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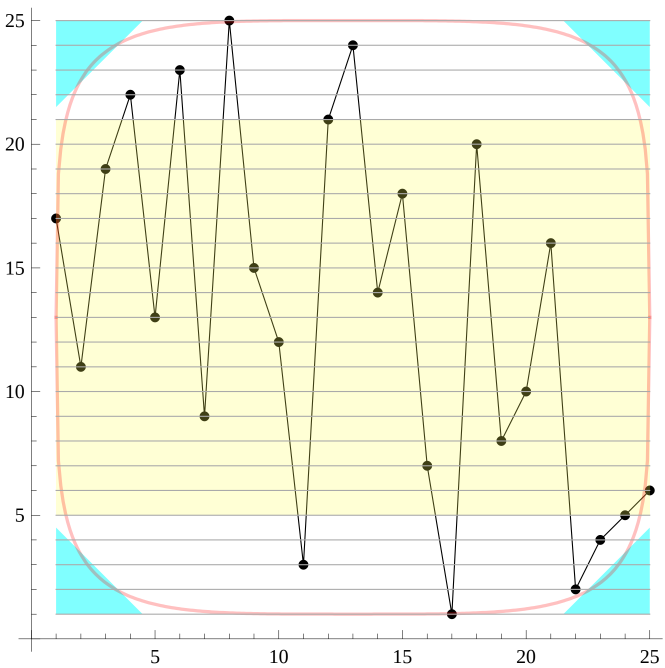
M500 305



$$\begin{bmatrix} 1 & 2 & 4 & 5 & 6 \\ 3 & 7 & 8 & 10 & 16 \\ 9 & 12 & 14 & 18 & 20 \\ 11 & 13 & 15 & 21 & 24 \\ 17 & 19 & 22 & 23 & 25 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 4 & 6 & 8 \\ 2 & 5 & 9 & 12 & 13 \\ 7 & 10 & 14 & 15 & 18 \\ 11 & 16 & 19 & 20 & 21 \\ 17 & 22 & 23 & 24 & 25 \end{bmatrix}$$

The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: `m500.org.uk`.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The Revision Weekend is a residential Friday to Sunday event providing revision and examination preparation for both undergraduate and postgraduate students. For details, please go to the Society's website.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, please go to the Society's website.

Advice for authors We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to the Editor, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. For more information, go to `m500.org.uk/magazine/` from where a LaTeX template may be downloaded.

M500 Revision Weekend 2022 We are hoping to run the Revision Weekend in 2022 over 13th–15th May at Kents Hill Park, Milton Keynes. For further details and an application form please check the website, or e-mail the Weekend Organizer, weekend@m500.org.uk.

Matter–antimatter asymmetry

The strange mystery: could it be a question of probabilities?

Colin P. George

The beginning of time as we know it is believed to be around 10^{-43} seconds. This is called Planck time and furthermore, to actually envisage a time when the age of the universe was less than 10 to the power of -43 seconds seems to be rather difficult to define in a meaningful way. Einstein's *General Relativity* and even *Quantum Theory* appear to break down before Planck time.

The Big Bang happened around 13.8 billion years ago. Time as we know it was a by-product of the Big Bang. Matter and antimatter were originally produced in equal measures too. In this new world there was a war of annihilation between matter and antimatter. Although matter and antimatter tend to cancel each other out violently there was always perfect matter–antimatter symmetry. This would concur with the laws of physics. As Chief Engineer Montgomery Scott used to say 'You can't change the laws of physics' (TV series *Star Trek*, 1966). What is rather strange is that at some point during this period normal matter dominated. The laws of physics appear to have been broken. No one really knows why this happened. This event is a mystery that has baffled mathematicians and physicists for many years.

What caused this imbalance? Now, some believe, just maybe, that all of this happened by pure chance, a bit like the toss of a coin. It could have gone either way ....

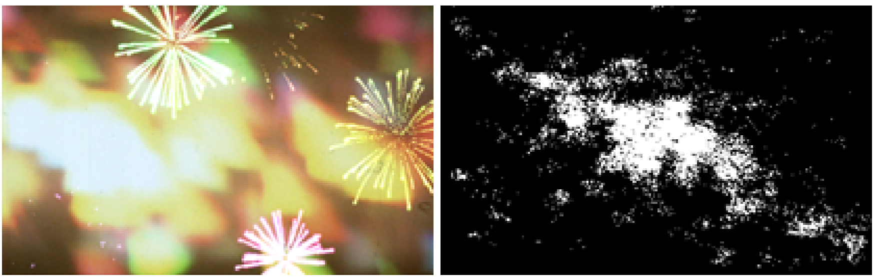


Fig 1 (Left) Long ago there was a time when there was perfect matter–antimatter symmetry. (Right) In our universe matter became dominant. In some other place physicists believe it could be the other way round.

Here is a little thought experiment. If we imagine well over a thousand pennies (let's say a very high quantity representing infinity as an analogy) on a flat piece of cloth measuring less than one square metre. We toss every one of these coins. Now, statistically speaking heads or tails is likely to have a 0.5 probability overall (here in this particular context we are using heads/tails as our matter/antimatter analogy). However, that piece of cloth is made out of a new form of spandex that can stretch to 10 square metres, then 20 square metres then 30 and beyond (here we are using expanding cloth as the expanding universe analogy). Let's assume no more coins are being produced. The cloth has now expanded to the point where there are only about 10 coins per square metre. Assuming each square represents or has manifested itself as a separate world or domain then surely getting heads or tails will begin to drift from that original 0.5 probability quoted earlier. Statistically speaking we will begin to get more heads than tails or vice versa and that is because a small number of coins in each individual square (or domain) will eventually lead to a large sampling error. It becomes more unlikely that we will get 5 heads and 5 tails in each of those individual square metre domains. However, that is not to say that it will never happen.

Fig 2 (Left) Large quantities of pennies nearing infinity gives a score of a 0.5 probability. (Right) As the quantities get smaller per unit area the possibility of getting heads is less likely to carry a probability of 0.5. The score is more likely to be more heads than tails or vice versa.

The expansion of the spandex cloth changed the dynamics, or put more accurately, it brought about a significant change in probabilities. Heads will dominate in some domains (the matter dominated world analogy), tails in others (the antimatter dominated world analogy), and in a minority of domains it may actually be evens (in such worlds mass scale annihilations may still exist). The notion that asymmetry in our universe may have happened by chance seems elegant, tidy and simple. No doubt this is what mathematicians and physicist like to see; the less messy and complicated the better. However, many physicists believe that there is a lot more to it than that. You see, when it comes to the subject of matter–antimatter symmetry or asymmetry, things may not be as beautifully simple and elegant as we may wish.

Mathematicians and physicists have theorized about this phenomenon for many years. A possible cause or at least a contributory factor regarding this imbalance could be to do with a thing called CP violation. However, that's another story . . . .

Further reading

Gleick, J., Richard Phillips Feynman, *Encyclopaedia Britannica*, Chicago, 2012.

Jones, M. H., Lambourne, R. J. A. and Serjeant, S. (eds), *Introduction to Galaxies and Cosmology*, Cambridge University Press, 2010.

Lambourne, R. J. A., *Relativity, Gravitation and Cosmology*, Cambridge University Press/The Open University, 2010.

Serjeant, S., *Observational Cosmology*, Cambridge University Press/The Open University, 2010.

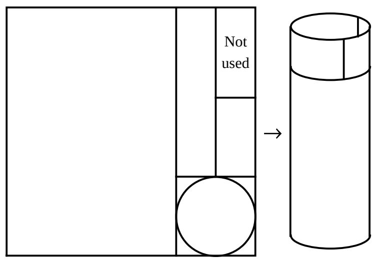
Sutton, C., Antimatter, *Encyclopaedia Britannica*, Chicago, 2012.

Sutton, C., CP violation, *Encyclopaedia Britannica*, Chicago, 2012.

Problem 305.1 – Cylindrical bin

Tony Forbes

You have a 1 metre square of sheet metal. From this you make a cylinder closed at one end to provide a facility for temporarily storing discarded office material. To keep welding costs down, the circular base must be cut out in one piece and you are allowed up to three rectangular pieces to stick together to form the cylinder. What's the maximum volume?



The example shows one way. Can you do better?

Iterating Euler's phi function

Tony Forbes

Recall that Euler's phi function counts the positive integers less than n that are co-prime to n; i.e. $\phi(n)$ is the number of positive integers m such that $\gcd(m, n) = 1$. Moreover, the function is completely determined by:

$$\begin{aligned} \phi(1) &= 1, \\ \phi(p^t) &= (p-1)p^{t-1} \qquad \text{if } p \text{ is prime and } t \geq 1, \\ \phi(mn) &= \phi(m)\phi(n) \qquad \text{if } \gcd(m,n) = 1, \end{aligned}$$

which also provides an efficient way of computing $\phi(n)$ provided you know the complete factorization of n. On the other hand, as is well known to people whose business it is to read other people's private messages, the determination of $\phi(n)$ when the factorization of n is not available can be rather difficult. If you don't believe me, try computing $\phi(2^{4096} + 1)$.

Since $\phi(1) = 1$ and $\phi(n) < n$ for $n > 1$, we can expect the sequence n, $\phi(n)$, $\phi(\phi(n))$, ... to converge to 1, the fixed point of ϕ. In an attempt to make things more exciting one can experiment with some 'linear' combination, such as $a\phi(n) + b$. After a certain amount of piddling[1] I decided that $a = 2$, $b = -1$ is worth exploring and even writing about. So let

$$F(n) \;=\; 2\phi(n) - 1$$

and let $F^d(n)$ be the result of iterating $x \mapsto F(x)$ d times, starting with $x = n$; i.e. $F^d(n) = F(F(\ldots F(n)\ldots))$ with d Fs and d pairs of brackets. Then we have $F(1) = F(2) = 1$, $F(3) = F(4) = 3$, and hence 1 and 3 are fixed points of the function F. Thereafter things get more interesting, and the table on the next page shows what happens to $F^d(n)$ when $5 \leq m \leq 31$ and $1 \leq d \leq 13$, revealing another fixed point, 15.

To summarize the situation so far, we know that $F(n)$ has fixed points 1 reached via $n \in \{1, 2\}$, 3 via $n \in \{3, 4, 6\}$ and 15 via $n \in \{15, 16, 20, 24, 30\}$. If you guess the next three numbers in the sequence 1, 3, 15, ... to be 255, $63355 = 2^{16} - 1$ and $4294967295 = 2^{32} - 1$, you would not be wrong. There are indeed further fixed points:

255 reached via $n \in \{255, 256, 272, 320, 340, 384, 408, 480, 510\}$,

65535 via $n \in \{65535, 65536, 65792, 69632, 69904, 81920, 82240, 87040, 87380, 98304, 98688, 104448, 104856, 122880, 123360, 130560, 131070\}$ and

[1] This seems to be the accepted technical term for non-purposeful experimental calculations with numbers, usually with the help of a computer.

4294967295 via $n \in \{4294967295, 4294967296\}$.

Thereafter the supply seems to dry up. The next three numbers, $2^{64} - 1$, $2^{128} - 1$ and $2^{256} - 1$, are not fixed points of $F(n)$. A very interesting and non-trivial problem suggests itself.

> *Prove that the fixed points less than* $2^{2^{34}} - 1$ *of the function* $n \mapsto 2\phi(n) - 1$ *are* 1, 3, 15, 255, 65535 *and* 4294967295*; or find another.*

The limit is there because I do not know whether $2^{2^{33}} + 1$, a factor of $2^{2^{34}} - 1$, is prime or composite. See `http://www.prothsearch.com/fermat.html` for the current status of $2^{2^{33}} + 1$ and other Fermat numbers.

$F^d(n)$	1	2	3	4	5	6	7	8	9	10	11	12	13
5	7	11	19	35	47	91	143	239	475	719	1435	1919	3599
6	3	3	3	3	3	3	3	3	3	3	3	3	3
7	11	19	35	47	91	143	239	475	719	1435	1919	3599	6959
8	7	11	19	35	47	91	143	239	475	719	1435	1919	3599
9	11	19	35	47	91	143	239	475	719	1435	1919	3599	6959
10	7	11	19	35	47	91	143	239	475	719	1435	1919	3599
11	19	35	47	91	143	239	475	719	1435	1919	3599	6959	13915
12	7	11	19	35	47	91	143	239	475	719	1435	1919	3599
13	23	43	83	163	323	575	879	1167	1551	1839	2447	4891	9503
14	11	19	35	47	91	143	239	475	719	1435	1919	3599	6959
15	15	15	15	15	15	15	15	15	15	15	15	15	15
16	15	15	15	15	15	15	15	15	15	15	15	15	15
17	31	59	115	175	239	475	719	1435	1919	3599	6959	13915	19359
18	11	19	35	47	91	143	239	475	719	1435	1919	3599	6959
19	35	47	91	143	239	475	719	1435	1919	3599	6959	13915	19359
20	15	15	15	15	15	15	15	15	15	15	15	15	15
21	23	43	83	163	323	575	879	1167	1551	1839	2447	4891	9503
22	19	35	47	91	143	239	475	719	1435	1919	3599	6959	13915
23	43	83	163	323	575	879	1167	1551	1839	2447	4891	9503	16127
24	15	15	15	15	15	15	15	15	15	15	15	15	15
25	39	47	91	143	239	475	719	1435	1919	3599	6959	13915	19359
26	23	43	83	163	323	575	879	1167	1551	1839	2447	4891	9503
27	35	47	91	143	239	475	719	1435	1919	3599	6959	13915	19359
28	23	43	83	163	323	575	879	1167	1551	1839	2447	4891	9503
29	55	79	155	239	475	719	1435	1919	3599	6959	13915	19359	25703
30	15	15	15	15	15	15	15	15	15	15	15	15	15
31	59	115	175	239	475	719	1435	1919	3599	6959	13915	19359	25703

Solution 299.5 – Integral

Let a be a positive integer. Show that

$$I(a) \;=\; \int_0^1 \frac{\sqrt{1-x^{1/a}}}{\sqrt{1+x^{1/a}}}\,dx \;=\; \frac{1}{B}-\frac{\pi a B}{2},$$

where

$$B \;=\; \begin{cases} \dfrac{1}{2^a}\dbinom{a}{a/2} & \text{if } a \text{ is even,} \\[2ex] \dfrac{-1}{2^{a-1}}\dbinom{a-1}{(a-1)/2} & \text{if } a \text{ is odd.} \end{cases}$$

What is interesting is the simplicity of this formula, and if you add them together in pairs and multiply, it gets even simpler. If $a>1$, then

$$(I(a)+I(a+1))(I(a)+I(a-1)) \;=\; \frac{\pi}{2a}.$$

Tommy Moorhouse

First we can rewrite the integrand by multiplying the numerator and denominator by $\sqrt{1-x^{1/a}}$ to get

$$I(a) \;=\; \int_0^1 \frac{1-x^{1/a}}{\sqrt{1-x^{2/a}}}dx.$$

Next set $x=y^{a/2}$ so that $dx = ay^{a/2-1}dy/2$. We arrive at the integrals

$$I_1(a) \;=\; \frac{a}{2}\int_0^1 y^{a/2-1}(1-y)^{-1/2}dy$$

and

$$I_2(a) \;=\; -\frac{a}{2}\int_0^1 y^{a/2-1/2}(1-y)^{-1/2}dy$$

with $I(a)=I_1(a)+I_2(a)$. Integrals of this type were first investigated by Euler and Legendre, and the standard definition is

$$\beta(p,q) \;=\; \int_0^1 y^{q-1}(1-y)^{p-1}dy$$

(observe that $\beta(p,q)=\beta(q,p)$.) Thus

$$I_1(a) \;=\; \frac{a}{2}\,\beta\left(\frac{1}{2},\frac{a}{2}\right),\;\; I_2(a) \;=\; -\frac{a}{2}\beta\left(\frac{1}{2},\frac{a}{2}+\frac{1}{2}\right).$$

Using

$$\beta(p,q) \;=\; \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

we see that

$$I(a) \;=\; \frac{a}{2}\,\Gamma(1/2)\left(\frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)} - \frac{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}+1\right)}\right).$$

We can use the standard gamma function identities $z\Gamma(z) = \Gamma(z+1), \Gamma(n+1) = n!$ and $\Gamma(1/2) = \sqrt{\pi}$ to work out the various cases and the result follows. For example, when $a = 2M$ for some integer M we have

$$I_1(a) \;=\; M\Gamma(1/2)\frac{\Gamma(M)}{\Gamma\left(M+\frac{1}{2}\right)} \;=\; \frac{2^M M!}{(2M-1)(2M-3)\cdot\cdots\cdot 3\cdot 1}.$$

Multiplying top and bottom by $2^M M!$ we find

$$I_1(a) \;=\; 2^{2M}\frac{M!^2}{(2M)!} \;=\; \frac{1}{B(a)}$$

with $B(a)$ as defined in the problem. The other cases are similar.

Problem 305.2 – Factorial ratio

Tony Forbes

Show that if n is a non-negative integer and $n\#$ denotes the product of the primes less than or equal to n, then

$$\frac{1}{(n!)^n}\,\frac{(n^2)!}{(n^2)\#\,(n^2/2)\#}$$

is a rational number with denominator n or 1 depending on whether n is or is not prime. For example, the first few numbers are

$$1,\ 1,\ \frac{1}{2},\ \frac{4}{3},\ 10,\ \frac{6048}{5},\ 26078976,\ \frac{375756226560}{7},\ \ldots.$$

Solution 300.5 – Sum

Show that

$$\frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{6\cdot 7\cdot 8\cdot 9}+\frac{1}{11\cdot 12\cdot 13\cdot 14}+\dots$$
$$= \frac{\sqrt{5}\pi}{150}\left(\sqrt{5+2\sqrt{5}}-3\sqrt{5-2\sqrt{5}}\right).$$

Tommy Moorhouse

First we use a standard result from complex analysis, namely that the partial fraction expansion of $\pi\cot\pi\tau$ is

$$\pi\cot\pi\tau \;=\; \frac{1}{\tau}+\sum_{m=-\infty}^{\infty *}\left(\frac{1}{\tau+m}-\frac{1}{m}\right),$$

where the star indicates that the term in $m=0$ is omitted from the sum. Next we write

$$\sum_{k=0}^{\infty}\frac{1}{(5k+1)(5k+2)(5k+3)(5k+4)}$$
$$= \frac{1}{5^4}\sum_{k=0}^{\infty}\frac{1}{(k+1/5)(k+2/5)(k+3/5)(k+4/5)}.$$

We can expand the general term using partial fractions:

$$\frac{1}{(k+1/5)(k+2/5)(k+3/5)(k+4/5)}$$
$$= \frac{25}{2}\left(\frac{1}{(k+1/5)(k+4/5)}-\frac{1}{(k+2/5)(k+3/5)}\right).$$

Taking the first term and expanding once more in partial fractions gives

$$\sum_{k=0}^{\infty}\frac{1}{(k+1/5)(k+4/5)} \;=\; \frac{5}{3}\sum_{k=0}^{\infty}\left(\frac{1}{k+1/5}-\frac{1}{k+4/5}\right)$$
$$= \frac{5}{3}\left(\frac{1}{1/5}+\sum_{k=1}^{\infty}\frac{1}{k+1/5}-\sum_{k=1}^{\infty}\frac{1}{k-1/5}\right).$$

On rearranging the terms, the bracketed expression on the right hand side becomes

$$\frac{1}{1/5}+\sum_{k=-\infty}^{\infty *}\left(\frac{1}{k+1/5}-\frac{1}{k}\right) \;=\; \pi\cot(\pi/5)$$

(we add and subtract the terms $1/k$ as the convergence of the sum requires the grouping given above). Similarly

$$\sum_{k=0}^{\infty} \frac{1}{(k+2/5)(k+3/5)} \;=\; 5\pi \cot(2\pi/5).$$

Thus

$$\sum_{k=0}^{\infty} \frac{1}{(5k+1)(5k+2)(5k+3)(5k+4)} \;=\; \frac{\pi}{50}\left(\frac{5}{3}\cot\frac{\pi}{5} - 5\cot\frac{2\pi}{5}\right).$$

To complete the solution we need expressions for $\cot(\pi/5)$ and $\cot(2\pi/5)$. One way to find them is to use $\cos(5\pi/10) = 0$, leading to a polynomial in $\cos(\pi/10)$ which we can use to find $\cos(\pi/5) = (1+\sqrt{5})/4$. Some fairly straightforward algebra involving trigonometric identities then gives

$$\cot(\pi/5) \;=\; \frac{1}{\sqrt{5}}\sqrt{5+2\sqrt{5}}$$

and

$$\cot(2\pi/5) \;=\; \frac{1}{\sqrt{5}}\sqrt{5-2\sqrt{5}}$$

leading to the desired form of the solution

$$\sum_{k=0}^{\infty} \frac{1}{(5k+1)(5k+2)(5k+3)(5k+4)} \;=\; \frac{\sqrt{5}\pi}{150}\left(\sqrt{5+2\sqrt{5}} - 3\sqrt{5-2\sqrt{5}}\right).$$

Problem 305.3 – 262144

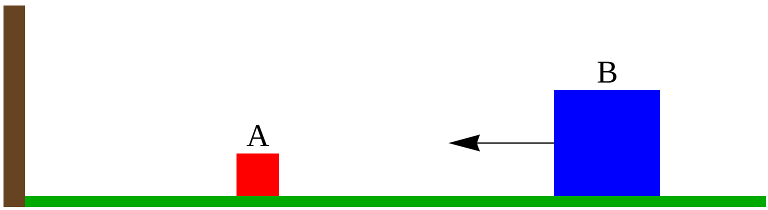
This was discovered on a social media platform by Jeremy Humphries:

$$\sqrt{2^{6^{2^{1^{4^4}}}}} \;=\; 262144.$$

Can you find more examples? (If you use a calculator and instead of 262144 you get something like 79228162514264337593543950336, it is possible that your device does not know how to do stacked exponents correctly.)

Problem 305.4 – Collisions

There is a wall on the left. To the right of the wall there is a stationary mass, A, of 1 kg, as in the picture. Mass B of m kg approaches from the right at non-zero speed and eventually collides with A. Mass A now moves towards the wall, rebounds from it and then heads towards B for another encounter. And so on. The experiment stops when B is travelling to the right with sufficient speed to avoid a further collision with A. Assume all this is taking place on level ground, there is no friction, and collisions are perfectly elastic. Also you might want to ignore mutual gravitational attraction between A, B and the wall.



Show that for large m, the total number of collisions is $\sqrt{m}\,\pi$, at least approximately.

Thanks to Piers Myers and the London South Bank University Maths Study Group,

`https://www.theoremoftheday.org/MathsStudyGroup/index.html`,

for introducing me (TF) and others to this problem. In particular, we, Piers and his audience, observed that if you put $m = 10^{200}$, the mechanics of the problem provides a convenient way of calculating π to about 100 decimal places in a few seconds and without the need to perform extensive arithmetical calculations on a computer. Just count the collisions.

Problem 305.5 – Ratio of two sums

Tony Forbes

Show that

$$\frac{1+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}-\frac{1}{13}-\frac{1}{17}-\frac{1}{19}-\frac{1}{23}+\frac{1}{25}+\frac{1}{29}+\frac{1}{31}+\frac{1}{35}-\ldots}{1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}+\frac{1}{19}-\frac{1}{23}+\ldots} = \sqrt{2}.$$

Re: Snooker without friction

Tony Forbes

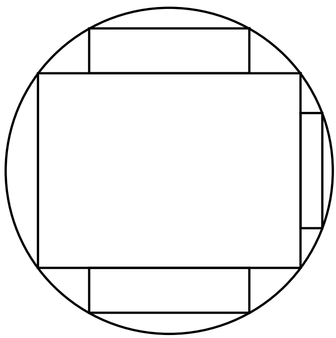
While I was editing 'Problem 305.4 – Collisions' (page 10) it occurred to me that I could simplify a similar problem that has been bothering me. Recall that Kim Forbes and I wrote 'Solution 266.2 – Snooker without friction', M500 **302**, where the original Problem 266.2 asked, 'Is it sensible to play snooker on a frictionless table? If it is, devise a strategy for winning a frame in a finite amount of time.'

The difficulty with striking a spherical object with a snooker cue is that it is not obvious (to me) what happens when there is no friction between the ball and the table. Does the ball roll or slide? Perhaps there is a combination of both, together with some spinning. I can imagine that it depends sensitively on exactly how the ball is stuck. I think we have to assume there is still friction between the cue tip and the surface of the ball. Goodness knows what might happen if there isn't. Unfortunately I don't think I can begin to work out how to set up and solve the relevant equations.

However, it seems to me that we can greatly simplify the situation if we use *cylindrical* balls instead. Without the possibility of unrestricted spinning, the behaviour of the (ball, cue) system should be much easier to model—provided we insist that the cylinder always has one of its flat surfaces in contact with the table. Then we need to consider only translation in the horizontal plane and spinning about a vertical axis. To maintain compatibility with the standard game, the height of the cylinder should 0.035 m so that it has the same weight as before.

Problem 305.6 – Four rectangles

Tony Forbes



This is like Problem 269.2 – Two rectangles except that the number of rectangles is different. Four rectangles are packed inside a circle of radius 1 according to the pattern indicated on the right. What is the largest area they can occupy? A numerical solution would be acceptable.

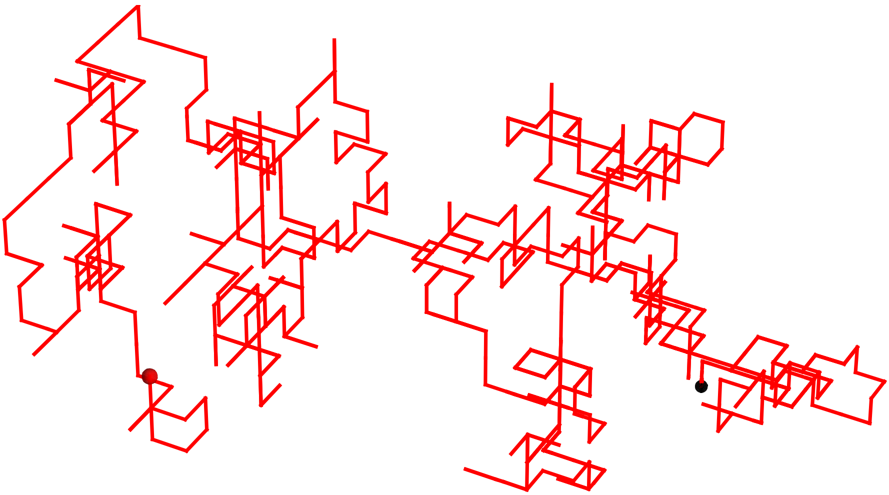
We would be interested if you can find a better way to arrange the rectangles.

Problem 305.7 – 3D Random walk

Tony Forbes

You start at $(0,0,0)$. At each tick of the clock you take one step in one of the directions $\{(-1,0,0),\ (1,0,0),\ (0,-1,0),\ (0,1,0),\ (0,0,-1),\ (0,0,1)\}$ chosen at random with probability $1/6$. This can be done by throwing a die marked {west, east, south, north, down, up}, say. Show that the probability of you eventually returning to $(0,0,0)$ is

$$1 - \frac{16\ \sqrt{2/3}\ \pi^3}{\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{5}{24}\right)\Gamma\left(\frac{7}{24}\right)\Gamma\left(\frac{11}{24}\right)} \ \approx\ 0.340537.$$



Compare with the situation in one or two dimensions. In each case the probability of returning to the origin after a finite number of steps is 1.

A slight generalization of the one-dimensional case might be of interest to those who think they can make money by predicting the outcome of coin tosses even when the payout is worse than 1:1. Starting with any positive integer, a random walk consisting of steps of ± 1 chosen uniformly at random will, with probability 1, eventually reach 0. The gambler will be ruined. On the other hand, the same applies if you start with a negative integer. The gambler will be delighted. *Warning.* Casinos do not offer unlimited credit. *In real life the scheme just described will not work!*

While we are on the subject of walking in three dimensions, here is something curious and interesting.

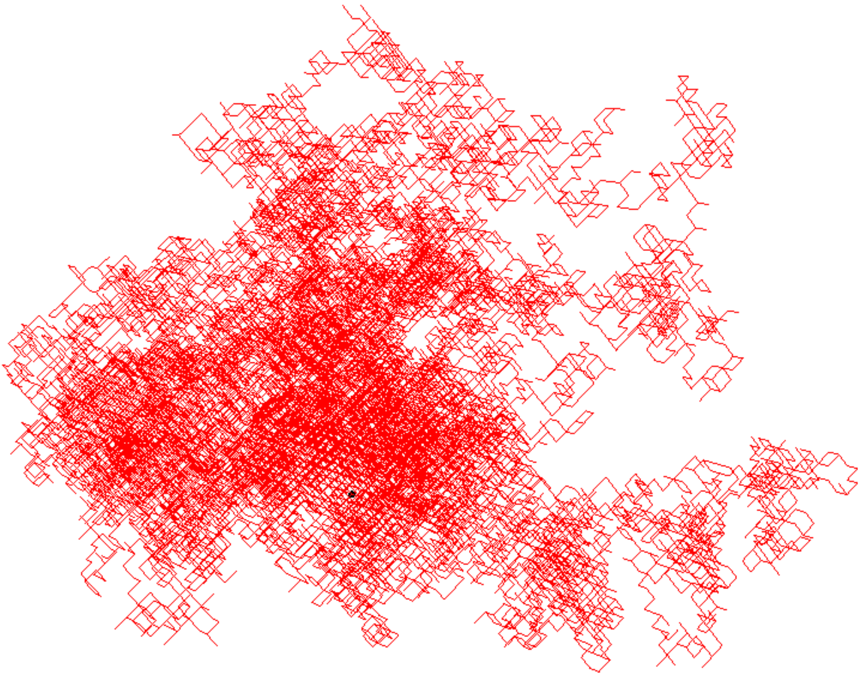
Start at $(0, 0, 0)$. Then for each p as p runs through the primes greater than 7, take a unit step in one the six axis directions chosen according to the value of p modulo 7:

(1, east), (2, west), (3, north), (4, south), (5, up), (6, down).

So the first twelve steps would take you to the following positions.

11 south	$(0, -1, 0)$	13 down	$(0, -1, -1)$	17 north	$(0, 0, -1)$
19 up	$(0, 0, 0)$	23 west	$(-1, 0, 0)$	29 east	$(0, 0, 0)$
31 north	$(0, 1, 0)$	37 west	$(-1, 1, 0)$	41 down	$(-1, 1, -1)$
43 east	$(0, 1, -1)$	47 up	$(0, 1, 0)$	53 south	$(0, 0, 0)$

Observe that there are returns to $(0, 0, 0)$ at steps 4, 6, 12, and if you continue, 46, 48, 52, 380 and 390. However, after 390 there is a surprisingly (to me) long wait until the next return, which occurs at 23750.



Solution 300.4 – Integral

Show that

$$\int_0^{\pi/2} \left((\sin x)^{4/5} + (\cos x)^{4/5}\right)^5 \, dx \;=\; \left(\frac{53\sqrt{5}}{40} - \frac{2}{5}\right)\pi.$$

According to MATHEMATICA, $\int_0^{\pi/2} \left((\sin x)^{(n-1)/n} + (\cos x)^{(n-1)/n}\right)^n dx$ probably has an elementary and not too complicated evaluation for positive integer n only when $n = 1$ (trivial), $n = 3$ (Problem 299.2), $n = 5$ and possibly $n = 7$.

Tommy Moorhouse

In this solution we make use of several gamma function identities. These can be found in many standard texts on complex analysis. We first find a general expression for integrals of this type. Expanding gives a sum of terms

$$\left(\cos(x)^{(n-1)/n} + \sin(x)^{(n-1)/n}\right)^n$$
$$= \sum_{k=0}^{n} \cos(x)^{k(n-1)/n} \sin(x)^{(n-k)(n-1)/n} \binom{n}{k},$$

which can be integrated using a result from the theory of the gamma function:

$$\int_0^{\pi/2} \cos(x)^{a-1} \sin(x)^{b-1} dx \;=\; \frac{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}(a+b)\right)}.$$

The integrated sum becomes

$$I_n \;=\; \frac{1}{2\Gamma\left(\frac{n+1}{2}\right)} \sum_{k=0}^{n} \binom{n}{k} \Gamma\left(\frac{1}{2}\Big(n - k\Big(1 - \frac{1}{n}\Big)\Big)\right) \Gamma\left(\frac{1}{2}\Big(1 + k\Big(1 - \frac{1}{n}\Big)\Big)\right).$$

As a check take I_1 ($n = 1$), when there are two terms in the sum, both equal:

$$\int_0^{\pi/2} 2dx \;=\; \frac{1}{2\Gamma(1)} \left(2\Gamma\Big(\frac{1}{2}\Big)^2\right) \;=\; \pi,$$

which is what we expect. The case $n = 2$ involves the term $\Gamma(3/4)^2$ which is not reducible to standard functions.

Turning to the case $n = 5$ we see that we have six terms in the sum, but that they pair up to give

$$\frac{1}{\Gamma(3)} \left(\Gamma\Big(\frac{5}{2}\Big)\Gamma\Big(\frac{1}{2}\Big) + 5\Gamma\Big(\frac{21}{10}\Big)\Gamma\Big(\frac{9}{10}\Big) + 10\Gamma\Big(\frac{17}{10}\Big)\Gamma\Big(\frac{13}{10}\Big)\right).$$

Now we can use some gamma function identities. First $z\Gamma(z) = \Gamma(z+1)$, so $\Gamma(5/2) = 3\Gamma(3/2)/2 = 3\Gamma(1/2)/4$. Similarly $\Gamma(21/10) = 11\Gamma(1/10)/100$, $\Gamma(13/10) = 3\Gamma(3/10)/10$ and so on. We also have $\Gamma(3) = 2$. Finally we need

$$\Gamma(z)\Gamma(1-z) \;=\; \frac{\pi}{\sin(\pi z)}.$$

Putting it all together we get

$$I_5 \;=\; \frac{\pi}{2}\left(\frac{3}{4} + \frac{55}{100\sin(\pi/10)} + \frac{21}{10\sin(3\pi/10)}\right).$$

We need an expression for $\sin(\pi/10)$ and we can use as a starting point the identity $\cos(5\pi/10) = 0$. Using $\cos(x) = (e^{ix} + e^{-ix})/2$ we find that $\cos^3(x) = (\cos(3x) + 3\cos(x))/4$ and $\cos^5(x) = (\cos(5x) + 5\cos(3x) + 10\cos(x))/16$. Substituting into $\cos(5x) = 0$ we find

$$\cos(x)\left(16\cos^4\left(\frac{\pi}{10}\right) - 20\cos^2\left(\frac{\pi}{10}\right) + 5\right) \;=\; 0.$$

The nontrivial solution we need is

$$\cos^2\frac{\pi}{10} \;=\; \frac{1}{8}(5+\sqrt{5}),$$

and we can use standard trigonometric identities to find

$$\sin\frac{\pi}{10} \;=\; \frac{1}{2\sqrt{2}}\sqrt{3-\sqrt{5}} \;=\; \frac{1}{4}(1-\sqrt{5}),$$
$$\sin\frac{3\pi}{10} \;=\; \frac{1}{2\sqrt{2}}\sqrt{3+\sqrt{5}} \;=\; \frac{1}{4}(1+\sqrt{5}).$$

This gives

$$I_5 \;=\; \frac{\pi}{2}\left(\frac{3}{4} + \frac{44}{20(\sqrt{5}-1)} + \frac{84}{10(\sqrt{5}+1)}\right)$$

which is

$$I_5 \;=\; \left(\frac{53\sqrt{5}}{40} - \frac{2}{5}\right)\pi.$$

The case $n = 7$ is quite similar to $n = 5$, in that the integral can be expanded in terms such as $1/\sin(\pi/14)$, but in this case the equation for $\cos^2(\pi/14)$ is a cubic. We know that the solution can be expressed in terms of radicals (square roots, cube roots, etc.) So in this sense I_7 is not too complicated. Greater values of n give rise to polynomials of greater degree whose solutions may not be expressible in terms of radicals.

Dice

Jeremy Humphries

Milena's cube problem [M500 **304**, Problem 304.4 – Cube shadow] reminded me that I was recently having a look at Martin Gardner's *The Ambidextrous Universe*, which of course I first read many years ago.

There's a stack of three dice, with a coin on top to hide the top face of the top one. You can see two adjacent sides of each die, as shown on the right. The task is to name the value of the top face of each die. Gardner says that not one person in a thousand can do this.

I was thinking a bit more about this, and I wonder if Gardner's 'not one person in a thousand' is a bit wide of the mark. There are only four options for each die, so you should get one correct answer in 64 from pure guesswork. I expect what he means, or what he takes a wild guess at, is that not one person in a thousand knows the rules for labelling the faces of dice.

Gardner says in the Preface to the First Edition that the first draft of the book was looked at by Richard P. Feynman, who made numerous good suggestions. One of which was not that 4^3 is 64, it seems.

Solution to 2B

Ted Gore

I seem to remember reading in a copy of M500 some time ago, the suggestion that quotes from Shakespeare might be put into mathematical form.

I decided to do it with "To be or not to be ..."

Let "To be" be represented by $2B$, "or" by $+$, following the usage in probability where $\Pr(X \text{ or } Y) = \Pr(X) + \Pr(Y)$, "not to be" by either the additive or multiplicative inverse of "to be", and the some one whose existence is being discussed by 1.

We obtain two possibilities:

$$1 = 2B + (-2B) \;\Rightarrow\; 1 = 0,$$
$$1 = 2B + 1/(2B) \;\Rightarrow\; 2B = 1/2 \pm i\sqrt{3/4}.$$

Taken together these suggest either that one is nothing or that living ("to be") is a complex process with an imaginary component.

Problem 305.8 – Integral

Tony Forbes

Show that if t is an odd positive integer, then

$$\int_0^1 \frac{\sqrt{1-x^{2/t}}}{\sqrt{1+x^{-2/t}}} \;=\; A + B\pi,$$

where A and B are rational. For example,

$$\int_0^1 \frac{\sqrt{1-x^{2}}}{\sqrt{1+x^{-2}}} = \frac{\pi}{4} - \frac{1}{2}, \qquad \int_0^1 \frac{\sqrt{1-x^{2/3}}}{\sqrt{1+x^{-2/3}}} = \frac{6}{4} - \frac{3\pi}{8},$$

$$\int_0^1 \frac{\sqrt{1-x^{2/5}}}{\sqrt{1+x^{-2/5}}} = \frac{5\pi}{8} - \frac{5}{3}, \;\ldots, \quad \int_0^1 \frac{\sqrt{1-x^{2/27}}}{\sqrt{1+x^{-2/27}}} = \frac{4608}{1001} - \frac{11583\pi}{8192}, \;\ldots.$$

Proverbs and well-known sayings

Continuing a trend we started some time ago (M500 **192**, **194**, **211**, **280**), here are some more well-known sayings and proverbs that we have translated for easier understanding by M500 readers.

1. It is not excessively difficult to recover the elements copper and zinc from sewage treatment facilities.
2. (a) It is unwise to travel with your cash, credit cards, travel documents, passport, driving licence, mobile telephone and possibly other valuable items all in the same handbag. (b) It is unwise not to have distinct passwords for accessing your various accounts. (c) It is unwise not to have more than one computer storage device for the safe keeping of all your machine-readable data. (d) It is unwise not to employ a variety of shopping bags when purchasing fragile consumables from supermarkets.
3. Accurate dessert appraisal can only be achieved by personal consumption.
4. It is impossible for persons reliant on solicited unconditional handouts to enjoy vacations on ocean-going vessels.
5. Give him 0.0254 m and he will take 1609.344 m.
6. Making repairs to perfectly functioning equipment is an absurdity.
7. Human–rodent collaboration frequently results in deviations from the intended strategy despite meticulous attention to detail.

Where there's muck there's brass. Don't put all your eggs in one basket. The proof of the pudding is in the eating. Beggars can't be cruisers. Give him an inch and he will take a mile. If it ain't broke, don't fix it. The best-laid schemes o' mice an' men gang aft agley.

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04.00

Front cover The graph of a permutation P of the integers $\{1, 2, \ldots, 25\}$ and the unique pair of Young tableaux with which P is associated by the Robinson–Schensted–Knuth correspondence. The tableaux are 5×5 matrices, and hence P has no monotonic subsequence of length 6. See *Theorem of the Day*, number 143, `https://www.theoremoftheday.org/GroupTheory/RSK/TotDRSK.pdf`. Can you work out how to construct the graph from the matrices?