &0.4792560938942377 \\ &\quad = \frac{1}{1.5} - \frac{0.5}{3.5} + \frac{0.5(0.5-1)}{5.5\cdot 2!} - \frac{0.5(0.5-1)(0.5-2)}{7.5\cdot 3!} + \ldots \\ &\quad = k\sum_{n=0}^{\infty} \frac{2(-1)^n\Gamma(1/2)}{(3+4n)\Gamma(1/2-n)n!} = \sqrt{\pi}k\sum_{n=0}^{\infty} t_n, \end{aligned}$$

where $t_0 = 2/(3\sqrt{\pi})$ and $t_n = \dfrac{(3+4(n-1))(2n-1)}{(3+4n)2n} t_{n-1}$.

Calculating 200,000,000 terms we get a partial sum ≈ 0.675956 and $k \approx 0.4$. We can split 0.4 into 0.2 and 2 so that

$$0.4792560938942377 = \frac{\sqrt{\pi}}{5} 1.351911.$$

The inverse of $1.351911 = 0.739694$.

While looking for results on the ratio of gamma functions I came across [1], which gives a result

$$\frac{\mathit{\Gamma}(x+1/2)}{\mathit{\Gamma}(x)} \approx \left(x - \frac{1}{4}\right)^{1/2} \left(1 + \frac{1}{64}x^{-2} + \frac{1}{128}x^{-3}\right).$$

We can try various values of x and $x = 3/4$ gives 0.715652, which is close to 0.739694. The inverse of this is $1.397326 = \dfrac{\mathit{\Gamma}(x)}{\mathit{\Gamma}(x+1/2)}$.

This suggests that the initial series approaches $\dfrac{\sqrt{\pi}}{5}\dfrac{\mathit{\Gamma}(3/4)}{\mathit{\Gamma}(5/4)}$ as the number of terms approaches infinity.

Reference

[1] A. Laforgia and P. Natalini, 'On the asymptotic expansion of a ratio of gamma functions', *J. Math. Anal. Appl.* **389** (2012), 833–837.

Problem 304.4 – Cube shadow

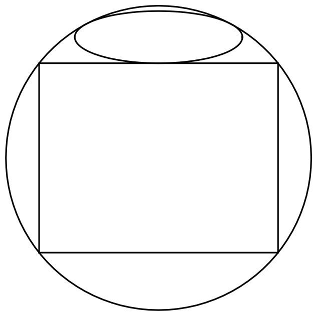
Milena Dragic

What is the largest possible area of shadow that a cube with 1 m sides can cast on a sunny day? For definiteness, assume that the shadow is cast (by the Sun) on the surface of planet Earth, which is of course a perfect sphere with circumference 40000000 m.

Too late now, but a (correct) solution could have entitled you to a £25 discount for the 2022 M500 Winter Weekend.

Problem 304.5 – A rectangle and an ellipse

Tony Forbes



This is like Problem 269.2 – Two rectangles, except that one of the rectangles is not a rectangle.

A rectangle and an ellipse are packed inside a circle of radius 1, possibly but not necessarily according to the pattern indicated on the right. What is the largest area they can occupy?

When you have solved the problem as stated, try again but this time with the extra condition that the rectangle must be larger than the ellipse. A numerical solution is acceptable. We would be very interested if there is a better way to arrange the two shapes.

Problem 304.6 – Trinomial factorization

Tony Forbes

Show that for every positive integer n except powers of 3, the polynomial $x^{2n} + x^n + 1$ has a non-trivial factorization into polynomials with integer coefficients. For example,

$$\begin{aligned}
x^4 + x^2 + 1 &= (x^2 - x + 1)(x^2 + x + 1), \\
x^8 + x^4 + 1 &= (x^2 - x + 1)(x^2 + x + 1)(x^4 - x^2 + 1), \\
x^{10} + x^5 + 1 &= (x^2 + x + 1)(x^8 - x^7 + x^5 - x^4 + x^3 - x + 1), \\
x^{12} + x^6 + 1 &= (x^6 - x^3 + 1)(x^6 + x^3 + 1), \quad \ldots .
\end{aligned}$$

Rayo's number

Jeremy Humphries

Hear my song – how two professors,
In the year two thousand-seven,
When the month was January,
And the day the 26th one,
Undertook to hold a contest:
Who could name the biggest number?

At the left side see defender,
Standing firm – AGUSTIN RAYO,
'Mexican and Multiplier',
'Plural Power', 'Ray gun' RAYO,
Out of MIT at Boston,
In the State of Massachusetts,
Where they staged the competition.

Challenger and at the right side,
'Dr Evil' – ADAM ELGA,
Ivy League was his location,
University of Princeton,
Situated in New Jersey.

Auditorium was crowded,
People struggled to get entry,
Poking heads through doors and windows,
Almost like the boxing matches
That they hold in New York City,
At the Madison Square Gardens.

Whence emerged these two professors?
From the World of Mathematics?
So you'd think for such a contest,
But reality corrects that.
Maths it seems was just their hobby,
Bit of fun to pass an hour.
In Philosophy, Linguistics,
That's where each one earned his living.

Board and chalk would be the weapons.
Rules were laid down at the outset.
As a gentlemen's agreement,
Each new try must break new ground, so
No referring to the other,
Adding one and saying 'Gotcha'.
That way lies an endless pathway.
Audience would soon get restive,

Drift away to watch the football.
Weird semantics too they outlawed –
Ditto all transfinite values.

First one up, Professor Elga,
Wrote a '1' upon the blackboard.
Didn't really think he'd win it,
But they had to start off somewhere.

Rayo now, who added more '1's,
String of them across the blackboard,
From the left side to the right side,
Didn't really count how many,
Maybe twenty-five or thirty,
So – a pretty big repunit.
Clear it beat the single digit.

Back came Elga, clever Elga,
Left the first two '1's untouched but
Through the rest he drew his finger
Near the bottom, rubbed the chalk off,
Turned the '1's into the symbol
Of the mark of exclamation,
Made factorials that nested,
Bang of bang of [dot dot dot] of
Bang of bang of bang eleven.
Pretty big, or even bigger.

Now it's back to Rayo, who then
Came up with the Busy Beaver,
Busy Beaver of a googol,
Conjured in the mind of Turing,
Dwelling in machines of Turing.

Elga thought, and went one better –
Not just Turing – super Turing –
Therefore super Busy Beaver.
Could they do that in this contest,
With the rules laid down at outset?
So it seems, as that's what happened.

Back and forth then went the battle –
Sleeves were rolled and brows were glistened –
Chalk dust billowed round the venue.
Somewhat more were the suggestions
Than I mention in these verses.
Just to give a few examples:
Donald Knuth supplied some arrows,
Ackermann supplied a function,

TREE of 3 (or more) was quoted.
I just got some bits from YouTube,
'Numberphile' the place I found them,
Mainly from the Prof. of Physics,
Called Antonio Padilla,
Known to the world as Tony.
Do your own research for details.

Finally along came Rayo:
Think of now the smallest number
Bigger than the largest number
(There he meant a finite number)
That can be expressed in symbols
Of set theory (first order),
When the tally of those symbols
Shall be ten raised to a power,
And that power be one hundred.
Or, to use the common parlance,
There shall be a googol symbols.

Elga looked and Elga pondered,
Crowd they waited, breath was bated,
More he looked, and more he pondered.
Is it paradox, the concept?
Maybe it would be, in English,
But set theory (first order)
Gives the language, and it's reckoned
Paradox is thus confounded.
Elga then at last conceded.
Rayo triumphed. Elga, vanquished,
Trudged away back home to Princeton.

Even now, the 2020s,
When the leaf is sere and fallen,
When the nights are cold and lengthy,
When along the Charles's river,
In the MIT environs,
Fires glow and people gather,
When professors drink their coffee,
When their students cluster round them,
Then are told the tales of legend,
Like the one that you've just finished:
How was fought the mighty battle
To devise the biggest number,
By the mighty two opponents –
Worthy loser, worthy winner,
'Dr Evil' – ADAM ELGA,
'Plural Power' – 'Ray gun' RAYO.

04.00

Problem 304.7 – Separated Roman integers

Characterize those positive integers n such that the number of symbols in the Roman representation of $n+1$ is -1, 0 or 1 more than the number of symbols in the Roman representation of n. For example, $n = 9 =$ IX ($n+1$ = X) but not $n = 8 =$ VIII ($n+1 =$ IX).

Front cover The deficiency graph of a pentagonal geometry PENT(5,21); it is 5-regular with 90 vertices and girth 5. See `https://arxiv.org/abs/2111.13599`.