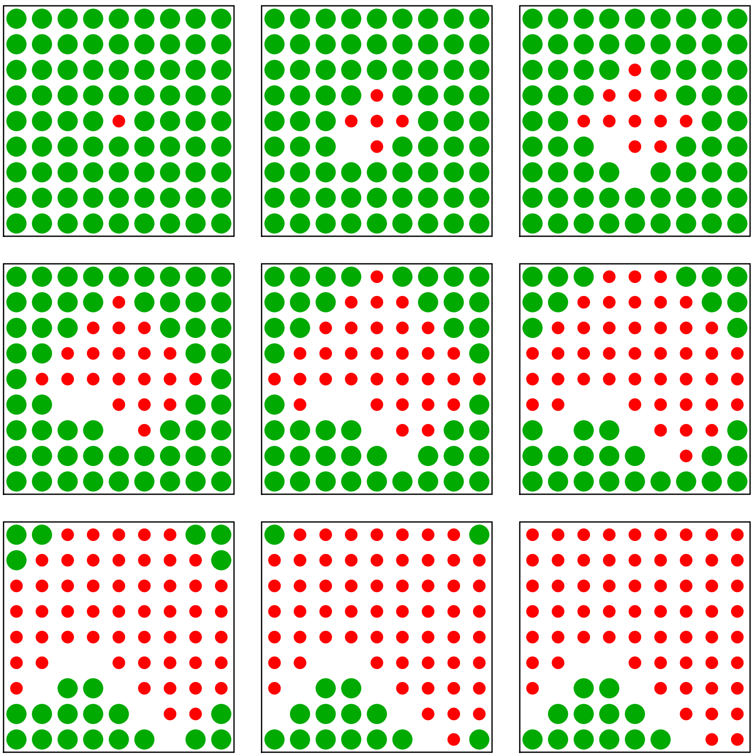
*

ISSN 1350-8539

M500 306



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The Revision Weekend is a residential Friday to Sunday event providing revision and examination preparation for both undergraduate and postgraduate students. For details, please go to the Society's website.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, please go to the Society's website.

Advice for authors We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to the Editor, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. For more information, go to m500.org.uk/magazine/ from where a LaTeX template may be downloaded.

Subscription renewal If your M500 Society membership expires on 31st July 2022, you will receive a subscription renewal form either separately by email or, if you do not have an email address, on paper. Please follow the instructions on the form to renew your subscription.

Beyond the fourth dimension: fact or fiction?

Colin P. George

A by-product of the Big Bang was one time dimension plus three spatial dimensions we designate as length (x), width (y) and height (z). This raises another question; could it be possible that there are several other dimensions beyond the four that we know, but that just happen to remain hidden? The sad reality is we simply do not know for sure.

There was a lecture by Stephen Hawking and an interesting book by Malcolm Hulke and Terrance Dicks called *The Making of Doctor Who*, which contained a very clever chapter about a race of people called Flatlanders (Stephen Hawking called them bugs) who lived in a two-dimensional world. Although this might sound rather far-fetched, we would like you to hold that thought, as we are going to extend this theme within the world of geometry.

How does a non-mathematician or layman or even the smartest person even begin to understand the notion of such higher dimensions? How does one even envisage something so far-out? We as humans do not appear have the cognitive machinery to visualise a 5-, 6- or 7-dimensional world.

Since string and M-theory appears to entertain the possibility of higher dimensions we're going to do something rather bold. To delve further we are going to take a trip to a land of fiction. What follows next is pure fiction and is purely hypothetical. It's a kind of thought experiment that shows how limited we really are as human beings.

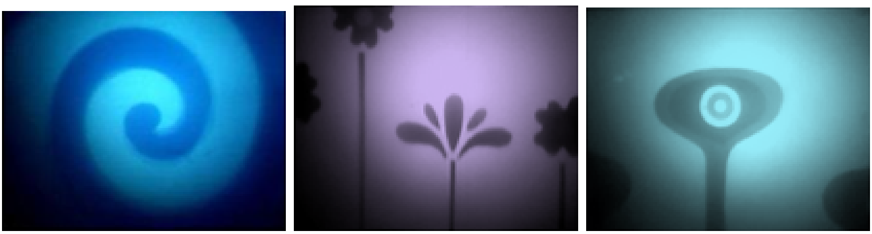


Figure 1: (a) Let us try to imagine a hypothetical world of two dimensions: (b) The two-dimensional world of the Flatlanders (Flatland): (c) Although they don't look like us, if such a world really exists, they probably have their own mathematicians, scientists, artists and philosophers.

Just as it is difficult for us to visualise and imagine a 5- or 6-dimensional curved space, one could say the same for someone in a 2-dimensional plane to actually imagine a 3-dimensional world. This story depicts a hypothetical world of mathematicians, scientists, abstract artists and philosophers who, unlike humans, are Flatlanders.

The Flatlanders live in a world of limited dimensions. Flatland seems to be a world of two spatial dimensions. However, that surface is actually part of a larger 3-dimensional object. This might be a cube, a cylinder or even a sphere or some other 3-dimensional object.

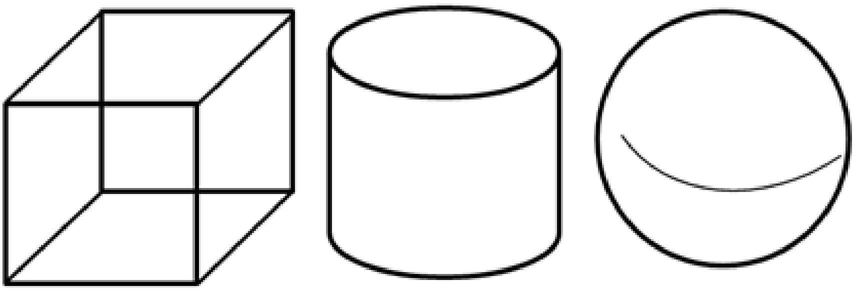


Figure 2: This two-dimensional world could be one plane of a cube or other three-dimensional object. You never know, the Flatlanders may even be aware of an extra amount of mass/weight they cannot fully explain; who knows?

Remember, the Flatlanders are only conscious of two spatial dimensions and can only theorise about this extra third spatial dimension in which their 2-dimensional world sits. One could say they are faced with the same predicament as we humans. You see, they don't have the cognitive machinery to 'visualise' that third dimension, but there is nothing to stop them from 'describing' it as a mathematical construct; the Flatlander's version of theoretical mathematics.

In their world, even the best abstract artist might not be able to imagine this third spatial dimension. Objects such as a cube, sphere or cylinder may be difficult to visualise or imagine. At some point the Flatlander mathematicians might describe and envisage this three-dimensional world interacting in some way with their world. After all, the Flatlander's two-dimensional world sits within the third dimension.

No doubt the Flatlander mathematicians, scientists, artists and philosophers will be familiar with two-dimensional geometry in order to calculate the area of a square, rectangle or circle. Three dimensions would be of no relevance in a two-dimensional world and would serve no practical purpose. The three-dimensional world is a world that is invisible to them. However, the discovery of three-dimensional mathematics using 'n dimensions' (a kind theoretical construct, in the same way string and M-theory is in ours) and knowing its possible relevance to their two-dimensional world could surely be a major breakthrough for the Flatlanders in solving many mysteries in which their world resides.

Although the world of the Flatlanders is nothing more than pure fiction, it may serve as a possible analogy when trying to explain to the non-mathematician, the layman or even the highly educated about a possible notion or potential for higher dimensions.

Albert Einstein's *General Relativity* (1916) and other respectable papers show that the geometry of the world is not always quite what it seems. Other forms of geometry are possible. Could it be the four dimensions that we know exist, in which we ourselves reside, actually sits within a world of several other dimensions? To understand those other dimensions (assuming they exist) may require mathematics that goes way beyond standard models we are taught at school.

References

Dicks, T. & Hulke, M. (1972), *The Making of Dr Who; could it all be true?*, pp. 100–103, Pan Books Ltd, London.

Green, B. R. (2012), String Theory, *Encyclopaedia Britannica*, Chicago.

Lambourne, R. J. A. (2010), *Relativity, Gravitation and Cosmology*, The Open University/Cambridge University Press, London.

Problem 306.1 – Number representation

Tony Forbes

Either show that all sufficiently large integers can be represented in the form

$$12ab + 20a + 9b, \qquad a \geq 3, \qquad b \geq 300,$$

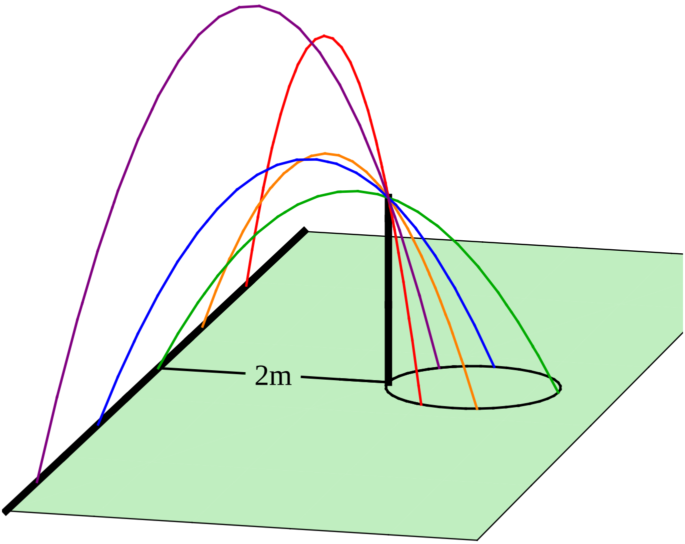
or find an infinite sequence of positive integers that can't.

(The numbers 12, 20, 9, 3 and 300 are not especially significant; they just happened to define a special case that was of interest to me.)

Solution 296.5 – A line and a pole

A vertical pole of height h m is separated by 2 m from the y-axis and has its base on the (x, y)–plane. A projectile is fired with horizontal velocity component v m/s from somewhere on the y–axis and just grazes the top of the pole.

Assuming that gravity, $g = 9.80665\,\mathrm{m/s^2}$, acts vertically downwards and that the atmosphere has been removed, show that the projectile lands on the (x, y)–plane somewhere on a circle with radius $hv^2/(2g)$ m.



Alan Davies

Preamble

The picture, above, shows a variety of possible trajectories covering a three-dimensional space. With that in mind it would be better to redefine the axes to coincide with the usual right-hand x, y, z system, origin O. The post is at the point P on the y–axis, distance d from O. Notice here that we treat the distance of the post from the projection line as a *parameter* of the problem. This gives much more clarity to the solution since it maintains the dimensional consistency of the equations. The distance 2m is data and can be substituted for d at the end, however there is no need for that here. For a good description of the differences between variables, parameters and data see MST204. One final comment: the question says to assume that g is known to six significant figures. If that level of precision is required then to ignore air resistance would not be a good modelling option.

Solution

The projection will be from a general point, A, on the x–axis distance D from P, as shown in figure 1. Suppose that the projectile hits the (x, y)–plane at B, distance L from P. The motion takes place in the vertical plane defined by the line APB and the post. The initial horizontal velocity is v in the direction AP. Suppose that the initial vertical velocity is w. For details, see figure 1.

Let the coordinate ξ, along the line APB, define the horizontal distance travelled during the motion. Then, using the usual projectile equations, at time t,

$$\xi = vt \quad \text{and} \quad z = wt - \frac{1}{2}gt^2. \tag{1}$$

It then follows that the time, t_1, to reach the post is given by

$$t_1 = \frac{D}{v}. \tag{2}$$

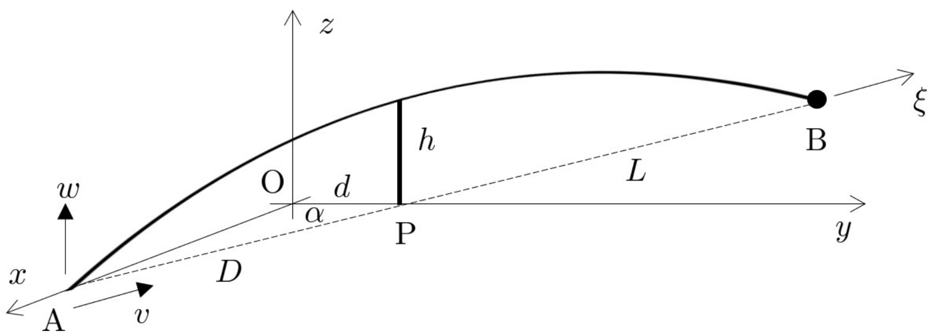


Figure 1: Motion in xyz-space.

Since the projectile just grazes the top of the post, equation 1 leads to

$$h = wt_1 - \frac{1}{2}g{t_1}^2$$

and equation 2 gives, after a little algebra,

$$w = \frac{vh}{D} + \frac{1}{2}g\frac{D}{v}. \tag{3}$$

When the projectile hits the (x, y)–plane, at B, $z = 0$ and equation 1 gives

$$0 = wt - \frac{1}{2}gt^2.$$

Since $t = 0$ corresponds to the initial conditions, the time, t_2, to reach B is given by

$$t_2 = \frac{2w}{g}. \qquad (4)$$

The horizontal distance travelled to reach B is $D + L$ so that equation 1 gives

$$\begin{aligned} D + L &= vt_2 \\ &= v\,\frac{2w}{g} \\ &= \frac{2v}{g}\left(\frac{vh}{D} + \frac{1}{2}g\frac{D}{v}\right), \quad \text{using (3)} \\ &= D + \frac{2v^2h}{gD}. \end{aligned}$$

Hence

$$L = \frac{2v^2h}{gD}, \qquad (5)$$

and it is convenient to write this as

$$L = \frac{2v^2h}{gd}\frac{d}{D}$$

so that

$$L = \frac{2v^2h}{gd}\cos\alpha, \quad \text{see figure 1.} \qquad (6)$$

Define a set of polar coordinates in the (x, y)–plane with pole at P and axis along the y–axis, as shown in figure 2. The landing point, B, has the polar coordinates (r, θ).

It is clear from figures 1 and 2 that at B, $r = L$. Hence

$$\begin{aligned} r^2 &= rL \\ &= r\,\frac{2v^2h}{gd}\cos\alpha, \quad \text{using (6).} \end{aligned}$$

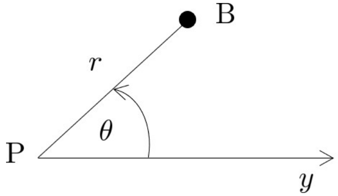


Figure 2: Plane polar coordinates with pole at P.

Now, write

$$a = \frac{v^2 h}{gd}$$

and noticing that at B, $\theta = \alpha$, it follows that

$$r^2 = 2ra\cos\theta, \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

the polar equation of a circle radius a, centre $(a, 0)$.

No matter where the projectile is launched on the x–axis it lands somewhere on a circle of radius v^2h/dg.

Problem 306.2 – Approximate roots

Let n be an integer greater than 1, let x be positive number and let

$$r = \lfloor \sqrt[n]{x} \rfloor.$$

Show that if x is not too small, then

$$\frac{(n+1)x + (n-1)r^n}{(n-1)x + (n+1)r^n}\, r$$

is nearly $\sqrt[n]{x}$.

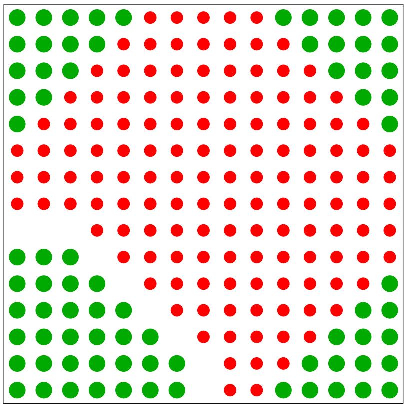
This is described in Section LXIV of *Short Methods and By-Ways in Arithmetic* by H. W. Dickie (W. R. Chambers, Ltd, 1912, price £0.05) as a quick way to get an approximate nth root. Clearly it is not much use if $x < 1$, but when x is sufficiently large it can be quite effective. For example, with $n = 3$ and $x = 1100$, so that $r = 10$, it gives 10.3226 whereas $\sqrt[3]{1100} = 10.3228$.

Problem 306.3 – Burning trees

Tony Forbes

On 13 May 2021 at the LSE Combinatorics Colloquium I attended an interesting lecture by Jessica Enright of Glasgow University, 'Firefighting on Graphs'. Here is a typical problem suggested by the material presented.

A forest consists of a 15×15 square lattice with 225 trees planted at locations $\{(x, y) : x, y = 1, 2, \ldots, 15\}$. At time $t = 0$ the central tree (at location (8,8)) starts burning. At each time $t = 1, 2, \ldots,$ until there are no more trees to burn: (i) you remove one non-burning tree from the forest, and then (ii) the fire spreads to all trees that are at distance 1 from a burning tree. How many live trees can you save?



For an example of a non-optimal scenario, which you might like to act out on a sheet of squared paper, remove trees at (5,5), (6,4), (4,6), (7,3), (3,7), (8,2), (2,7), (1,7), (8,1) to get the result shown on the right. You have prevented the fire from reaching the 32 trees in the south-west corner, and a further three can be rescued with fire-breaks at (14,2), (13,1), (15,3) making a total of 35 trees saved.

The situation is improved significantly if you are allowed to remove a fixed number $d \geq 2$ trees at each stage. For instance, when $d = 4$ you get the best possible result, 5 trees lost, 220 saved. For yet another variation, suppose that you are allowed to remove up to t trees at time t, the logic being that as time passes more and more resources can be organized to fight the fire. We would be particularly interested in this last option.

Problem 306.4 – Sets

Tony Forbes

There are t sets, each containing u elements. In each set there are precisely r elements each of which occurs in precisely s other sets such that the rs sets in which they occur are distinct. How many elements are there?

Solution 296.3 – Elliptic curve

Let a be a positive real number. Then the elliptic curve $y^2 = x(x^2 - a^2)$ has two components, an unbounded curve that passes through $(a, 0)$ and a closed 'bubble' that passes through $(0, 0)$ and $(-a, 0)$. What area does the bubble enclose?

Tommy Moorhouse

This problem has been approximately solved by Ted Gore (M300 **302** and M500 **304**) but an exact solution is not hard to find. The problem involves finding the area A enclosed by the closed component of the elliptic curve defined by

$$y^2 = x(x^2 - a^2).$$

The relevant integral is

$$A = 2\int_{-a}^{0} \sqrt{x(x^2 - a^2)}dx.$$

Making the substitution $x = a \sin u$ we transform the integral to

$$A = 2a^{5/2} \int_0^{\pi/2} (\cos u)^2 (\sin u)^{1/2} du.$$

Happily, this integral can be done using our old favourite, the Beta function:

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)},$$

which holds when both m and n have a positive real part, as here. In this case we substitute $x = \cos^2 u$ to find that

$$\int_0^{\pi/2} (\cos u)^{2m-1}(\sin u)^{2n-1}\, du = \frac{\Gamma(m)\,\Gamma(n)}{2\Gamma(m+n)}.$$

By inspection we have $2m = 3$, $2n = 3/2$ and the full area integral is

$$A = a^{5/2}\, \frac{\Gamma(3/2)\,\Gamma(3/4)}{\Gamma(9/4)}.$$

We can now use the standard identities $\Gamma(1 + z) = z\Gamma(z)$ and $\Gamma(1/2) = \sqrt{\pi}$ to arrive at

$$A = \frac{2}{5} a^{5/2} \sqrt{\pi}\, \frac{\Gamma(3/4)}{\Gamma(5/4)}.$$

Solution 296.4 – Cubic curve

Let a be a positive real number. Then the cubic curve

$$y^2 \;=\; \frac{x(x-a)^2}{3a}$$

has a loop that passes through $(0,0)$ and $(a,0)$. What is its length and what area does it enclose?

Unlike in Problem 296.3 (see Solution 296.3 on the previous page) the curve is not elliptic. Curiously, the denominator $3a$ in the definition has some significance. Remove it, and the loop length of the curve $y^2 = x(x-a)^2$ becomes much more difficult to compute. You are welcome to try!

Tommy Moorhouse

We are tasked with finding the length L of, and the area A enclosed by, the loop of the cubic curve $y^2 = x(x-a)^2/(3a)$. First we consider the curve length. The strategy is to describe the curve as a path in the plane, find the velocity and hence the speed s and integrate this w.r.t. time between the appropriate limits to find the length. One parametric expression for the path is given by

$$\gamma(t) \;=\; \left(t, \sqrt{t}(t-a)/\sqrt{3a}\right), \;\; t \in [0,a].$$

The image of γ for t from 0 to a is one branch of the cubic curve loop. The velocity of $\gamma(t)$ in the $x-y$ plane is

$$\dot\gamma(t) \;=\; \left(1, \frac{1}{2\sqrt{3at}}(3t-a)\right).$$

The speed $s(t)$ is the square root of

$$|\dot\gamma(t)|^2 \;=\; 1 + \frac{1}{12at}\left(9t^2 - 6at + a^2\right)$$

which is

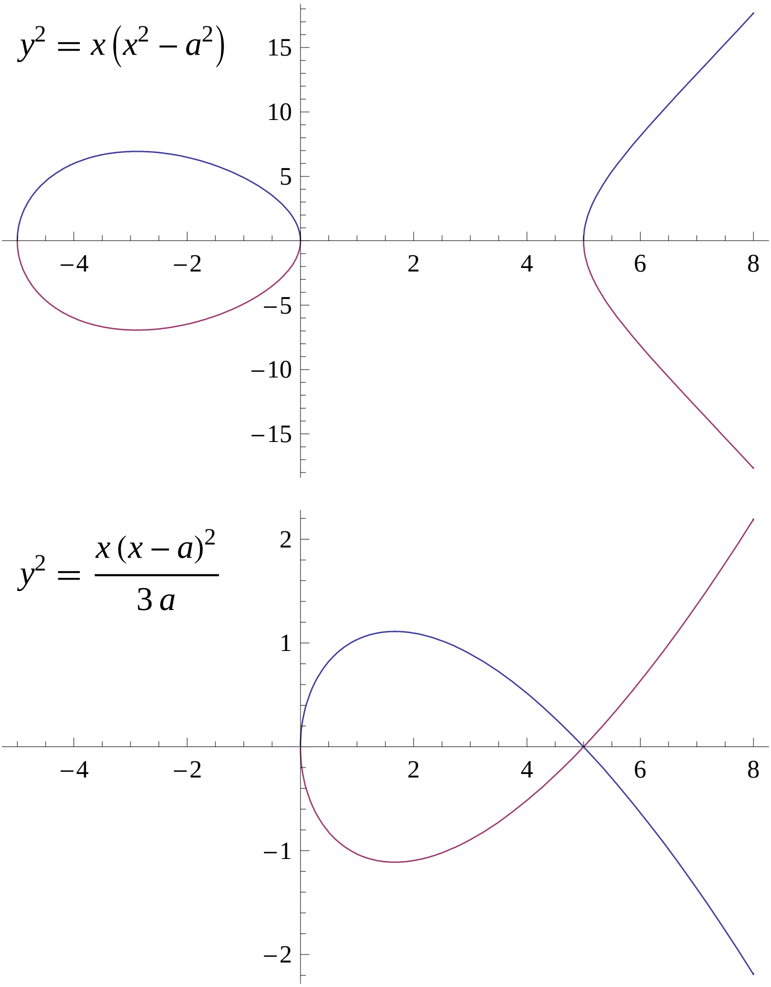
$$s(t) \;=\; \frac{1}{2\sqrt{3a}}\left(3\sqrt{t} + \frac{a}{\sqrt{t}}\right).$$

The denominator $3a$ has allowed us to find a simple expression for $s(t)$, giving a perfect square in the numerator of $|\dot\gamma(t)|^2$. This really does simplify matters. Now we integrate over t:

$$L \;=\; 2\int_0^a s(t)\,dt \;=\; \frac{4a}{\sqrt{3}}.$$

The area integral is more straightforward, and we just need to choose the negative square root for y to get a positive area:

$$A = \frac{2}{\sqrt{3a}} \int_0^a (ax^{1/2} - x^{3/2})\, dx = \frac{8a^2}{15\sqrt{3}}.$$



Set sharing

Tamsin Forbes

Start with $n \geq 2$ singleton sets, which we can assume are $\{1\}$, $\{2\}$, ..., $\{n\}$. The objective is to transform them into n sets each of which contains all n elements,

$$\{1, 2, \ldots, n\}, \ \{1, 2, \ldots, n\}, \ \ldots, \ \{1, 2, \ldots, n\},$$

by a finite number of special constructions that we call 'sharings', defined as follows.

> A *sharing* is the process where two sets A and B are replaced by two identical sets, $A \cup B$ and $A \cup B$. That is, we *share* A and B to get two copies of $A \cup B$.

To see how it works, let $n = 8$. Then, starting with $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$, $\{8\}$, and noting that 8 is a power of 2, we see that 12 sharings suffice.

Share 1, 5	→ 15, 15, 2, 3, 4, 6, 7, 8.
Share 2, 6	→ 15, 15, 26, 26, 3, 4, 7, 8.
Share 3, 7	→ 15, 15, 26, 26, 37, 37, 4, 8.
Share 4, 8	→ 15, 15, 26, 26, 37, 37, 48, 48.
Share 15, 37	→ 1537, 1537, 15, 26, 26, 37, 48, 48.
Share 15, 37	→ 1537, 1537, 1537, 1537, 26, 26, 48, 48.
Share 26, 48	→ 1537, 1537, 1537, 1537, 2648 2648, 26, 48.
Share 26, 48	→ 1537, 1537, 1537, 1537, 2648 2648, 2648, 2648.
Share 1537, 2468 4 times	→ eight instances of 12345678.

For brevity we have omitted set brackets and commas; so we have written 15 instead of $\{1, 5\}$, etc. Also it is important to point out that the items in the table should be treated as instances of sets. For example, the sets $\{1, 5\}$ and $\{1, 5\}$ that occur in the first row are really distinct entities even though they look identical. To make this clear(er) we could index them, $\{1, 5\}_1$ and $\{1, 5\}_2$, but that would just create notational clutter of questionable necessity.

For each n, we are interested in the smallest number of sharings required to achieve the stated objective. Call this number $S(n)$.

We have shown that $S(8) \leq 12$. However, we can prove that

$$S(n) \ \leq \ 2n - 4 \quad \text{for all } n \geq 4$$

by the following general strategy, which you might like to verify on a computer using your favourite programming language.

1. Let $h = \lfloor n/2 \rfloor$.
2. Share $\{1, 2, \ldots, i\}$ with $\{i+1\}$, $i = 1, 2, \ldots, h-1$, to create, amongst other things, two sets $\{1, 2, \ldots, h\}$.
3. Share $\{h+1, h+2, \ldots, i\}$ with $\{i+1\}$, $i = h+1, h+2, \ldots, n-1$, to create two sets $\{h+1, h+2, \ldots, n\}$.
4. Share $\{1, 2, \ldots, h\}$ and $\{h+1, h+2, \ldots, n\}$ twice to get four complete sets, $\{1, 2, \ldots, n\}$.
5. Mop up the $n-4$ remaining incomplete sets by sharing each one with a complete set.

If you add up the numbers of sharings, you get

$$(h-1) + (n-1-h) + 2 + (n-4) \; = \; 2n-4,$$

as claimed.

However, what we really want is either a proof that $S(n) = 2n-4$ for all $n \geq 4$, or a better bound. If this general problem is too difficult we would still be interested if anyone can determine the exact value of $S(n)$ for special cases—for example, small n, or n that are powers of 2, or perhaps $n = 7622014$.

Fans of *Dune* might recognize this last case. A Bene Gesserit Reverend Mother is able to access the memories of her maternal ancestors, and two Reverend Mothers may 'share' their entire set of 'other lives'. When the Bene Gesserit library planet Lampadas was attacked by the Honored Matres it seemed very likely that its resident population of 7622014 Reverend Mothers was doomed. The Reverend Mothers' best defence was to share their identities in the manner outlined above. Fortunately only 7622013 perished in the conflict. Reverend Mother Lucilla escaped, carrying this 'horde of Lampadas', the combined knowledge of all, and by sharing with others she was able to ensure it reached her sisters on Chapterhouse [Frank Herbert, *Chapterhouse: Dune*].

I showed the problem to the Editor of this magazine, and after a while he confidently asserted that he had completely solved it. Upon interrogation, however, he confessed that he had solved a different and much easier problem, which might actually be more useful to the Reverend Mothers of Lampadas.

By a *session* we mean the simultaneous sharing in parallel of any number of pairs of distinct instances of sets. Now we want to compute $T(n)$, the minimum number of sessions to create n complete sets from n singleton sets.

Obviously the algorithm for $S(n)$ on page 13 isn't going to work very well for large n. If you look at step 2, for instance, you will see that $\lfloor n/2 \rfloor - 1$ sessions are needed because the sharings have to be performed in sequence. We cannot share $\{1, 2\}$ with $\{3\}$ until we have shared $\{1\}$ and $\{2\}$. However, we think $T(n)$ can be determined by a different procedure that aims to do as many parallel sharings as possible in each session. Suppose $n \geq 2$ and assume the initial collection of sets is $(P_1, P_2, \ldots, P_n) = (\{1\}, \{2\}, \ldots, \{n\})$ ordered as shown.

If n is even:

for $s = 1,\ 2,\ \ldots,\ \lceil (\log n)/(\log 2) \rceil$:

for $i = 1,\ 2,\ \ldots,\ n/2$ (simultaneously, in parallel):

share P_i with $P_{n/2+i}$ to get Q_{2i-1} and Q_{2i};

for $j = 1,\ 2,\ \ldots,\ n$: $P_j = Q_j$.

If n is odd:

for $s = 1,\ 2,\ \ldots,\ \lceil (\log n)/(\log 2) \rceil + 1$:

for $i = 1,\ 2,\ \ldots,\ (n-1)/2$ (simultaneously, in parallel):

share P_i with P_{n+1-i} to get Q_{2i-1} and Q_{2i};

$Q_n = P_{(n+1)/2}$;

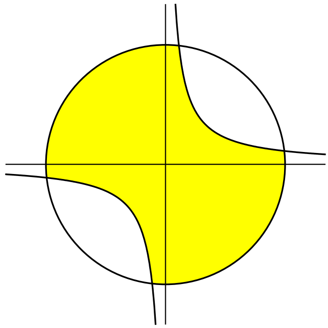
for $j = 1,\ 2,\ \ldots,\ n$: $P_j = Q_j$.

The sessions are indexed by s and we cannot reduce their number any further. The fastest way to construct a complete set from the initial singletons is to double the set size whenever possible. If n is even, that will take at least $\lceil (\log n)/(\log 2) \rceil$ sessions. And when n is odd the initial sharings will leave at least one unpaired singleton; hence we will need one extra session. Thus we have

$$T(n) = \begin{cases} \left\lceil \dfrac{\log n}{\log 2} \right\rceil & \text{if } n \text{ is even,} \\[2ex] \left\lceil \dfrac{\log n}{\log 2} \right\rceil + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Solution 302.1 – A circle and a hyperbola

The circle has radius r and the hyperbola is $y = 1/x$. What's the area of the yellow (light grey) part?



Ted Gore

We have $x^2 + y^2 = r^2$ for the circle and $y = 1/x$ for the hyperbola.

We want to find where the two curves intersect in the first quadrant so we combine them to get

$$x^2 + 1/x^2 = r^2.$$

This can be reorganized to $(x^2)^2 - r^2x^2 + 1 = 0$ from which we get

$$x = \sqrt{\frac{r^2 \pm \sqrt{r^4 - 4}}{2}}.$$

Letting

$$x_1 = \sqrt{\frac{r^2 - \sqrt{r^4 - 4}}{2}} \quad \text{and} \quad x_2 = \sqrt{\frac{r^2 + \sqrt{r^4 - 4}}{2}}$$

we get that the area of the shaded part of quadrant 1,

$$A(r) = \int_0^{x_1} + \int_{x_2}^{r} \sqrt{r^2 - x^2}\, dx + \int_{x_1}^{x_2} \frac{dx}{x}.$$

Integrating this we get

$$A(r) = \Big[(r^2 \arcsin(x/r) + x\sqrt{r^2 - x^2})/2\Big]_0^{x_1} + \Big]_{x_2}^{r} + \Big[\ln(x)\Big]_{x_1}^{x_2}.$$

The total area of the shaded region is

$$T(r) = 2A(r) + \pi r^2/2.$$

For $r = 2$, $x_1 = 0.5176381$ and $x_2 = 1.9318517$ so that

$$A(2) = 1.0235988 + 0.0235988 + 1.3169579 = 2.3641554$$

and $T(2) = 11.0114962$. As r increases, $T(r)$ approaches $\pi r^2/2$.

Richard Gould

As presented, the integral limits looked a bit daunting so I decided to first rotate the hyperbola anticlockwise through 45°. Denoting the old coordinates by (x', y') and the new coordinates by (x, y) we have

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

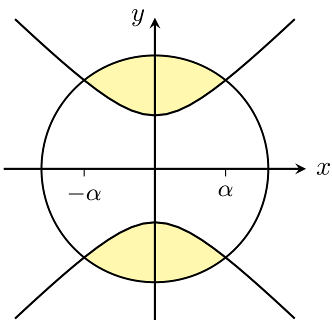
So

$$x'y' = \frac{1}{\sqrt{2}}(x+y) \times \frac{1}{\sqrt{2}}(-x+y) = 1,$$

giving

$$y^2 - x^2 = 2$$

as the equation of the rotated hyperbola. The problem now looks like this:



where we subtract the shaded regions from πr^2 to give the required result. The limits of integration for these regions are

$$x = \pm\,\alpha = \pm\,\sqrt{(r^2-2)/2}.$$

The area of each shaded region is given by

$$\int_{-\alpha}^{\alpha} \int_{\sqrt{2+x^2}}^{\sqrt{r^2-x^2}} dy\,dx = \int_{-\alpha}^{\alpha} \sqrt{r^2-x^2}\,dx - \int_{-\alpha}^{x=\alpha} \sqrt{2+x^2}\,dx$$

$$= 2\int_0^{\alpha} \sqrt{r^2-x^2}\,dx - 2\int_0^{\alpha} \sqrt{2+x^2}\,dx$$

$$= I_1 - I_2.$$

Substituting $x = r\sin\theta$ in I_1 gives

$$\begin{aligned}
I_1 &= 2r^2 \int_{x=0}^{x=\alpha} \cos^2\theta \, d\theta \\
&= r^2 \left[\theta + \frac{1}{2}\sin(2\theta)\right]_{x=0}^{x=\alpha} \\
&= \left[r^2 \arcsin\left(\frac{x}{r}\right) + x\sqrt{r^2 - x^2}\right]_0^\alpha \\
&= r^2 \arcsin\left(\frac{\alpha}{r}\right) + \alpha\sqrt{r^2 - \alpha^2} \\
&= r^2 \arcsin\left(\sqrt{\frac{r^2-2}{2r^2}}\right) + \sqrt{\frac{r^2-2}{2}}\sqrt{r^2 - \frac{r^2-2}{2}} \\
&= r^2 \arcsin\left(\sqrt{\frac{r^2-2}{2r^2}}\right) + \frac{1}{2}\sqrt{r^4 - 4}, \quad r > \sqrt{2}.
\end{aligned}$$

For some reason MAPLE prefers the arctan function for this integral so to facilitate subsequent checks I used

$$\arcsin x = \arctan(x/\sqrt{1 - x^2})$$

to give

$$I_1 = r^2 \arctan\left(\sqrt{\frac{r^2-2}{r^2+2}}\right) + \frac{1}{2}\sqrt{r^4 - 4}.$$

Setting $x = \sqrt{2}\sinh u$ in I_2 gives

$$\begin{aligned}
I_2 &= 4\int_{x=0}^{x=\alpha} \cosh^2 u \, du \\
&= 2\left[u + \frac{1}{2}\sinh(2u)\right]_{x=0}^{x=\alpha}
\end{aligned}$$

$$\begin{aligned}
&= 2\,\mathrm{arcsinh}\left(\frac{\alpha}{\sqrt{2}}\right) + \frac{2\alpha}{\sqrt{2}}\sqrt{1 + \frac{\alpha^2}{2}} \\
&= 2\,\mathrm{arcsinh}\left(\frac{1}{2}\sqrt{r^2-2}\right) + 2\sqrt{\frac{r^2-2}{4}}\sqrt{1 + \frac{r^2-2}{4}} \\
&= 2\,\mathrm{arcsinh}\left(\frac{1}{2}\sqrt{r^2-2}\right) + \frac{1}{2}\sqrt{r^4 - 4}, \quad r > \sqrt{2}.
\end{aligned}$$

Putting all this together, the required area is $\pi r^2 - (2I_1 - 2I_2)$, giving

$$\pi r^2 - 2r^2 \arctan\left(\sqrt{\frac{r^2-2}{r^2+2}}\right) + 4\,\mathrm{arcsinh}\left(\frac{1}{2}\sqrt{r^2-2}\right).$$

Finally, noting that a direct approach may give logarithmic rather than hyperbolic functions, we can rewrite the final term to give

$$\pi r^2 - 2r^2 \arctan\left(\sqrt{\frac{r^2-2}{r^2+2}}\right) + 4\ln\frac{1}{2}\left(\sqrt{r^2-2}+\sqrt{r^2+2}\right), \quad r > \sqrt{2}.$$

Note: The author (TF) suggested that this might have been a suitable problem for someone learning how to calculate areas by integration. Even without the opening bit of linear algebra, I suspect this remark to have been made somewhat tongue-in-cheek!

Problem 306.5 – Sorting an array

Take a rectangular $r \times c$ array, A, of distinct numbers where each row is sorted in increasing order. Create array B by sorting each column of A into increasing order.

Show that the rows of B are sorted in increasing order.

$$A = \begin{bmatrix} 8 & 12 & 13 \\ 2 & 6 & 7 \\ 5 & 11 & 14 \\ 3 & 4 & 9 \\ 1 & 10 & 15 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 2 & 6 & 9 \\ 3 & 10 & 13 \\ 5 & 11 & 14 \\ 8 & 12 & 15 \end{bmatrix} = B.$$

Now create another array, C, as follows. Start with $C = A$. Then for $j = 1$, $2, \ldots, c$, rearrange the rows of the subarray of C defined by columns j to c so that they are in increasing order of the element in column j.

Show that $C = B$.

$$\begin{bmatrix} 8 & 12 & 13 \\ 2 & 6 & 7 \\ 5 & 11 & 14 \\ 3 & 4 & 9 \\ 1 & 10 & 15 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 10 & 15 \\ 2 & 6 & 7 \\ 3 & 4 & 9 \\ 5 & 11 & 14 \\ 8 & 12 & 13 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 9 \\ 2 & 6 & 7 \\ 3 & 10 & 15 \\ 5 & 11 & 14 \\ 8 & 12 & 13 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 2 & 6 & 9 \\ 3 & 10 & 13 \\ 5 & 11 & 14 \\ 8 & 12 & 15 \end{bmatrix}.$$

Solution 301.9 – Integral

Show that

$$\int_0^1 \frac{\sqrt{1-x^2}}{\sqrt{1+x^2}}\,dx \;=\; \frac{\Gamma(1/4)^4-8\pi^2}{4\sqrt{2\pi}\,\Gamma(1/4)^2}.$$

Ted Gore

Let

$$y \;=\; \int_0^1 \frac{\sqrt{1-x^2}}{\sqrt{1+x^2}}dx.$$

We can multiply both top and bottom by $\sqrt{1-x^2}$ so that

$$y \;=\; \int_0^1 \frac{1}{\sqrt{1-x^4}}dx - \int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx.$$

But

$$\int_0^1 \frac{1}{\sqrt{1-x^4}}dx \;=\; \frac{G\pi}{2},$$

where G is Gauss's constant, $G=\Gamma(1/4)^2/(2\sqrt{2\pi^3})$. Then

$$\int_0^1 \frac{1}{\sqrt{1-x^4}}dx \;=\; \frac{\Gamma(1/4)^2}{4\sqrt{2\pi}} \;=\; \frac{\Gamma(1/4)^4}{4\sqrt{2\pi}\Gamma(1/4)^2}.$$

From *Wikipedia* I obtained the result

$$B(v,w) \;=\; n\int_0^1 x^{nv-1}(1-x^n)^{w-1}dx,$$

where B is the beta function. Applying this we see that

$$\int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx \;=\; B(3/4,1/2)/4 \;=\; \frac{\Gamma(3/4)\,\Gamma(1/2)}{4\Gamma(5/4)}.$$

Now

$$\Gamma(3/4) \;=\; \frac{\pi\sqrt{2}}{\Gamma(1/4)} \quad\text{and}\quad \Gamma(5/4) \;=\; \frac{\pi}{2\sqrt{2}\Gamma(3/4)}.$$

After some manipulation, we get

$$\int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx \;=\; \frac{8\pi^2}{4\sqrt{2\pi}\Gamma(1/4)^2}$$

and

$$y \;=\; \frac{\Gamma(1/4)^4-8\pi^2}{4\sqrt{2\pi}\,\Gamma(1/4)^2}.$$

Problem 306.6 – Pistachio nuts

Tony Forbes

Pistachio nuts can be an important component of a well-balanced diet and have featured significantly in past issues of this magazine as Problems 275.6, 278.1, 290.5 and 283.6 as well as in some cases their solutions. Just when I was thinking that the subject was exhausted I discovered an interesting simple variation that I had hitherto overlooked.

There is a bowl containing n pistachio nuts. How may times would you expect to perform the following procedure in order to consume all of the edible material in the bowl?

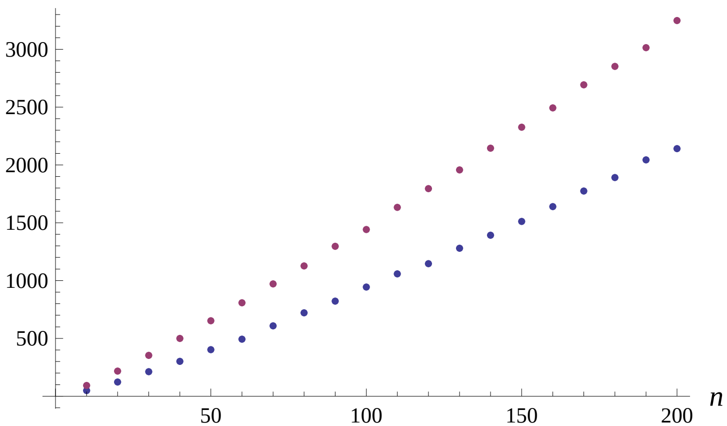
(i) You select uniformly at random one object from the bowl. It might be a whole pistachio nut in its shell, or just a pistachio nut kernel without its shell, or just half of a pistachio nut shell.

(ii) If it is a half-shell, you return it to the bowl.

(iii) If it is a naked pistachio nut kernel, you consume it.

(iv) If it is a pistachio nut in its shell, you split it into its three components, kernel and two half-shells, which you return to the bowl.

The graph shows the results averaged over 1000 trials for each of $n = 10, 20, \ldots, 200$ (upper plot). Compare with the original Problem 275.6, where in (iv) you eat the kernel instead of returning it to the bowl (lower plot).



Problem 306.7 – Quintic roots

Let a and b be positive numbers. Show that the quintic

$$2x^5 - 5a^3x^2 + 3b^5 \;=\; 0$$

has $2 + \operatorname{sign}(a - b)$ real roots.

Zeros

Jeremy Humphries

I saw a comment on social media aimed at certain MPs, which said:

> What people really want is for their elected representatives to actually represent them, not represent whoever can get them the most zeroes on their pay slip.

Now, this little thought is not about the current parliament, and whether it will go down in history as the 'Sleaze Parliament', alongside others that are notable enough to be named—the 'Long Parliament', the 'Barebones Parliament', the 'Cavalier Parliament' and so on. No. It's about the strange use of 'zeroes' to imply big money numbers.

You hear and see it so often. All those lovely zeroes on the cheque, or on the bottom line, or whatever. Why? We've got the digits 0 to 9, and the smallest of those is the 0. So why is it chosen to stand for large? Seems weird to me. If a sum of money represented by a string of digits of a certain length is to come into my hands, and I get a say in the matter, I don't want any of the digits in the string to be 0. In fact I want them all to be as far from 0 as possible. I want them all to be 9.

Two-letter words

Tony Forbes

Whilst seeking nonsense-eradication opportunities in something I had written I noticed this sentence: 'Construct a graph and discard it if it is or if it is not suitable.' A challenging problem is suggested.

> *Write a sensible English sentence that contains many consecutive two-letter words, the more the merrier.*

Did you hear about the mathematician who really didn't like negative numbers? He would stop at nothing to avoid them. —Jeremy Humphries

04.00

Problem 306.8 – Floor, ceiling and square root

Tony Forbes

If n is a positive integer that is not an integer square, show that

$$\left\lceil \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rceil - \lfloor \sqrt{n} \rfloor \;=\; \frac{3}{2} - \frac{1}{2}\bigl(\operatorname{sign}\sin(2\pi\sqrt{n})\bigr).$$

Front cover Burning trees on a 9×9 grid; see page 8.