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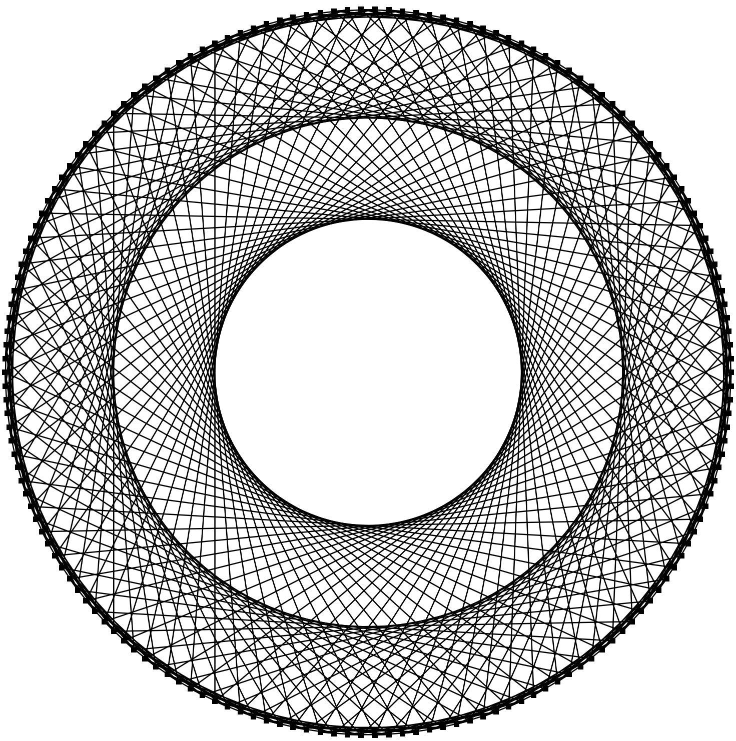
M500 295



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The Revision Weekend is a residential Friday to Sunday event providing revision and examination preparation for both undergraduate and postgraduate students. For details, please go to the Society's web site.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, please go to the Society's web site.

Advice for authors We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to the Editor, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. For more information, go to m500.org.uk/magazine/ from where a LaTeX template may be downloaded.

M500 Winter Weekend 2021

In view of the Covid-19 pandemic the M500 Society has regretfully decided to cancel the Winter Weekend 2021. We have every hope and expectation of running it in January 2022

Measuring the speed of light

Alan Davies

1 Introduction

Einstein's famous equation, $E = mc^2$, relates energy and mass. To calculate the energy we need to know c, the speed of light. Most early observers thought that light was transmitted instantaneously, i.e. with infinite speed. Galileo was the first to attempt to measure the finite value. He was hopelessly wrong but it started a story which we shall follow. After some discussion of the early ideas we shall describe the first reasonably accurate measurement by Ole Rømer using astronomical observations and follow with the first time-of-flight experiments of Fizeau *et al.* These then lead on to the nineteenth century and James Clerk Maxwell's development of light as an electromagnetic wave. The next part of the story then uses ever more accurate measurements of the electromagnetic parameters leading to the super accurate measurements using the wave nature of light and interferometry. We arrive in 1983 when the measurement of the speed was more accurate than the value of the standard metre of platinum-iridium held in Paris. Consequently the value was set at 299,792,458 $\mathrm{m\,s^{-1}}$ and the metre was redefined. Usually c is given using $\mathrm{km\,s^{-1}}$ and that's what we shall use rather than the SI value.

2 Early thoughts on the speed of light

Prior to the seventeenth century most observers thought that light was transmitted instantaneously. However, some observers thought that light speed was finite. The Greek philosopher Empedocles (495BC–435BC) is best known as the Western World's originator of the classical elements earth, air, fire and water. He suggested "it must take time for light to reach the Earth from the Sun" and so implies finite light speed. Euclid (*c.*325BC–*c.*265BC) is best known for his work on geometry as presented in the *Elements*. However, he also wrote a text, *Optics*, in which he assumes that light travels in straight lines and he can use geometrical ideas to describe perspective. He didn't have anything to say about the speed of light. Like many of his contemporaries, he assumed that light travels from the eye to the object so that it can be seen. Ptolemy (*c.*85–*c.*165) built on Euclid's ideas producing a text on optics in which he describes the processes of reflection and refraction. Hero (*c.*10–*c.*70) also believed that light travelled from the eye and that since we can see stars as soon as we open our eyes, light speed must be infinite. He proposed that light travels along the shortest path between two points so concluding that it travels in straight lines.

We move on some nine hundred years to scholars of the Golden Age of Islamic science such as Ibn al-Haytham (Alhazan) (*c*.965–*c*.1040) who believed that light did not come from the eye but from a source which illuminates the object. He was the originator of what we call the *Scientific Method* – systematic observation, measurement and experiment together with formulation, testing and modification of hypotheses. He used this approach to show that light travelled in straight lines from an object to the eye. Alhazan believed that light speed is finite and that it travels more slowly in water than in air.

The Franciscan friar, Roger Bacon (1214–1292), probably best known for measuring the angle of the rainbow as 42^0, built on Alhazan's work and also proposed that light speed is finite. The thirteenth century Polish physicist Witelo (*c*.1230–*c*.1275) was largely unknown in Western Europe but he made important contributions to optics. He believed that light travelled at infinite speed in a vacuum and slowed down when entering a denser medium.

The seventeenth century saw superstars such as Johannes Kepler (1571–1630) and René Descartes (1596–1650) proposing that the speed of light was infinite. Kepler argued that since there was nothing in space to slow it down, light would travel with infinite speed. Descartes had philosophical arguments as to why the speed of light was infinite. He also believed that light increased speed as it entered denser media. It was Galileo Galilei (1564–1642) who believed that light speed was finite and he described an experiment to measure this finite speed. Two lantern bearers would stand on hilltops some distance apart, the lanterns being covered. The first lantern bearer would uncover the lantern and on seeing the signal the second bearer would uncover that one. The time difference would allow the calculation of the speed of light. Galileo claimed to have performed the experiment but was unable to determine the speed. He did, however, conclude that the speed of light must be extraordinarily rapid, at least ten times the speed of sound. Similar attempts by the Dutch astronomer Isaac Beeckman (1588–1637) using a gunpowder flash were equally unsuccessful. The French mathematician Pierre de Fermat (1607–1665) certainly believed that light speed was finite and depended on the material through which it passed. It is this property which is responsible for refraction and he determined the appropriate law using his *Principle of Least Time*, which states that 'light travels along that path for which the time of travel is a minimum'. He was able to use this principle to develop the law in the well-known form

$\sin i / \sin r = k$, where k is the refractive index, the ratio of the speeds of light in the two media.

It is worthwhile saying something here about refraction. Ptolemy developed a quadratic law which was not very accurate. The concept of trigonometric ideas was just developing in his time and he knew them but couldn't relate them to refraction. Kepler had an approximation to the sine law in 1620 which was good for small angles and Willebrord Snell (1580–1626) used geometrical arguments to obtain the eponymous sine law in 1621. It is interesting to note that the English mathematician Thomas Harriot (1560–1621) developed the sine law in 1601 but he did not publish it. Also the Islamic scholar Abu Said Ibn Sahl (940–1000) published *On the burning instruments* in which he developed what we would recognise as the sine law. This manuscript was translated by Alhazan but he missed the correct law. He also translated Ptolemy's work and perpetuated the wrong law for some six hundred years.

3 Measuring the speed of light in the 17th and 18th centuries

Terrestrial measurements would always be very difficult because distances are too short for times to be measured with devices available at the time. Light takes about eight minutes to travel from the Sun to the Earth and so the Sun-Earth distance provides a suitable baseline for light speed measurements.

A very serious navigation problem at this time was the necessity to measure longitude at sea and it was suggested that one possible method could use the eclipses of Jupiter's innermost moon, Io. It was known that Io orbits once every $42\,{}^{1}\!/_{2}$ hours and it was thought that a table of eclipses could be used as a navigational aid. The Danish astronomer Ole Christensen Rømer (1644–1710) was working at the Royal Observatory in Paris observing these eclipses. Rømer noticed that when the Earth is between the Sun and Jupiter the time of eclipses is shorter than when the Earth is on the opposite side of its orbit. The period of Io was well-known at that time so the discrepancy must be due to something else. Since the distances d_1 and d_2, see Figure 1, are such that $d_1 < d_2$ the only explanation is that light speed is finite. Observing Io's eclipses has many difficulties, e.g. Jupiter is a large planet and often obscures the view from Earth, so Rømer needed about thirty observations before he was convinced and presented his results to the Paris Academy of Sciences in 1676. He didn't actually give a value for the speed of light but he did say that it takes about twenty-two minutes to cover the diameter of the Earth's orbit; i.e. it takes light eleven minutes to travel from the Sun to the Earth.

Rømer's findings were by no means universally accepted. In particular the Director of the Observatory, Giovanni Cassini (1625–1712), was very sceptical and he never accepted that light speed is finite. However, Rømer did have one very strong supporter, Christiaan Huygens (1629–1695). Using Rømer's data, Huygens calculated the light speed to be $16\,^2\!/_3$ Earth diameters per second. With the then-known value of the Sun-Earth distance he obtained the approximate value 214,000 $\mathrm{km\,s^{-1}}$. Compared with the modern values this is in error by more than twenty-five percent and this error is undoubtedly due to the error in the accepted value for the distance from the Sun to the Earth. Nevertheless, Huygens was pleased with the result in that he was interested only in the order of magnitude.

Over the next few decades many more observations were made and in 1817 the French mathematician Jean Baptiste Delambre (1749–1822) published a summary of about a thousand measurements to arrive at a value 303,000 $\mathrm{km\,s^{-1}}$.

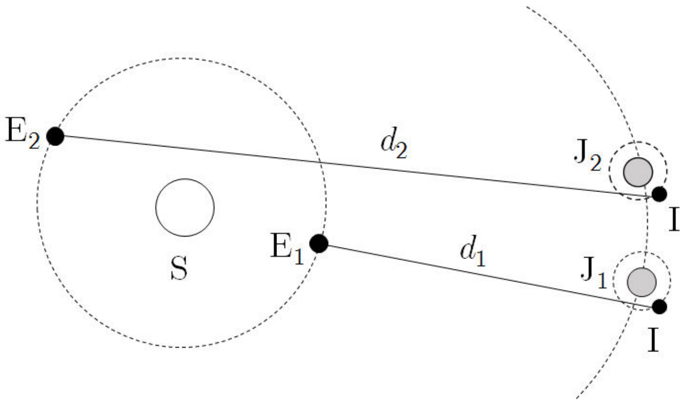


Figure 1: Earth and Jupiter at two points six months apart

Rømer's work was well-accepted in the early eighteenth century. Improvements were made using the transits of Venus in 1761 and 1769 but these were overshadowed by another completely independent form of extra terrestrial measurement in 1728 courtesy of James Bradley (1693–1762). The technique is often compared with walking in the rain. If the rainfall is vertical then, due to horizontal motion, the rain appears to come at an angle. Bradley realised that because the Earth is in motion about the Sun the light from the star would come from an apparent position, see Figure 2.

This apparent motion is called *aberration*. He based his work on the

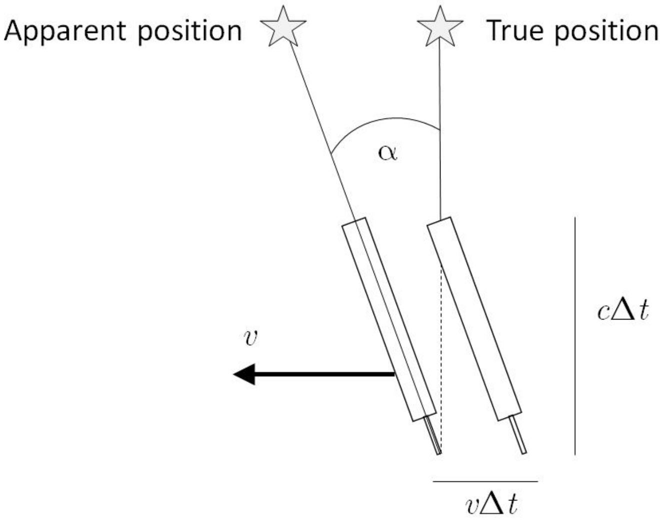


Figure 2: True and apparent position of γ-Draconis

star γ-Draconis and was able to measure the angle α to an accuracy which was amazing for the time. Bradley took observations over a whole year, the value of α varies depending on the relative direction of the Earth's orbital motion compared with the true direction of the star. His value of 20.25 seconds of arc was his best estimate of the average value. *c.f.* the modern value 20.47 seconds of arc, his error is approximately one percent.

If we consider the diagram in Figure 2 we see that

$$\tan\alpha \;=\; \frac{v}{c} \;=\; \beta. \tag{1}$$

At that time the interest seemed to be finding the time, T_{SE}, that it takes light to travel from the Sun to the Earth and Bradley had the brilliant idea that this time could be calculated without knowing the speed of light nor the Sun-Earth distance, d. The calculation goes as follows:

$$T_{ES} \;=\; \frac{d}{c}. \tag{2}$$

The speed, v of the Earth in its orbit about the Sun is given by

$$v \;=\; \frac{2\pi d}{n}, \tag{3}$$

where the number of seconds in a year is given by n.

Hence from equations (1), (2) and (3) it follows that

$$T_{ES} = \frac{1}{c}\frac{nv}{2\pi} = \frac{\beta n}{2\pi}$$

and with $\beta = \tan\alpha \approx 9.817 \times 10^{-5}$ and $n = 3.536 \times 10^{7}$ we find

$$T_{ES} = 492,$$

i.e. 8 min. 12 sec., an astoundingly accurate result compared with the modern value 8 min. 20 sec. With the modern value of d (1.493×10^{11} m) we obtain the speed of light as 303,500 $\mathrm{km\,s^{-1}}$.

4 Into the 19$^{\mathrm{th}}$ century

During the seventeenth and eighteenth centuries timing devices were not accurate enough to make a terrestrial measurement of the value of c. Distances would be just a few kilometres and the so-called terrestrial time-of-flight methods would require measuring times of the order of 10^{-5} s. It took a brilliant idea by the French physicist Hippolyte Fizeau (1819–1896) to develop a device which, while not actually measuring a time interval, enabled him, in 1847, to calculate the time taken for a beam of light to travel across the rooftops of Paris. The basic idea is that a beam of light is sent passing through a gap in a rotating toothed wheel. This rotating wheel 'chops' the light into short pulses and each pulse is reflected back. The rotation is set so that the returning pulse is blocked by the adjacent tooth. The apparatus was set up in his father's house in Suresnes and the pulse was reflected from a mirror in Montmartre 8.63 km away. There are certain technicalities which don't affect us here, e.g. the light beam is concentrated using a lens system and the outgoing and incoming rays are separated using a half-silvered mirror.

Fizeau's wheel had 720 teeth rotating at 12.6 rps so the time for the return journey is given by

$$\frac{1}{2 \times 720 \times 12.6} \approx 5.5 \times 10^{-5}, \tag{4}$$

i.e. a time of approximately five one-hundred-thousandths of a second. No timing device of that era could come close to measuring that.

Since the distance between the mirrors is 8.63 km, (4) gives the speed of light as 313,000 $\mathrm{km\,s^{-1}}$.

The idea of using a rotating device with rotational speeds in excess of 500 rps to 'chop' the light beam was further developed in 1862 by Léon Foucault (1819–1896). He used rotating mirrors to obtain a value 298,000 $\mathrm{km\,s^{-1}}$. In 1879 Albert Michelson (1852–1931) used a rotating prism to obtain a value 299,910 $\mathrm{km\,s^{-1}}$. Foucault and Michelson are probably better known for the *Foucault Pendulum* and the so-called *Michelson Morley experiment* respectively. By the end of the nineteenth century a breakthrough in the understanding of electromagnetic theory led to the possibility of indirect ways of obtaining light speed values. Nevertheless Michelson carried on with time-of-flight measurements obtaining a value 299,796 $\mathrm{km\,s^{-1}}$ (*c.* 1930) in a one-mile long vacuum tube.

5 Electromagnetic waves

In 1821 Michael Faraday (1791–1867) proposed the idea of lines of force leading to the concept of the magnetic field. This was a revolutionary idea in which a magnet affects all space and was in direct opposition to what was then the conventional wisdom: action at a distance in which forces acted instantaneously at the point of application. Many of his contemporaries were sceptical but his 1851 experiment with iron filings provided a very compelling demonstration (Faraday 1851). Usually at this stage of discussion of the development of the understanding of electromagnetism we would jump directly to James Clerk Maxwell (1831–1879). However, we acknowledge the pivotal work of George Green (1793–1841). He took the geometrical ideas of the field and cast them in a calculus setting, ripe for exploitation by Maxwell.

In 1861 Maxwell published, in the *Philosophical Magazine*, a set of twenty equations written in component form. He developed the ideas further in his monumental *Treatise* (1873). The four equations, in free space, that are familiar to us now is due to Oliver Heaviside (1850–1925), who wrote them in vector calculus terms as follows:

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \nabla\cdot\mathbf{H} = 0,$$

$$\nabla\times\mathbf{E} = -\mu_0\frac{\partial\mathbf{H}}{\partial t}, \qquad \nabla\times\mathbf{H} = \epsilon_0\frac{\partial\mathbf{E}}{\partial t},$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field vectors respectively and ϵ_0 and μ_0 are the electric permittivity and magnetic permeability respectively of free space.

It is a standard piece of undergraduate work using the properties of the vector operators to show that these equations lead to the following equation:

$$\nabla^2 \mathbf{E} \;=\; \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{5}$$

together with a similar equation for **H**. We see that we have a wave equation with wave speed $a = \left(\sqrt{\epsilon_0 \mu_0}\right)^{-1}$.

At this point we need to say something about units. At Maxwell's time there were two systems of units for electricity, electrostatic and electromagnetic. The wave equation developed by Maxwell involved a speed which is given by $a^2 = k_e / k_m$. This ratio of the electrostatic to the electromagnetic units of electric charge was measured in an 1856 experiment by Wilhelm Weber (1804–1891) and Rudolf Kohlrausch (1809–1858). With their measured values Maxwell found his wave speed to be 310,740 $\text{km}\,\text{s}^{-1}$. He was aware of Fizeau's measurement of c as 313,000 $\text{km}\,\text{s}^{-1}$ and deduced that his wave speed was so close to that of Fizeau that a must be the speed of light. There has been much written about the vagaries of the different systems and we shall not pursue that further here; the interested reader can see the article by Clarke (2019). It is interesting to note here that even though infrared and ultraviolet were discovered by William Herschel (1738–1822) in 1800 and Johann Wilhelm Ritter (1776–1810) in 1801 respectively, Maxwell considered only the speed of visible light.

Radio waves were discovered by Heinrich Hertz (1857–1894) in a series of experiments between 1887 and 1889 and hence provided the evidence needed to establish the wave nature of electromagnetism.

As we move into the twentieth century, measuring the parameters in equation (5) would yield a value for c: μ_0 is defined from the definition of the ampere as it stood prior to 20 May 2019 giving $\mu_0 = 4\pi \times 10^{-7}$ with the SI unit $\text{H}\,\text{m}^{-1}$ and ϵ_0 is measured experimentally with SI unit $\text{F}\,\text{m}^{-1}$. More than twenty different experiments were performed and published using a variety of capacitors to yield values for ϵ_0. In particular in 1907 Edward Rosa (1873–1921) and Noah Dorsey (1873–1959) at the *National Bureau of Standards* (NBS) in Maryland, USA, ran a series of very precise experiments and produced the value 299,788 $\text{km}\,\text{s}^{-1}$, the most accurate value at that time.

The speed c of an electromagnetic wave of frequency f and wavelength λ is given by

$$c \;=\; \lambda f. \tag{6}$$

The so-called cavity resonator working at microwave frequencies was used in the mid twentieth century at the *National Physical Laboratory* (NPL) in Teddington by Louis Essen (1908–1997) and A. C. Gordon–Smith. Using waves of a known frequency and measuring the wavelength of standing waves in the resonator, equation (6) yields a value for c. They obtained a value 299,792 $\text{km}\,\text{s}^{-1}$. A very good description of these methods is given by Essen (1952).

Improved values of wavelength can be obtained by the technique of interferometry in which waves interact and form interference patterns from which very accurate measurements can be obtained. In 1958, Keith Froome (1921–1995) at the NPL used this technique to obtain a value 299,792.5 $\text{km}\,\text{s}^{-1}$. With the development of lasers and atomic clocks in the nineteen-sixties even more accurate values were obtained. In particular, in 1973, Kenneth Evanson (1932–2002) *et al.* at the NBS, obtained the value 299,792.4574 $\text{km}\,\text{s}^{-1}$. In Table 1 we give a brief chronology of the development of the value of the speed of light.

Date	Author	Method	Result ($\text{km}\,\text{s}^{-1}$)	Error ($\text{km}\,\text{s}^{-1}$)
1676	Rømer	Eclipse of Io	214,000	
1726	Bradley	Stellar aberration	301,000	
1849	Fizeau	Toothed wheel	315,000	
1862	Foucault	Rotating mirror	298,000	500
1879	Michelson	Rotating prism	299,910	50
1907	Rosa & Dorsey	Electromagnetic constants	299,788	30
1926	Michelson	Rotating prism	299,796	4
1947	Essen & Gordon–Smith	Cavity resonator	299,792	3
1958	Froome	Interferometer	299,792.5	0.1
1973	Evanson *et al.*	Lasers/atomic clocks	299,792.4574	0.001

Table 1: A brief chronology of the progress, from 1676, in measuring c

The list in Table 1 is a list of the major contributions in terms of new approaches. Raynaud (2013) describes 268 different experiments arranged in twelve categories.

6 Post 1983

In 1967/68, the SI unit of time, the second, which had been defined in terms of the Earth's orbital motion around the Sun, was redefined using a quantum mechanical description of the caesium-133 atom transition frequency, $\Delta\nu$. Up until the nineteen-seventies the uncertainty in the measurement of c was greater than that in the measurement of the metre which was defined by the length of a platinum-iridium bar in Paris. However, by the late nineteen-seventies that had changed and the possibility of using a new definition became possible. By 1983 the following values of c had been obtained:

US: NBS	$299{,}792.4574 \pm 0.0011\ \mathrm{km\,s^{-1}}$,
UK: NPL	$299{,}792.4590 \pm 0.0008\ \mathrm{km\,s^{-1}}$

and the SI value 299,792,458 $\mathrm{m\,s^{-1}}$ was adopted.

This value of c means that we can now define the metre in terms of two non-terrestrial constants, c and $\Delta\nu$. Moving forward, physicists wished to define the seven SI base units in terms of universal constants and these were announced at the *Convocation de la Conférence générale des poids et mesures (26e réunion)*, Versailles, 13–16 novembre 2018. They came into force on 20 May 2019 and that's another story.

7 References

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If you are interested in what happened on 20 May 2019, have a look at [M500 **260**, 14–15] and [M500 **287**, 7] — TF.

Solution 280.6 – Four numbers

Find all solutions in positive integers a, b, c, d of

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} = d.$$

Peter Fletcher

If we run the following MATLAB® code

```
n = 30;
for a=1:n
    for b=a:n
        for c=b:n
            d = a/(b+c)+b/(a+c)+c/(a+b);
            if d-floor(d)<eps;
                [a,b,c,d]
            end
        end
    end
end
```

the output is

$$\begin{array}{cccc} 1 & 1 & 3 & 2 \\ 2 & 2 & 6 & 2 \\ 3 & 3 & 9 & 2 \\ & \vdots & & \vdots \\ 10 & 10 & 30 & 2 \end{array}$$

from which we can see that since the labelling of a, b and c is arbitrary, for $n = 1, 2, \ldots,$

$$\begin{aligned} a = b = n,\ c = 3n,\ d = 2, &\quad \text{or} \\ a = c = n,\ b = 3n,\ d = 2, &\quad \text{or} \\ b = c = n,\ a = 3n,\ d = 2. & \end{aligned}$$

Solution 259.3 – Discriminants

The discriminant of the cubic

$$f(x) \;=\; x^3 + ax^2 + bx + c$$

is

$$\Delta \;=\; a^2b^2 - 4b^3 - 4a^3c + 18abc - 27c^2.$$

Show that the discriminant of

$$g(x) \;=\; 3x^4 + 4ax^3 + 6bx^2 + 12cx + 4ac - b^2$$

is $-6912\Delta^2$.

Peter Fletcher

There is more than one way of finding the discriminant of a polynomial; for example, from

`http://mathworld.wolfram.com/PolynomialDiscriminant.html`,

we can write down the discriminant of $g(x)$ as

$$\Delta_g \;=\; \frac{(-1)^{4\cdot 3/2} R\bigl(g, g'\bigr)}{3} \;=\; \frac{R\bigl(g, g'\bigr)}{3},$$

where the numerator is the resultant of $g(x)$ and its derivative. There's an explanation and another example of the resultant's use here:

`https://www.encyclopediaofmath.org/index.php/Resultant` (the third equation should begin '$g(x) = \ldots$' rather than '$fg(x) = \ldots$'). That page also gives a matrix whose determinant is the resultant.

In our case, $g'(x) \;=\; 12x^3 + 12ax^2 + 12bx + 12c$, so we can write

$$R\bigl(g, g'\bigr) \;=\; \begin{vmatrix} 12 & 12a & 12b & 12c & 0 & 0 & 0 \\ 0 & 12 & 12a & 12b & 12c & 0 & 0 \\ 0 & 0 & 12 & 12a & 12b & 12c & 0 \\ 0 & 0 & 0 & 12 & 12a & 12b & 12c \\ 3 & 4a & 6b & 12c & 4ac - b^2 & 0 & 0 \\ 0 & 3 & 4a & 6b & 12c & 4ac - b^2 & 0 \\ 0 & 0 & 3 & 4a & 6b & 12c & 4ac - b^2 \end{vmatrix}.$$

We could go through a series of row and column operations to make this easier to evaluate, but instead we shall cheat and use Maple. The answer is a horrible-looking eighth-order polynomial in a, b, and c with 13 mixed terms, which we can factor in Maple to give

$$R(g, g') = -20736\bigl(4a^3c - a^2b^2 - 18abc + 4b^3 + 27c^2\bigr)^2.$$

The squared term is obviously $\bigl(-\Delta_f\bigr)^2$, so therefore

$$\Delta_g = \frac{-20736\Delta_f^2}{3} = -6912\Delta_f^2.$$

Problem 295.1 – Sequences

(i) Two persons, A and B, play a game. Player A creates a uniformly distributed random sequence, S_A, of three symbols from $\{0, 1\}$ (by tossing a coin, for example). Player B creates S_B in the same manner. If $S_A = S_B$, the game is void. Otherwise the game's referee generates a long, uniformly distributed random sequence, S, of 0s and 1s (by tossing a coin, for example). The winner is the player whose sequence first occurs in S.

For example, suppose $S_A = (0, 1, 1)$, $S_B = (1, 1, 0)$, which are obviously not equal. So the host generates a random sequence, say $S =$ (0, 1, 1, 0, 1, 0, ...), and the winner is A because $(0, 1, 1)$ occurs in S before $(1, 1, 0)$, but only just. In fact, B is doomed from the first 0, except that both players will die before the outcome is resolved if the pair $(0, 1)$ repeats for too long. Anyway, it's a fair game, would you not agree? But now we change the rules.

(ii) As **(i)** except that the players are allowed to choose S_A and S_B, revealing them only to the referee.

Is the game still fair? In case we have not achieved very much, we change the rules again. ...

(iii) As **(ii)** except that A chooses his[1] sequence first and is obliged to show it to B before B makes his[2] choice.

... And again.

(iv) As **(iii)** except that after B has made his choice, he shows S_B to A, who can then choose a different S_A if he wishes to do so.

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[1] Let's assume A is male ...

[2] ... and B

A child's 'proof' that most numbers are irrational

Mike Grannell

This note was prompted by Ben Mestel's article in M500 **294** concerning proofs of irrationality. There are proofs that almost all real numbers are irrational, but these depend on things like cardinal numbers and measure theory. So how would you convince someone with only basic arithmetic skills and no advanced mathematical knowledge? Well, here is a fairly compelling argument that only relies on the person being able to convert fractions to decimals. Because this is aimed at these people, I will refer to rational numbers as fractions, and real numbers as decimals (I can almost hear you wince). So here is how you should be able to convince a child (or an adult) that most decimals are not fractions in disguise.

Let's start with the example of converting the fraction $\frac{79}{7}$ to decimal form. We begin by calculating the integer part, which in this case is 11 (7s into 79 go 11 times with remainder 2). Thus $\frac{79}{7} = 11 + \frac{2}{7}$. It should be clear that we can always do this sort of thing, so the only fractions that we really need to consider are proper fractions, in this case $\frac{2}{7}$. To convert this proper fraction to a decimal, we write the 2 as 2.00000000... and start the division process, in this case dividing by 7. We get

$$\frac{2}{7} = 0.285714\,285714\,285714\ldots$$

Of course this is a recurring decimal and the reason becomes clear with a bit of thought. There are only 7 possible remainders at each stage in this division, namely $0, 1, 2, 3, 4, 5$ and 6. Once you get a remainder that you have previously encountered, the decimal will start to recur. Here are the details for this particular example. The initial remainder was 2 (that's how we got the fraction $\frac{2}{7}$). Then 7 into 20 gives 2 with remainder 6, 7 into 60 gives 8 with remainder 4, 7 into 40 gives 5 with remainder 5, 7 into 50 gives 7 with remainder 1, 7 into 10 gives 1 with remainder 3, and 7 into 30 gives 4 with remainder 2, which is the remainder with which we started, so the pattern now repeats with quotients $2, 8, 5, 7, 1$ and 4.

The same sort of thing will happen with any denominator, so a denominator 139 would have 139 possible remainders and the decimal will recur in blocks of length at most 139. If a remainder of 0 is ever encountered, all the subsequent quotients and remainders will also be 0, and so the decimal will 'terminate', but we can count this as recurring with digits 0. Thus every fraction can be represented as a recurring decimal.

But now imagine that we have a machine capable of emitting an infinite string of purely random digits from 0 to 9. Put '0.' in front of such a string to get a decimal between 0 and 1. What's the chance that this would be a recurring decimal? Surely it is exceedingly unlikely. So the decimals you get from this process will almost never correspond to proper fractions. Thus most, indeed almost all, decimals are not fractions in disguise. Convinced?

Of course this isn't a proof in the precise mathematical sense. It does rely on a hypothetical random digit generator, which begs for an explanation of what we might mean by an infinite string of random digits. But even so, it is now easy to specify some irrational numbers such as 0.1010010001 . . . . You just need a pattern that doesn't recur in fixed length blocks.

Problem 295.2 – States

Tony Forbes

What is the probability of winning a game of *Hangman* where the words are restricted to the names of six-letter USA states. Assume only one life. Assume also that you and your opponent always play sensibly.

(At the start you choose a 6-letter USA state, S, say, and maybe draw six dashes, _ _ _ _ _ _. Then (∗) your opponent chooses a letter, α, say. If α does not occur in S, the game ends and you win. Otherwise you reveal the position(s) of α in S. If S is identified, the game stops and you lose. Otherwise and the game continues from (∗).)

This is another watered-down version of Problem 184.9 – States; see M500, **184**, **187** and **276**. Apologies for recycling the subject yet again—and it looks like I am not going to shut up until the original problem in issue **184** (where you can choose any USA state, not just 6-letter ones) is completely solved.

For the general problem, I believe we have a complete solution in only the following cases.

(i) You lose if there is only one USA state that matches the word pattern.

(ii) You lose if there are common letters that the solver can safely utilise to home in on the answer, as is illustrated by the 9-letter states (Louisiana, Minnesota, Tennessee, Wisconsin).

(iii) With _ _ _ _ you win with probability 1/3 (Iowa, Ohio, Utah).

(iv) With _ _ _ _ _   _ _ _ _ _ _ you win with probability 1/2 (North Dakota, South Dakota, Rhode Island).

The only unresolved cases occur when there are 6, 7 or 8 letters. For the current problem the states are Alaska, Hawaii, Kansas, Nevada, and just when you are thinking they all have an 'a' in them, Oregon.

Planar graph smash

Tommy Moorhouse

This puzzle explores the smash product between graphs examined in a previous article [M500 **294**]. In particular we consider which smash products give planar graphs. Reference should be made to the earlier article for the terminology required. For convenience we recall that the smash product $G \vee H$ between two graphs G and H is obtained by connecting every vertex of G to every vertex of H by an edge, making no other changes. $G + H$ is the disconnected graph consisting of G and H. To denote that g is a subgraph of G we write $g \lhd G$.

If you have tried constructing smash products you may have noticed that a lot of them can't be drawn on the page without some of the edges crossing. Graphs that can't be so drawn are called nonplanar. We wish to understand which smash products give rise to planar graphs. To this end we need the lemma

Lemma 1 If $g \lhd G$ and $h \lhd H$ then $g \vee h \lhd G \vee H$.

Proof of Lemma 1 Since $g \vee h$ consists of g and h with all the vertices of g joined by an edge to those of h we see that $g \vee h$ is a subgraph of $G \vee H$.

Now we make use of Kuratowski's Theorem (see Chapter 3 of the reference, for example) that states that any nonplanar graph is a supergraph of an extension of the 'utility graph' U or of the complete graph K_5.

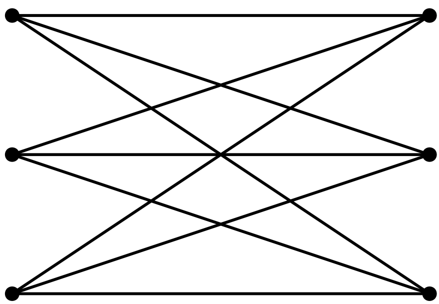


Figure 1: The graph U

We don't worry too much about the extension part here. Now we look at smash products that give rise to U or to K_5. Previously we established that $K_n \vee K_m \simeq K_{n+m}$, so $K_5 \simeq K_1 \vee K_4$ and $K_5 \simeq K_2 \vee K_3$. Observe that $K_1 \simeq N_1$ where N_n is the graph with n vertices and no edges.

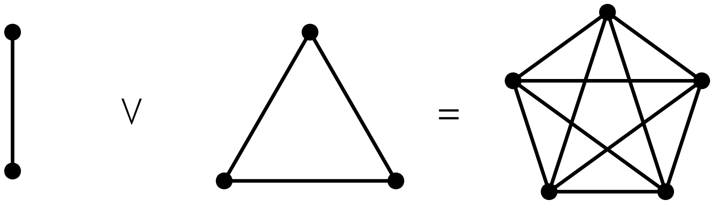


Figure 2: Smash product $K_2 \vee K_3 = K_5$

Prove now that

$$U \simeq N_3 \vee N_3.$$

Deduce that if G and H both have three or more vertices then $G \vee H$ is nonplanar. Thus the planar smash products are among those where at least one of the factors has two or fewer vertices. Can you find a more precise statement? It will be necessary to consider several sub-cases. For example, if T_4 is the graph having one vertex of order three connected to three vertices of order one (see below) then $N_2 \vee T_4$ is nonplanar.

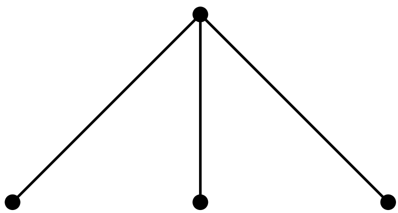


Figure 3: The graph T_4

Reference

Richard J. Trudeau, *Introduction to Graph Theory*, Dover, 1993.

Problem 295.3 – Integers

Let a, b, k be positive integers with $b \geq a \geq k-2 \geq 1$. Show that

$$\frac{(ka)!\,(kb)!}{k^k\,a!\,b!\,(a+b)!}$$

is an integer except possibly when $k = 3$ and $a = 1$. Or find a counter-example.

Solution 228.2 – Arithmetic progression

An arithmetic progression contains only positive integer terms. The sum of the first three terms is 51. The sum of the last four terms is 332. Show that only two arithmetic progressions satisfy these conditions and list those two progressions.

Peter Fletcher

Let the first term be a, the common difference be d and the number of terms be n. Then

$$51 = a + (a + d) + (a + 2d) = 3(a + d)$$

so that $a + d = 17$ and $a = 17 - d$. Also,

$$\begin{aligned} 332 &= (a + [n - 4]d) + (a + [n - 3]d) + (a + [n - 2]d) + (a + [n - 1]d) \\ &= 4a + 4nd - 10d. \end{aligned}$$

Substituting for a, we get

$$332 = 4(17 - d) + 4nd - 10d = 68 + 4nd - 14d$$

so that

$$264 = 2d(2n - 7) \quad \text{and} \quad 132 = d(2n - 7),$$

which gives

$$d = \frac{132}{2n - 7}.$$

The denominator here is odd, so we are looking for odd factors of $132 = 2 \cdot 2 \cdot 3 \cdot 11$, which are 3, 11 and 33.

If $2n - 7 = 3$ then $2n = 10$ and $n = 5$. Also $d = 44$ and $a = 17 - 44 = -27$, which is not allowable because we are told that the arithmetic progressions we want only have positive terms.

If $2n - 7 = 11$ then $2n = 18$ and $n = 9$. Also $d = 12$ and $a = 5$, giving

$$5, 17, 29, 41, 53, 65, 77, 89, 101.$$

If $2n - 7 = 33$ then $2n = 40$ and $n = 20$. Also, $d = 4$ and $a = 13$, giving

$$13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57, 61, 65, 69, 73, 77, 81, 85, 89.$$

As a check, $5 + 17 + 29 = 13 + 17 + 21 = 51$ and $65 + 77 + 89 + 101 = 77 + 81 + 85 + 89 = 332$.

Solution 289.6 – 52 Cards

I take a standard deck of 52 playing cards, randomized and placed on the table face down. One by one I turn over the cards. Before a card is turned over I invite you to guess what it is. Assuming you play intelligently, how many do you expect to get right?

The game is played again, but this time you are to guess only the rank of the card. The suit is irrelevant. Again, how many do you expect to get right?

Ted Gore

For the first part of the question where both the rank and suit of the card is guessed the best strategy is to keep track of all the cards that have been disclosed up to now and to choose at random an undisclosed card.

The expected number of successful guesses would then be

$$\sum_{n=1}^{52} \frac{1}{n} = 4.538044.$$

Five simulations, each of a million repetitions of the game, were carried out on a computer giving

$$4.535365, \ 4.533909, \ 4.525787, \ 4.538353, \ 4.527825.$$

The second part is more complicated since it depends on the order in which cards are disclosed and there are 52! ways of that happening.

The best strategy is to keep a count of how many cards are left to be disclosed for each rank and to choose a rank with the maximum.

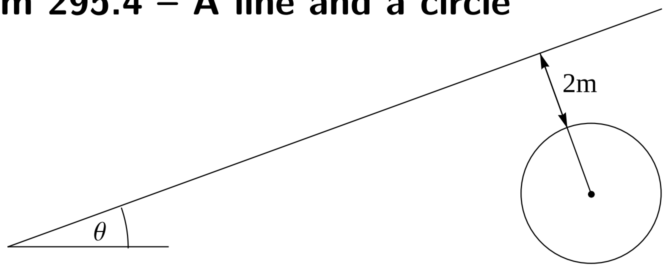
For those sequences in which the first 13 cards disclosed are all of different ranks, and likewise the second, third and fourth blocks of 13 then the expected number of successes would be

$$\sum_{n=40}^{52} \frac{4}{n} + \sum_{n=27}^{39} \frac{3}{n} + \sum_{n=14}^{26} \frac{2}{n} + \sum_{n=1}^{13} \frac{1}{n} = 6.86408.$$

Five simulations, without this assumption, were carried out giving

$$6.847271, \ 6.863124, \ 6.85604, \ 6.865375, \ 6.86261.$$

Problem 295.4 – A line and a circle



There is a line L inclined at angle θ to the horizontal, $0 < \theta < \pi/2$, and below L there is a circle C separated from L by 2 metres. A particle P is dropped from somewhere on L that is vertically above C. Determine the fastest time for P to reach the circumference of C under the action of gravity, and hence or otherwise show that it is independent of the size of the circle. Gravity is constant and acts vertically downwards.

Problem 295.5 – Graphs with girth at least 5

Given $k \geq 3$ and sufficiently large v, it is often not too difficult[1] to find a k-regular graph with v vertices and girth at least 5 (i.e. contains no triangles or 4-cycles). Show that there is no such graph that has a cyclic automorphism of order v (i.e. can be drawn with v-fold rotational symmetry). Or find one.

Note that the condition on k is necessary. For example, a hexagon is 2-regular, has girth at least 5 and when drawn in the usual manner has 6-fold rotational symmetry. Just in case anybody suggests it, the dodecahedron doesn't work; it is 3-regular, and has girth 5 but it does not have a cyclic automorphism of order 20.

Problem 295.6 – Divisibility test

Tony Forbes

Show that the following divisibility test works in any number base, b. Convert the base b digits of n to base c and add them up in base c. If the sum is divisible by $b-1$ in base c, then n is divisible by $b-1$ in base b.

For example, take $n = 9999999999999999999999995_{14}$. Convert the digits to binary (base 2) and add them up. There are $24_{10} = 11000_2$ instances of the symbol 9 and one 5; so we have $11000_2 \times 1001_2 + 101_2 = 11011101_2$, which is obviously divisible by 1101_2. Hence n is divisible by $\mathrm{D}_{14} = 1101_2 = 13_{10}$.

[1] It is actually impossible if kv is odd

Mathematics in the kitchen – XI

Tony Forbes

Here is another in our series of mathematical experiments that one can perform within the confines of one's kitchen. Actually this one is more like a problem the solution of which has more than academic interest.

A 0.56826125-litre glass drinking vessel of the type one can usually find in one's local pub is to be cleaned thoroughly in order to remove all trace of viruses and other harmful living matter. It is filled with household bleach, a solution of 50 g/l of sodium hypochlorite, left for a few hours after which the bleach is poured away. Then the glass is repeatedly filled with tap water and emptied.

How many times must this last operation be performed to be sure that the glass is fit for use?

I have stated the capacity because I think it is relevant. You should get different answers for different sizes due to the non-linear ratio of volume to internal surface area.

WARNING. Do not try this at home. Sodium hypochlorite is a corrosive substance, which can cause severe skin burns and eye damage. Never allow household bleach to enter your body, and please ignore anybody (even someone in high office) who advises otherwise.

Letters

I (TF) am reminded that 2020 is the 151st anniversary of the Periodic Table. So here is a possible answer to 'Letters' on page 20 of M500 **294**. The letters `a a a a a a a a A A A A A A A A b b b b b b b B B B B B B B c c c c C C C C C C C C C C C C d d d d d D D D e e e e e e e e e e E E E f f f F F F F F g g g g g g G G G h h h h H H H H H H i i i i i I I I k K K l l l l L L L L L m m m m m m M M M M M M n n n n n n N N N N N N N N N o o o o o O O O p P P P P P P P P P r r r r r r r r r r r R R R R R R R R s s s s s s s S S S S S S S S S t t t T T T T T T T T T u u u u u u U v V W X y Y Y Z Z` are arranged thus.

```
H                                                  He
Li Be                                B  C  N  O  F  Ne
Na Mg                                Al Si P  S  Cl Ar
K  Ca Sc Ti V  Cr Mn Fe Co Ni Cu Zn Ga Ge As Se Br Kr
Rb Sr Y  Zr Nb Mo Tc Ru Rh Pd Ag Cd In Sn Sb Te I  Xe
Cs Ba La Ce Pr Nd Pm Sm Eu Gd Tb Dy Ho Er Tm Yb Lu
         Hf Ta W  Re Os Ir Pt Au Hg Tl Pb Bi Po At Rn
Fr Ra Ac Th Pa U  Np Pu Am Cm Bk Cf Es Fm Md No Lr
         Rf Db Sg Bh Hs Mt Ds Rg Cn Nh Fl Mc Lv Ts Og
```

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04.01

Problem 295.7 – Divisibility

For positive integer n, show that $n^n + (n+1)^{n-1} - 1$ is divisible by $n(n+1)$ if and only if n is even.

Front cover A 5-regular graph with 166 vertices and girth 6, the deficiency graph of a recently discovered pentagonal geometry PENT$(5, 40)$ [`https://arxiv.org/abs/2006.15734`].