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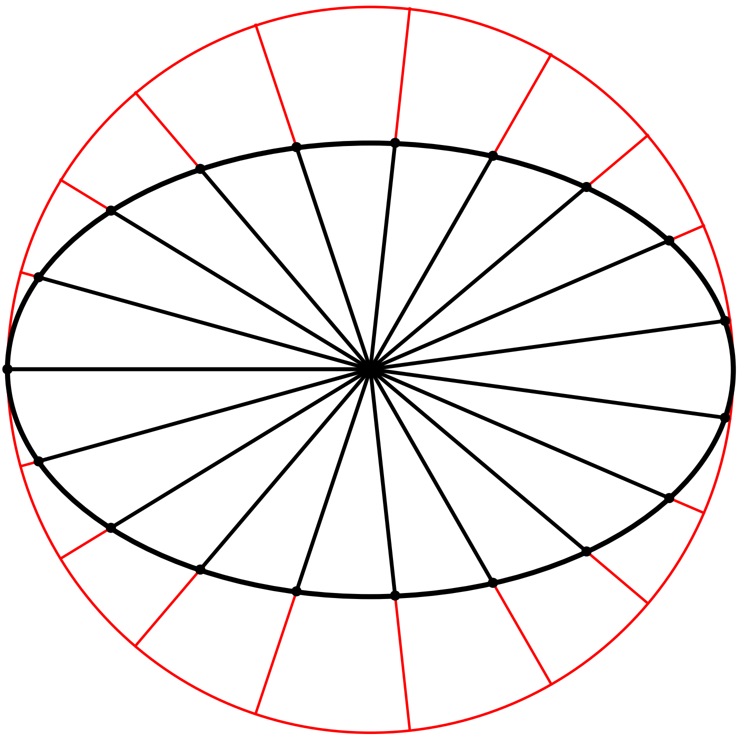

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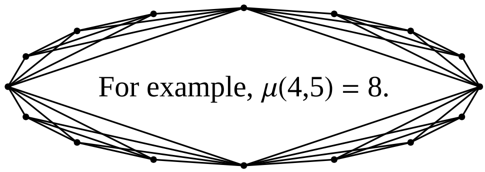
Solution 302.5 – Eigenvalues

(i) Find a closed formula for the function $\mu(m, n)$ defined for integers $m, n \geq 3$ by

$$\mu(m, 3) \;=\; 0, \qquad \mu(m, n+1) \;=\; \mu(m, n) + m, \quad n \geq 3.$$

(ii) Let $m, n \geq 3$ be integers. Take m copies of the complete graph K_n and join them together to form a cycle where two adjacent K_n graphs have precisely one vertex in common. Show that the multiplicity of eigenvalue -1 of the graph's adjacency matrix is $\mu(m, n)$. Or find a counter-example.



Tommy Moorhouse

The solution to the recurrence relation is easily deduced, namely $\mu(m, n) = (n-3)m$. The second part of the problem describes the construction of a family of graphs. We can iteratively construct the incidence matrices of subfamilies as follows. Fix m. Each graph of the family, denoted $K(m, n)$ for given n, is based on a cyclic graph C_m, which we will call the skeleton of $K(m, n)$. The incidence matrix of C_m will be denoted M_0, and this is shown below.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

We label the vertices of the skeleton clockwise from 1 to m. We build the graph $K(m, n)$ in stages. First we construct $K(m, 3)$ by joining new vertex $m+1$ to vertex 1 and vertex 2, joining new vertex $m+2$ to vertex 2 and vertex 3 and so on, finishing by joining new vertex $2m$ to vertex m and vertex 1. The new vertices are connected only to the adjacent vertices of the skeleton. The incidence matrix of $K(m, 3)$ is then, in block form,

$$\begin{bmatrix} M_0 & h \\ h^t & \mathbf{0} \end{bmatrix}.$$

Here h stands for the $m \times m$ matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & \cdots & 0 \\ \cdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

and h^{t} stands for its transpose. This can be deduced from the structure of the graph around the skeleton. Here and in what follows $\mathbf{1}$ and $\mathbf{0}$ stand for the $m \times m$ unit and zero matrices respectively.

We now add another vertex between each of those of the skeleton. Vertex $2m+1$ is connected to vertex 1, vertex 2 and vertex $m+1$ but to no others, and similarly for vertices $2m+2, \ldots, 3m$. In general vertices congruent to $k \pmod m$ are connected to each other and to vertex $k+1$ of the skeleton. This leads to the block incidence matrix $M(m,4)$ (for $K(m,4)$)

$$\begin{bmatrix} M_0 & h & h \\ h^{\mathrm{t}} & \mathbf{0} & \mathbf{1} \\ h^{\mathrm{t}} & \mathbf{1} & \mathbf{0} \end{bmatrix}.$$

The next iteration is

$$\begin{bmatrix} M_0 & h & h & h \\ h^{\mathrm{t}} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ h^{\mathrm{t}} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ h^{\mathrm{t}} & \mathbf{1} & \mathbf{1} & \mathbf{0} \end{bmatrix}$$

and for $K(m,n)$ we have the $m(n-1) \times m(n-1)$ incidence matrix $M(m,n)$

$$\begin{bmatrix} M_0 & h & h & h & \cdots & h \\ h^{\mathrm{t}} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\ h^{\mathrm{t}} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{1} \\ \cdots \\ h^{\mathrm{t}} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{0} \end{bmatrix}.$$

We could proceed by finding the eigenvalue equation $\det(M - \lambda) = 0$ for λ and showing that it has multiple factors of $\lambda + 1$. However, the block form allows us to take a different approach. We take the simplest case first, computing

$$(M(m,3)+1)V \;=\; 0$$

where V is a column vector of length $2m$, with the first m rows forming a vector denoted u and the last m rows being denoted v. Multiplying out the blocks we find

$$\begin{aligned} M_0 u + hv &= -u, \\ h^{\mathrm{t}} u &= -v. \end{aligned}$$

Multiplying the second equation by h and substituting for hv in the first equation gives $(M_0 - hh^{\mathrm{t}} + 1)u = 1u = 0$. Here hh^{t} is

$$\begin{bmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 \\ \cdots & & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix}.$$

Thus $u = v = 0$ and so $V = 0$ and $M(m,3)$ has no repeated eigenvectors associated with the eigenvalue -1.

In the general case if we write the $m(n-1)$ vector V as the 'block' vector $(u, v^{(1)}, v^{(2)}, \cdots, v^{(n-2)})$ the eigenvalue equation $M(m,n)V = -V$ gives rise to $n-1$ equations for the block vectors. In fact only two of these are independent:

$$\begin{aligned} M_0 u + h \sum_{i=1}^{n-2} v^{(i)} &= -u, \\ h^{\mathrm{t}} u + \sum_{j>1} v^{(j)} &= -v^{(1)}. \end{aligned}$$

Reasoning as above this leads to the conclusion that $u = 0$ and that the $n-2$ m-vectors $v^{(i)}$ are linearly dependent (as m-vectors). Since there are only m linearly independent m-vectors the vectors $v^{(i)}$ can be arranged to form the $m(n-3) = \mu(m,n)$ linearly independent $\lambda = -1$ eigenvectors of $M(m,n)$.

To see how this works consider $K(m,4)$, when we find $u = 0$, $v^{(1)} = -v^{(2)}$. Thus the eigenvectors associated with $\lambda = -1$ form an m-dimensional subspace of $\mathbb{R}^{3m}$. Putting it all together we conclude that the number of repeated $\lambda = -1$ eigenvectors of $M(m,n)$ is $\mu(m,n)$.

Wanchancy and Game of Primes

Douglas Clarkson

Introducing Wanchancy

The game of ‘Wanchancy’ is based on a 16×16 matrix play board which is used to set down integer number sequences in a manner resembling a crossword puzzle. Thus sequences should therefore interlock appropriately without disturbing previous sequences. A total of 100 number tiles with distribution of values between 1 and 37 inclusive are available for play. During rounds of the game up to four players each use nine tiles to place integer sequences on the play board and where the minimum length of a sequence thus played is three tiles. The scoring of each sequence is based on the sum of the individual tile values of the sequence, and where also the play board can include factors to multiply the individual tile values (double, triple, quad) and also the entire sum of the sequence (double, triple).

Not being a mainstream mathematician, but rather with experience in applied physics, I restricted the scope of integer sequences to arithmetic and geometric types and also to polynomials of a single variable. This was, however, a vast underestimation of the complexity of the world of integer sequences. Limiting factors relating to allowed sequences include the fact that the maximum tile value is 37 and the restriction on the number of tiles of any specific value available. The game itself has developed incrementally over a number of years and with the various physical components of the game—the play board, number tiles and tile holders—being obtained from diverse sources. The documentation has also become more complex. Figure 1 indicates an example of a completed Wanchancy game.

Table 1 outlines the structure of a subset of sequences based on arithmetic and geometric relationships and Table 2 sequences based on polynomial of a single integer variable. The scope for geometric sequences is in fact rather limited compared with arithmetic types.

Sequence	Type
13, 17, 21, 25, 29, 33	arithmetic
16, 22, 28, 34	arithmetic
5, 10, 15, 20, 25	arithmetic
1, 3, 9, 27	geometric
2, 4, 8, 16, 32	geometric

Table 1: Examples of arithmetic and geometric sequences

Formula	Sequence	n
$n^2 + 1$	5, 10, 17, 26, 37	n = 2, 3, 4, 5, 6
$n^2 - 1$	3, 8, 15, 24	n = 2, 3, 4, 5
$n^2 - 3$	1, 6, 13, 22	n = 2, 3, 4, 5
$2n^2 - 3n + 7$	6, 9, 16, 27	n =1, 2, 3, 4
$n^3 + 4n - 4$	1, 12, 35	n = 1, 2, 3
$n^3 - n^2 + n + 7$	8, 13, 28	n = 1, 2, 3

Table 2: Examples of polynomial sequences of a single integer variable

It is certainly possible to define sequences based on relationships between the terms of an integer sequence. A well-known example of this is the Fibonacci sequence, which can be defined as the sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, etc. where $a(n) = a(n-1) + a(n-2)$. A related sequence is that known as Narayana's Cows sequence of 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, etc. where $a(n) = a(n-1) + a(n-3)$. A rule is suggested that sequences are structured in this way using at most two sequence elements as in the Fibonacci and Narayana's Cows sequence since it is almost possible to structure any integer sequence based on specific relationships between included individual elements if enough are included. A considerable number of integer sequences of this type are possible with definition of the first two or three terms and application of rules to combine them – as indicated in Table 3.

First terms	rule	sequence
1, 4	$a(n+1) = a(n) + a(n-1)$	1, 4, 5, 9, 14, 23, ...
1, 4	$a(n+1) = a(n) + 2a(n-1)$	1, 4, 6, 14, 26, ...
1, 5, 6	$a(n+1) = 2a(n-1) + (n-3)$	1, 5, 6, 13, 31, ...
3, 5, 8	$a(n+1) = 3a(n-3) + a(n-2)$	3, 5, 8, 14, 23, ...

Table 3: Additional notional sequences based on term combinations

As the game developed, however, it was considered appropriate to include integer sequences which would reflect more diverse maths properties and this would significantly expand the scope of the game. A key example of this would be prime numbers—with a total of 12 prime numbers being available in the number range of 1 to 37. Prime numbers lend themselves to subgroups with specific identities such as twin primes, cousin primes and permutable primes as outlined in Table 4.

Of the various 'families' of primes, perhaps the most intriguing is that of the Mersenne Primes, based on formula $2^n - 1$, named after the French monk

Marin Mersenne, who proposed in 1644 that the sequence would include values of n of 31, 67, 127 and 257. It was subsequently discovered, however, that the next terms of the sequence would continue 31, 61, 89, 107, 127 and 521. Currently Mersenne primes lead the charge to discover larger and larger new prime numbers based on the Great Internet Mersenne Prime Search (GIMPS) as coordinated by the University of Tennessee at Martin in the USA. It is perhaps the present lack of understanding of the theory of prime numbers that still makes progress in searches such as GIMPS a necessity. Annoyingly simple sequences can also be derived from combinations of prime number elements, such as the sum of successive pairs of prime numbers.

Description	Sequence Elements
Prime Numbers	2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37
Mersenne Primes	3, 7, 31
Twin Primes	(3, 5), (5, 7), (11, 13), (17, 19), (29, 31)
Cousin Primes	(3, 7), (7, 11), (13, 17), (19, 23)
Permutable Primes	2, 3, 5, 7, 13, 17, 37
Circular Primes	2, 3, 5, 7, 11, 13, 17, 37
Pierpont Primes	2, 3, 5, 7, 13, 19, 37
Emirps	13, 17, 31, 37
Lucas Primes	2, 3, 7, 11, 29
Fermat Primes	3, 5, 17
Sophie Germain Primes	2, 3, 5, 11, 23, 29

Table 4: Summary of 'interesting' Sequences within the Wanchancy number range based on prime numbers

The history of the Mersenne primes also touches upon the contributions of the Ancient Greeks, who were well versed in their structure and recognised the number 127 as a Mersenne prime. However, the discovery and unravelling of the workings and function of the Antikythera mechanism, estimated to date from around 2200 BC, and identified now as an accurate gear analogue computer based on an array of astronomical cycles, has led to a complete revision of both the Ancient Greeks' understanding of the physical world and the means to compute cycles and parameter relationships.

In the subset of primes known as 'emirps', an 'emirp' has the property of remaining a prime when its digits are reversed. The first few 'emirps' in this subset are 13, 17, 31, 37, 73, etc. Another interesting prime number subset is that of 'unique primes', where the reciprocal of a prime number expressed as a decimal fraction has a unique repeat period. There are surprisingly few primes with this property, and investigation has indicated there are 18 unique primes less than 1050.

Permutable primes are primes which remain prime for every rearrangement of their digits. Circular primes are primes which remain prime after cyclic rotation of their digits. Pierpont primes are based on the numbers of the format $2^k 3^l + 1$ where k, l are integers. The sequence reflects the number of ways in which a circle can be divided using origami paper folding techniques. Lucas primes are closely aligned with the Fibonacci series where the Lucas number sequence is given by $a(0) = 2$, $a(1) = 1$, $a(2) = 3$ and $a(n+1) = a(n) + a(n-1)$.

Other 'interesting' integer sequences have also been identified for use within Wanchancy and are indicated in Table 5. This would be, however, but a small fraction of the possible sequences that could be included.

Description	Sequence Elements
Powerful Numbers	1, 4, 8, 9, 16, 25, 27, 32, 36
Thabit Numbers	2, 5, 11, 23
Fortunate Numbers	3, 5, 7, 13, 17, 19, 23, 37
Palindromic Numbers	2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33
Square Pyramidal	1, 5, 14, 30
Triangular Tetrahedral	1, 4, 10, 20, 35
Toothpick Sequence	1, 3, 7, 11, 15, 23, 35
Diagonal Intersects	1, 5, 13, 35
Pronic Numbers	2, 6, 12, 20, 30
Partitions	1, 2, 3, 5, 7, 11, 15, 22, 30
Narayana's Cows	1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28
Pythagorean Triples	3, 4, 5; 5, 12, 13; 8, 15, 17; 7, 24, 25; 20, 21, 29
Semiprimes	4, 6, 9, 10, 14, 15, 21, 22, 25, 26, 33, 34, 35
Fibonacci Numbers	1, 1, 2, 3, 5, 8, 13, 21, 34
Abundant Numbers	12, 18, 20, 24, 30, 36

Table 5: Summary of additional 'interesting' sequences within the Wanchancy number range

These sequences are, for the most part, relatively straightforward in their structure. One of the more famous integer sequences is that of 'partitions', which is based on the number of ways to express a given number. As an example, the number 4 has five partitions since it can be represented as $(1+1+1+1)$, $(2+1+1)$, $(2+2)$, $(3+1)$ and (4). This particular topic was given prominence in the film *The Man Who Knew Infinity*, based on the life of the Indian mathematician Srinivasa Ramanujan and his cooperation with the Cambridge mathematician G. H. Hardy as documented in the book by Robert Kanigel of the same title as the film.

Figure 1: Image of completed game of Wanchancy

A positive integer n is described as powerful if for every prime p dividing n, p^2 also divides n. Powerful numbers can also be expressed as a^2b^3 where a and b are positive integers. Thabit numbers (also called 321 numbers) are given by the expression $3 \cdot 2^n - 1$ where n is the series of integers 1, 2, etc. This set of numbers was first studied in the 9th century by the Arabic scholar Thâbit ibn Quarra (836–901), who was a significant innovator in the world of mathematics centred in Baghdad. He also made important contributions to Arabic translations of Greek manuscripts, notably Euclid's *Elements*. The diagonal intersect sequence relates to the number of diagonals of regular polygons. Pronic numbers are the product of two consecutive integers and were initially studied by Aristotle. Semiprimes are numbers which are the product of two prime numbers, allowing also for squares of a single prime. An abundant number is described as a number where the sum of its (proper) divisors is greater than the number itself. As an example, in the first abundant number, 12, the sum of divisors $(1 + 2 + 3 + 4 + 6)$ is 16. The literature on the properties of numbers is extensive. The publication *The Magic Numbers of Dr Matrix* by Martin Gardner, once long-term columnist of Scientific American, gives the appearance of an honest account of dealings with a mysterious Dr Matrix, though the entire account is purely fictional though nonetheless intriguing. The extensive series of books published by Ian Stewart is especially useful since they are of excellent academic pedigree.

Enter the OEIS

Any investigation of integer number sequences sooner or later comes across the OEIS – the On-Line Encyclopaedia of Integer Sequences. This resource was founded by the British–American mathematician Neil Sloane and has grown from an initial list of 6000 sequences to a resource which currently identifies in excess of 340,000 sequences. This on-line resource has a very useful feature where the terms of a sequence can be entered into an enquiry box and the system will attempt to provide a reference number for the sequence or at least identify an equation to define it. The resource provides therefore a convenient method of checking if an integer sequence can be expressed as a valid numerical equation or as a recognised mathematical identity—these being the requirements for sequences played in Wanchancy. This allows sequences set down on the play board by one player to be expanded subsequently by other players where this knowledge of the derivation of a sequence is required to be shared as a feature of the game. When using the OEIS to identify a sequence, it is of benefit to include additional terms beyond the value 37 since this will assist in the process of sequence identification. The resource of the OEIS is in a way an expression of the eternal allure of integer number sequences within mathematics. The enthusiasm for this can be observed in the various video clips in YouTube featuring Neil Sloane. As a convenience, OCR images of the OEIS website and a Wolfram Alpha web page are included in the game documentation to allow ease of access to these sites using a smartphone.

Documentation has been a key element of the Wanchancy project where the game requires relevant guidance to allow players to play the game appropriately. The resource of Amazon's Kindle Direct Publishing has been used to provide affordable paperback copies of such instructions. Over time the page count has grown from a mere 20 pages to now over 50 and with the potential to keep expanding. There is a more general lesson here, for all those individuals with unpublished manuscripts, where with Kindle Direct Publishing there are no up-front costs to publish a paperback (and now hardback) edition. Just supply a file in appropriate format in Microsoft Word and a pdf cover design. Individual paperback author copies can be produced for use with Wanchancy by Amazon's Kindle Direct Publishing for less than the cost of a high street coffee. Potential authors seeking measurable return in sales of such self published material should be aware of the brutal reality of such publishing in this way without a plan for paid advertising as offered by Kindle Direct Publishing or a well configured social media exposure. Looking back on the history of mathematics, however, a very real factor limiting its progress has been the loss or delay in publishing of key manuscripts. Modern publishing technology, as demonstrated by Kindle Direct Publishing, however, identifies a very practical way to reduce this loss of resources. Such material, however, is of course not peer reviewed.

Enter Game of Primes

It was during the development of the board game Wanchancy that the idea of 'Game of Primes' emerged. Rather than place integer sequences on the play board, this game would focus instead on prime numbers—using number digits 0 to 9. Based on the selected board size, this allowed the placement of prime numbers up to 16 digits long. With a set of 7 tiles in range 0 to 9 available for each player, this allowed successively large prime numbers up to a length of 16 digits to be established on the play board. The use of either smartphone apps or Internet resources are required to check presented numbers for prime number status. For each prime number presented, the potential for scoring increases with the length of the specific prime number where the basic score is the sum of all of the digits in the prime number. Rather than adopt a conventional board scoring pattern as used with Wanchancy, a scoring pattern based on the Ulam Spiral has been adopted where higher scoring patterns are identified with the sequence of prime numbers between 1 and 256. The Ulam Spiral is of course based on the 'doodle' created by the Polish born mathematician Stanislav Ulam during a boring lecture. For a tile number placed on a square which is a prime number, a 'simple' prime adds + 10 to the score, while a twin prime adds + 20 and a cousin prime adds + 30. A cousin prime which is a neighbour of a twin prime adds + 40 to the score. This in a way communicates the apparent randomness of the distribution of prime numbers.

Suitable smartphone Apps for checking of prime numbers are available with free download for both Android and iPhone devices. As regards the Internet, the facility available through the University of Tennessee at Martin is a convenient site `https://primes.utm.edu/curios/includes/primetest.php`, which can be used for numbers up to the 16 digit number 9007199254740991. For values greater than this the resource created by Dario Alpern can be used `https://www.alpertron.com.ar/ECM.HTM`. Basic documentation about playing the game is available in paperback form, again using the convenience of Kindle Direct Publishing to produce copies economically. The documentation includes references to a subset of 'interesting' prime number sets essentially as outlined previously in Table 4. Figure 2 indicates an example of a completed game of Game of Primes.

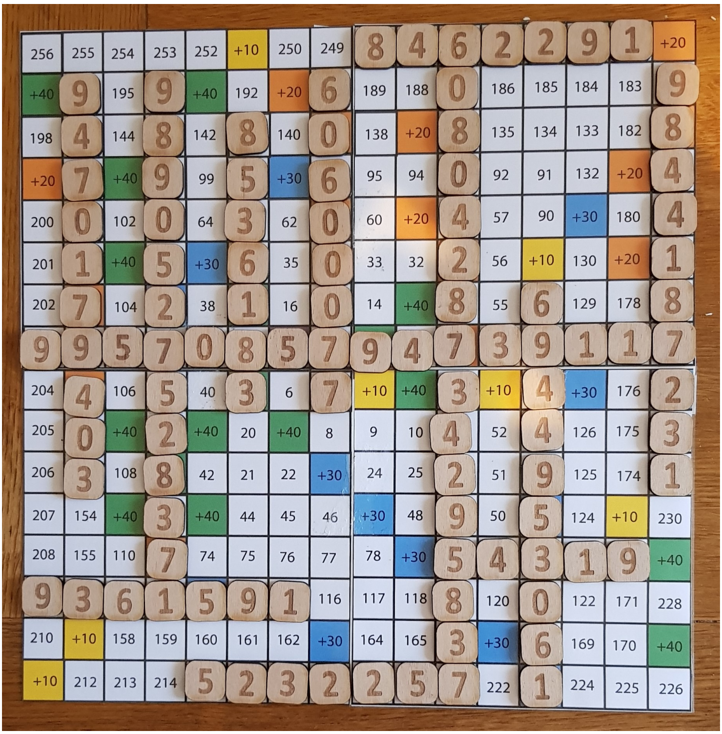


Figure 2: Image of completed game of Game of Primes

In Conclusion

It has been an interesting process developing both games and along the way various fascinating areas of mathematics have been encountered. As two novel games have been 'discovered', there is curiosity if a third one awaits uncovering.

References

Robert Kanigel, *The Man Who Knew Infinity*, Abacus, 2014.

Martin Gardner, *The Magic Numbers of Dr Matrix*, Prometheus Books, 1985.

Ian Stewart, *Significant Figures*, Profile Books, 2018.

Solution 303.7 – Thirtieth powers

Either show that the only solution in positive integers a, b and c of

$$a(a+b)(a+2b)+1 \;=\; c^{30}$$

is $a = 331$, $b = 1028$, $c = 2$, or find another.

Dave Wild

Lat n be a positive integer and a and b the positive integers 2 and $4n^3 + 6n^2 + 3n - 1$ respectively. Then

$$a(a+b)(a+2b)+1 \;=\; (2n+1)^6.$$

If $c > 2$ is an odd integer then we can find an n such that $2n+1 = c^5$. In this case

$$a(a+b)(a+2b)+1 \;=\; (2n+1)^6 \;=\; c^{30}.$$

Therefore a solution can be found if c is an odd integer greater than 1.

Problem 307.1 – Eigenvalues of graphs

Let G be a simple graph with n vertices, $n \geq 2$. Remove a vertex and its incident edges from G to get a graph H with $n-1$ vertices. Let λ_1, λ_2, $\ldots$, λ_n be the eigenvalues in non-decreasing order of the adjacency matrix of G. Let μ_1, μ_2, $\ldots$, μ_{n-1} be the eigenvalues in non-decreasing order of the adjacency matrix of H. Show that

$$\lambda_1 \;\leq\; \mu_1 \;\leq\; \lambda_2 \;\leq\; \mu_2 \;\leq\; \ldots \;\leq\; \lambda_{n-1} \;\leq\; \mu_{n-1} \;\leq\; \lambda_n.$$

Solution 270.4 – Limit

Show that

$$\frac{(2\,n!)^{1/n}}{n} \;\to\; \frac{1}{e} \qquad \text{as } n \to \infty.$$

Peter Fletcher

Let

$$L \;=\; \lim_{n\to\infty} \left(\frac{(2\,n!)^{1/n}}{n} \right)$$

so that

$$\log(L) \;=\; \lim_{n\to\infty} \left(\frac{1}{n} \log\,(2\,n!) - \log(n) \right).$$

We can write $\log(n)$ as $(1/n)\log\,(n^n)$. Doing this, we get

$$\log(L) \;=\; \lim_{n\to\infty} \left(\frac{1}{n} \Big(\log(2) + \log(n!) - \log(n^n) \Big) \right).$$

Since

$$\lim_{n\to\infty} \left(\frac{\log(2)}{n} \right) \;=\; 0,$$

expanding the remaining terms gives

$$\log(L) \;=\; \lim_{n\to\infty} \left(\frac{1}{n} \Big(\log(1) + \log(2) + \cdots + \log(n) - \log(n) - \log(n) - \cdots - \log(n) \Big) \right).$$

This can now be written as

$$\log(L) \;=\; \lim_{n\to\infty} \left(\frac{1}{n} \Big(\log(1/n) + \log(2/n) + \cdots + \log(n/n) \Big) \right),$$

which is just the definition of the area between the x-axis and $\log(x)$ from $x = 0$ to $x = 1$, i.e.

$$\log(L) \;=\; \int_0^1 \log(x)\,\mathrm{d}x \;=\; \Big[x\log(x) \Big]_0^1 - \int_0^1 \mathrm{d}x \;=\; -1.$$

Therefore

$$L \;=\; \frac{1}{e}.$$

Solution 281.3 – Powers of 2

Show that for any positive integer there is a non-negative power of two which starts with this integer. For example, we have the following.

Integer	1	2	3	4	5	6	7	8	9	10	500
Power of 2	0	1	5	2	9	6	46	3	53	10	1650

This problem was mentioned in the *Coursera* course 'What is a Proof?'

Dave Wild

Let M be a positive integer which is not a power of 10. If there is a positive integer n such that M^n starts with a positive integer N then there is a non-negative integer T such that $N \le M^n 10^{-T} < N + 1$.

Let x be a real number which satisfies $1 < x < 1 + 1/N$. Then there is a positive power of x which lies in the interval $[N, N + 1)$, since

$$N \cdot x \;<\; N \cdot (1 + 1/N) \;=\; N + 1.$$

If an x can be found in this range which has the form $M^m \cdot 10^t$ where m is a positive integer, and t is an integer then we know there are values n and T that satisfy the above inequality.

To find an appropriate value of x use the following algorithm.

1. Let the upper limit $U = M^1 \cdot 10^{-k}$ where k is an integer chosen so that $1 < U < 10$. Let the lower limit $L = U/10$ so $0.1 < L < 1$. Both U and L are of the form $M^i \cdot 10^j$ where i is a positive integer and j is an integer.

2. If $U < 1 + 1/N$ then $x = U$ is the value sought and a value of M^n which starts with N exists. Otherwise go to the next step.

3. Calculate LU. This is of the same form as L and U. Also $0.1 < L < LU < U < 10$. LU cannot equal 1 since this would imply a positive power of M equals a power of 10. Therefore either $U < LU < 1$ or $1 < LU < U$.

4. If $LU < 1$ increase the lower bound to LU. If $LU > 1$ decrease the upper bound to LU. Therefore on every iteration one of the bounds is changed. Both will eventually be changed. If $L = 1 - \delta_L$ and $U = 1 + \delta_U$ consider what happens to their product when $\delta_L \ll \delta_U$ and $\delta_U \ll \delta_L$. Return to step 2.

As U tends to 1 then we will eventually find a value less than $1 + 1/N$ whatever the value of N. Therefore there is always a power of M which starts with N. In fact we have shown there is at least one solution for all the integers from 1 to N.

As an example, a power of 2 will be found which starts with 38. Clearly, $1 + 1/38$ is approximately 1.026. The values of the lower and upper bounds will be

$$(0.2,\ 2.0),\ (0.4,\ 2.0),\ (0.8,\ 2.0),\ (0.8,\ 1.6),\ (0.8,\ 1.28),\ (0.8,\ 1.024).$$

As the upper bound is less than 1.026 then a value of the required form has been found. In this case it is $2^{10} \cdot 10^{-3}$. If we now solve

$$x^k \;=\; (2^{10} \cdot 10^{-3})^k \;=\; 38$$

and round k up to the next integer, then $k = 154$. So 2^{1540} starts with 38; 2^{85} also starts with 38.

Problem 307.2 – Angles in an ellipse

Tony Forbes

An ellipse with radii a and b is centred at the origin and has its $2a$ diameter on the x-axis. Let X, P and Y be three points on the circumference of the ellipse such that the distances from P to X and from P to Y along the ellipse are each $d/2$. That is, XPY is an arc of length d centred on P. Find a formula for the function $A(P, d)$ that gives the angle subtended at the origin by X and Y.

A search on the internet reveals that this is a well-asked question, usually expressed as: 'How can I have equally-spaced points around the perimeter of an ellipse?' However, as far as I can see—and I would not be the last to admit that it is not very far—all of the offered solutions—if indeed there actually exist any—are far too shrouded in obfuscation for me to readily understand—on the assumption that I can sometimes readily understand things communicated to me without excessive use of long words. The front cover illustrates an ellipse with 19 points.

If one can't avoid non-elementary functions for an exact answer, a good approximation will do, sufficient for drawing pictures when $|\log(a/b)|$ is not too large. Note that this is not the same as drawing an equilateral polygon with all of its vertices on the ellipse. That was M500 Problem 275.2 – Elliptic polygon.

Re: Problem 303.3 – Bin packing

There are infinitely many empty bins, each of capacity 100. At each tick of the clock you are presented with a random integer x in the range $[1, 100]$. You scan the partially filled bins that can accommodate an extra x. If there are none, you put x into an empty bin. Otherwise you put x into a bin that leaves the smallest unused capacity when x is added to it.

For example, you can verify that the sequence

$$79, 21, 68, 90, 1, 1, 33, 78, 30, 65, 21, 10, 34, 96, 68, 59, 99, 24, 56, 42$$

requires 11 bins of which 2 are full and 9 are partly filled thus: 98, 92, 98, 99, 96, 92, 59, 99, 42.

If n is the number of trials, let $b(n)$ be the number of bins required and $f(n)$ the number of full bins. What are the expected values of $b(n)/n$ and $f(n)/n$ as n tends to infinity?

Ted Gore

If there are n trials then each number will on average occur $n/100$ times. No two numbers greater than 50 can be in the same bin so there must be at least $50n/100 = n/2$ bins.

In addition, there are $n/100$ occurrences of the value 50. These cannot be placed in any of the bins mentioned above but it would be possible to have two occurrences in one bin so we require another $n/200$ bins. The minimum number of bins required is

$$\frac{n}{2} + \frac{n}{200} \;=\; \frac{101n}{200} \;=\; 0.5050n.$$

I ran a simulation three times for each of the n values in the table on the next page and averaged the results. In the table, b^* and f^* are the number of bins and full bins beyond $0.5050n$.

From the table we can make the following observations:

(i) $b(n)/n$ is slowly decreasing as n increases; but it cannot get smaller than 0.5050;

(ii) $f(n)/n$ is increasing as n increases; but it cannot get larger than $b(n)/n$;

(iii) b^*/b is decreasing as n increases; b^* is making a smaller contribution to $b(n)/n$;

(iv) $f(n)/b(n)$ seems to be approaching 1 as n increases, confirming that eventually $f(n)$ will equal $b(n)$.

Conclusion: $\dfrac{f(n)}{n}$ and $\dfrac{b(n)}{n}$ both tend to 0.5050 as n tends to infinity.

n	$b(n)$	$\frac{b(n)}{n}$	$f(n)$	$\frac{f(n)}{n}$	$\frac{f(n)}{b(n)}$	$b^*(n)$	$f^*(n)$	$\frac{b^*}{b}$
3000	1595	0.5316	1004	0.3348	0.6295	80	10	0.0501
5000	2594	0.5198	1756	0.3513	0.6769	69	11	0.0266
10000	5209	0.5209	3884	0.3911	0.7456	159	30	0.0305
20000	10314	0.5150	7527	0.3763	0.7298	214	61	0.0207
50000	25723	0.5145	21118	0.4224	0.8210	473	131	0.0184
100000	51204	0.5120	43539	0.4354	0.8503	704	216	0.0137
200000	102278	0.5114	90188	0.4509	0.8818	1278	515	0.0125
500000	255617	0.5112	235313	0.4706	0.9206	3117	1528	0.0122

Problem 307.3 – Partitioned permutations

Tony Forbes

Given positive integers r and s, how many of the $(rs)!$ permutations of the integers 1, 2, ..., rs can be partitioned into r sets of s elements, where each set covers all residue classes modulo s?

For example, when $r = 3$ and $s = 2$ we are asking for the number of permutations of the form $(a_1, a_2, b_1, b_2, c_1, c_2)$, where each of $\{a_1, a_2\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$ consists of an even–odd pair.

The example illustrates a common real-life situation when $s = 2$. Suppose you want to pair off your socks after taking them out of the dryer. If you do so at random, you might be interested in the probability of getting the left–right pairings correct. Of course, we are assuming that the socks are identical except that half are marked 'P' and the other half 'S'. Thanks to **Jed Baxter** for suggesting the problem.

Solution 304.6 – Trinomial factorization

Show that for every positive integer n except powers of 3, the polynomial $x^{2n}+x^n+1$ has a non-trivial factorization into polynomials with integer coefficients. For example,

$$\begin{aligned}
x^4+x^2+1 &= (x^2-x+1)(x^2+x+1),\\
x^8+x^4+1 &= (x^2-x+1)(x^2+x+1)(x^4-x^2+1),\\
x^{10}+x^5+1 &= (x^2+x+1)(x^8-x^7+x^5-x^4+x^3-x+1),\\
x^{12}+x^6+1 &= (x^6-x^3+1)(x^6+x^3+1),\ \ldots.
\end{aligned}$$

Stuart Walmsley

For convenience, the polynomials will be denoted by T_n. It is noted that

$$(x^{2n}+x^n+1)(x^n-1) = x^{3n}-1.$$

In this way, if $U_n = x^n-1$ is a second set of functions, then

$$T_n = U_{3n}/U_n.$$

The functions U_n are well known, the roots of the equation $U_n = 0$ being n distinct roots of 1, which are in general complex having the form $\exp(2\pi ij/n)$.

It is known that if attention is restricted to the field of rational numbers, U_n may be factored into polynomials with integer coefficients: the cyclotomic polynomials, C_n. If each function U_n is treated in turn starting with $n=1$, a new polynomial emerges at each step. In this way

$$\begin{aligned}
U_1 &= (x-1), & U_1 &= C_1,\\
C_1 &= x-1,\\[4pt]
U_2 &= (x^2-1) = (x-1)(x+1), & U_2 &= C_1C_2,\\
C_2 &= x+1,\\[4pt]
U_3 &= (x^3-1) = (x-1)(x^2+x+1), & U_3 &= C_1C_3,\\
C_3 &= x^2+x+1,\\[4pt]
U_4 &= (x^4-1) = (x^2-1)(x^2+1), & U_4 &= C_1C_2C_4,\\
C_4 &= x^2+1,\\[4pt]
U_5 &= (x^5-1) = (x-1)(x^4+x^3+x^2+x+1), & U_5 &= C_1C_5,\\
C_5 &= x^4+x^3+x^2+x+1.
\end{aligned}$$

In general,

$$U_n = C_1 \ldots C_j \ldots C_n,$$

where j runs over all the factors of n. In the step by step procedure mentioned here $C_1, \ldots, C_{n-1}$ are known so that C_n may be determined.

Returning to the trinomials T_n,

$$T_n = \frac{U_{3n}}{U_n} = \frac{C_1 \ldots C_j \ldots C_{3n}}{C_1 \ldots C_k \ldots C_n},$$

where j runs over the factors of $3n$ and k runs over the factors of n. Thus T_n is a cyclotomic polynomial if all terms cancel except C_{3n}. This only happens if n is 3^j, $j = 0, 1, 2, \ldots$, so that $3n$ is 3^{j+1}.

In general, all terms which do not cancel correspond to multiples of 3. In this way

$$\begin{aligned}
T_1 &= C_3,\\
T_2 &= C_3 C_6,\\
T_3 &= C_9,\\
T_4 &= C_3 C_6 C_{12},\\
T_5 &= C_3 C_{15},\\
T_6 &= C_9 C_{18},\\
T_7 &= C_3 C_{21},\\
T_8 &= C_3 C_6 C_{12} C_{24}.
\end{aligned}$$

The trinomials provide a means of determining the cyclotomic polynomials corresponding to multiples of 3.

$$\begin{array}{rcllcl}
C_3 &=& T_1 & &=& (x^2 + x + 1),\\
C_6 &=& T_2/C_3 & &=& (x^2 - x + 1),\\
C_9 &=& T_3 & &=& (x^6 + x^3 + 1),\\
C_{12} &=& T_4/C_3C_6 & &=& (x^4 - x^2 + 1),\\
C_{15} &=& T_5/C_3 & &=& (x^8 - x^7 + x^5 - x^4 + x^3 - x + 1),\\
C_{18} &=& T_6/C_9 & &=& (x^6 - x^3 + 1),\\
C_{21} &=& T_7/C_3 & &=& (x^{12} - x^{11} + x^9 - x^8 + x^6 - x^4 + x^3 - x + 1),\\
C_{24} &=& T_8/C_3C_6C_{12} & &=& (x^8 - x^4 + 1),\\
C_{27} &=& T_9 & &=& (x^{18} + x^9 + 1).
\end{array}$$

A similar scheme can be devised to determine the cyclotomic polynomials corresponding to multiples of any prime number.

Hearst Castle

Jeremy Humphries

I was reading the Wikipedia entry for Hearst Castle, the former home of the US publishing tycoon William Randolph Hearst, and I saw the following statement:

> In 1919 Hearst inherited some \$11 million (equivalent to \$171, 925,144 in 2021) ...

Since I get annoyed when I see 2 s.f. approximations converted into 9 s.f. equivalents, I thought I would edit the 2021 value to something more sensible, like \$172 million. But when I went to the edit page, I found this:

> In 1919 Hearst inherited some \$11 million ({{Inflation | US | 11000000 | 1919 | fmt=eq}}) ...

Ah, not so simple. I looked up that thing and I found out it's the Wikipedia inflation template. I also saw that you can add a parameter for rounding, the one for 'millions' being 'r=-6'. So I changed it to:

> In 1919 Hearst inherited some \$11 million ({{Inflation | US | 11000000 | 1919 | fmt=eq | r=-6}}) ...

Now the entry page said:

> In 1919 Hearst inherited some \$11 million (equivalent to \$172,000,000 in 2021) ...

Well, better than it was, but I didn't like the mismatch in how the sums were expressed. So I went back again and changed '\$11 million' to '\$11,000,000', and I was done:

> In 1919 Hearst inherited some \$11,000,000 (equivalent to \$172,000,000 in 2021) ...

Tony Forbes

While we are on the subject of significant figure inflation, I am reminded of a ridiculous example in the book *Recreations in Science and Natural Philosophy. Dr. Hutton's Translation of Montucla's Edition of Ozanam. New Edition Revised and Corrected, with Numerous Additions, by Edward Riddle*, by Jacques Ozanam (William Tregg & Co., London, 1851).

In Problem XII, *Calculation of the time which Archimedes would have required to move the earth, with the machine of which he spoke to Hiero* (page 202), Ozanam writes the following.

> The expression which Archimedes made use of to Hiero, king of Sicily, is well known, and particularly to mathematicians, "Give me a fixed point," said the philosopher, "and I will move the earth from its place." This affords matter for a very curious calculation, viz. to determine how much time Archimedes would have required to move the earth only one inch, supposing his machine constructed and perfect; that is to say, without friction, without gravity, and in complete equilibrium.
>
> For this purpose, we shall suppose the matter of which the earth is composed to weigh 300 pounds the cubic foot, ... If the diameter of the earth be 7930 miles, the whole globe will be found to contain 261107411765 cubic miles, which make 1423499120882544640000 cubic yards, or 38434476263828705280000 cubic feet; and allowing 300 pounds to each cubic foot, we shall have 11530342879148611584000000 for the weight of the earth in pounds.

In the first step of Ozanam's very curious calculation it is clear that the whole globe is assumed to be a perfect one and that $\pi = 3.14160000000$. Thus we have a sad case of scientific as well as mathematical nonsense. However, we can eliminate the latter by using the correct value of π to obtain 11530315916263181014451372.3 pounds. Unfortunately it is still scientifically absurd. See also Eddie Kent, A simple lever, M500 **269**.

Incidentally, in M500 I usually adopt the rule that a quantity with a decimal point is an approximation to the stated number of significant figures. Conversely, if there is no decimal point, the value is often—but not always—assumed to be exact, as in Ozanam's assumptions concerning the density and diameter of the Earth. This can sometimes lead to an element of either clumsiness or confusion. The Earth's radius, according to *Wikipedia*, varies from 3950 to 3963 miles. As far as I can see, the only reasonable way to express this as a number correct to 2 significant figures is to write '4.0×10^3 miles'. If you write '4000 miles' without qualification, the number of significant digits is either 1 (assuming only the 4 is significant), or $z + 1$ (assuming z zeros are also significant, $z \in \{1, 2, 3\}$), or infinity, depending of how the reader chooses to interpret it. Also incidentally, I think M500 should adopt the standard that the planet is 'Earth'. On the other hand, an expression like 'the weight of the earth in pounds' would be appropriate, for example, when referring to a delivery of soil for someone's garden.

04.00

Problem 307.4 – Ant

Tony Forbes

An elastic rope has length 1 m at time $t = t_0$. It is being stretched in such a manner that the velocity of one end relative to the other end is $v(t)$ m/s. At time $t = t_0$ an ant starts at one end of the rope and walks towards the other end at a constant velocity u m/s relative to the rope.

It is well known that when $v(t)$ is constant the ant will eventually reach the other end. However, if u is small, say $u = 0.01$, and $v(t)$ is not small, say $v(t) = 100$, it will take quite a long time. You might like to determine how long.

On the other hand, if the stretching is accelerating sufficiently rapidly, the ant will never reach the other end.

So what we are asking for is a simple function $v(t)$ where the said insect only just manages to complete its journey.

Front cover Equally spaced points on an ellipse—see page 15.