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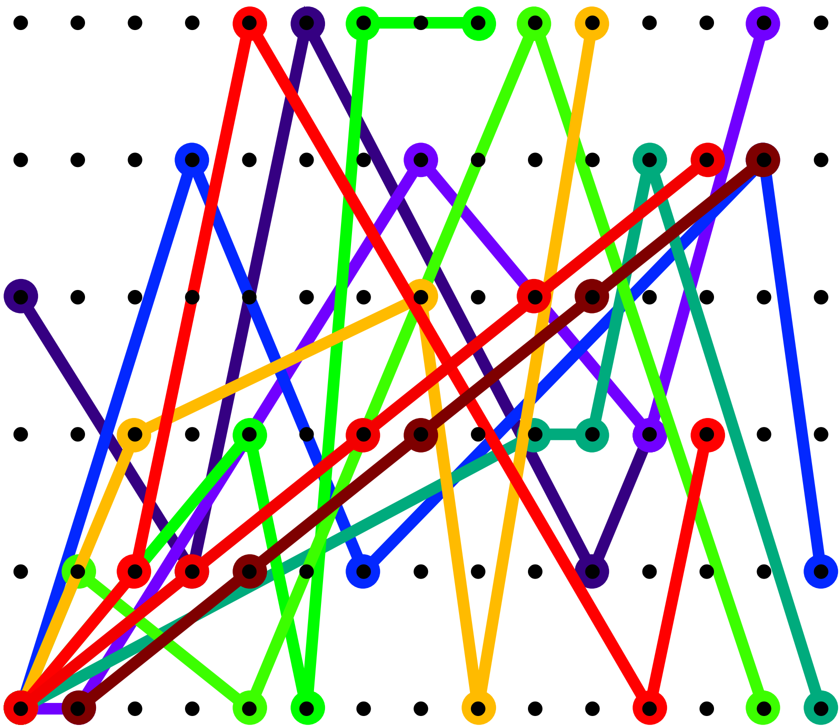
M500 308



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The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: m500.org.uk.

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Friday 13th – Sunday 15th January 2023
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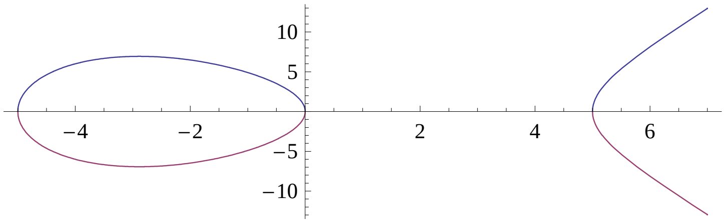
m500.org.uk/winter-weekend/

Solution 296.3 – Elliptic curve

Let a be a positive real number. Then the elliptic curve $y^2 = x(x^2 - a^2)$ has two components, an unbounded curve that passes through $(a, 0)$ and a closed bubble that passes through $(0, 0)$ and $(-a, 0)$. What area does the bubble enclose?



J. M. Selig

In the solution to Problem 296.3 – Elliptic curve, by Ted Gore, published in M500 **302**, TF comments that the exact solution is

$$\frac{\sqrt{\pi}}{5}\frac{\Gamma(3/4)}{\Gamma(5/4)},$$

but declares that he doesn't know where the exact value comes from. I think I may be able to explain.

By symmetry, the area under the curve $y = \sqrt{x(x^2 - a^2)}$ gives half the area of the bubble. So the area, A, is given by the integral

$$A = 2\int_{-a}^{0} \sqrt{x(x^2 - a^2)}\, dx.$$

Now, we perform a couple of simple substitutions to put this into a standard form. First, let $z = -x/a$. This turns the integral into

$$A = 2a^{5/2}\int_{0}^{1} \sqrt{z(1 - z^2)}\, dz.$$

Next, we set $z^2 = t$, and the integral becomes

$$A = a^{5/2}\int_{0}^{1} t^{-1/4}(1 - t)^{1/2}\, dt.$$

This standard integral is known as the beta function $B(x,\, y)$, and in general we have
$$B(x,\, y) \;=\; \int_0^1 t^{x-1}(1-t)^{y-1}\, dt.$$
Hence the area of the bubble is just
$$A \;=\; a^{5/2} B\left(\frac{3}{4},\, \frac{6}{4}\right).$$
Of course, this doesn't really solve the problem; we have just recognised the coefficient of $a^{5/2}$ has a nice name. What we really want to find is a value for the number. We could just use a computer application such as MATHEMATICA or MATLAB to evaluate this number:
$$B\left(\frac{3}{4},\, \frac{6}{4}\right) \;=\; 0.95851218778847376595.$$
This doesn't help TF though. Note, however, that the beta function has several useful properties. The two we need are
$$B(x,\, y) \;=\; \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$
and a recurrence relation,
$$B(x+1,\, y) \;=\; \frac{x}{x+y} B(x,\, y).$$
This is easily derived from the relation $\Gamma(z+1) = z\Gamma(z)$. The beta function is also symmetric in its arguments, so that $B(x,\, y) = B(y,\, x)$, and hence we also have that
$$B(x,\, y+1) \;=\; \frac{y}{x+y} B(x,\, y).$$
See the solution to Problem 299.2 by Tommy Moorhouse in M500 **303** for a similar application of the beta and gamma functions.

We get that
$$B\left(\frac{3}{4},\, \frac{6}{4}\right) \;=\; B\left(\frac{3}{4},\, \frac{1}{2}+1\right) \;=\; \frac{1/2}{5/4} B\left(\frac{3}{4},\, \frac{1}{2}\right) \;=\; \frac{2}{5}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}.$$
Finally, since $\Gamma(1/2) = \sqrt{\pi}$ we get the stated solution,
$$A \;=\; a^{5/2} B\left(\frac{3}{4},\, \frac{6}{4}\right) \;=\; \frac{2\sqrt{\pi}}{5}\, \frac{\Gamma(3/4)}{\Gamma(5/4)} a^{5/2}.$$

(I think TF has possibly forgotten to multiply by two, to get the area of the bubble above and below the x-axis.)

I suspect that TF found this solution by using a symbolic algebra program such as MATHEMATICA to find the original integral. There are several other ways to express this result and it is not clear how these programs make their choice, at least not clear to me. For example, applying the recurrence relation once more and Euler's reflection formula,

$$\Gamma(1-z)\Gamma(z) \;=\; \frac{\pi}{\sin(\pi z)}, \qquad \text{when } z \text{ is not an integer,}$$

the area can be written using one application of the Gamma function,

$$A \;=\; \frac{4(2\pi)^{3/2}}{5\,\Gamma(1/4)^2}a^{5/2}.$$

In fact, we can write the area as an elliptic integral and in this form the area can be evaluated very simply. To see how to do this we go back and write the area in terms of the beta function,

$$A \;=\; \frac{2^{7/2}\pi}{5\,B(1/4,\,1/4)}a^{5/2},$$

since $\Gamma(1/4+1/4) = \sqrt{\pi}$.

Now,

$$B(1/4,\,1/4) \;=\; \int_0^1 t^{-3/4}(1-t)^{-3/4}\,dt \;=\; 2\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta\cos\theta}}.$$

The final relation here is the result of substituting $t = \sin^2\theta$ in the first integral. Next we can make the substitution, $\tan\theta = \tan^2(\phi/2)$. This is facilitated by using trigonometric identities to write the integral as

$$B(1/4,\,1/4) \;=\; 2\int_0^{\pi/2} \frac{\sec\theta}{\sqrt{\tan\theta}}\,d\theta,$$

and the derivative,

$$\frac{d\theta}{d\phi} \;=\; \cos^2\theta\tan(\phi/2)\sec^2(\phi/2).$$

The integral becomes

$$B(1/4,\,1/4) \;=\; 2\int_0^{\pi} \cos\theta\sec^2(\phi/2)\,d\phi \;=\; 2\int_0^{\pi} \frac{d\phi}{\sqrt{\cos^4(\phi/2)+\sin^4(\phi/2)}}.$$

Using Pythagoras' theorem and a double angle formula the expression under the square root sign can be simplified,

$$B(1/4,\,1/4) \;=\; 2\int_0^{\pi} \frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}.$$

Apart from the value of the top limit, this is the standard form of an elliptic integral of the first kind,

$$K(k) \;=\; \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}.$$

Here, k is known as the elliptic modulus. Notice that the graph of the integrand $1/\sqrt{1-k^2\sin^2\phi}$ over the range 0 to π has a reflection symmetry in the line $\phi = \pi/2$. Hence our integral must be twice the standard integral, the one with top limit $\pi/2$. That is we have that

$$B(1/4,\,1/4) \;=\; 4\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}} \;=\; 4K\left(\frac{1}{\sqrt{2}}\right).$$

Only certain values of k give elliptic integrals $K(k)$ that can be expressed in terms of gamma functions. See the *Wikipedia* article 'Elliptic integral'.

The area of the bubble is thus

$$A \;=\; \frac{2^{3/2}\pi}{5\,K(1/\sqrt{2})}a^{5/2}.$$

The point of these gymnastics is that there is a very efficient way to compute the elliptic integral using the arithmetic-geometric mean. We have

$$K(k) \;=\; \frac{\pi}{2\,\mathrm{agm}(1,\,\sqrt{1-k^2})}.$$

Here, the arithmetic-geometric mean $\mathrm{agm}(a_0,\,g_0)$ of a pair of numbers $a_0 > g_0$ is given by repeatedly taking the arithmetic mean and geometric mean of the numbers and the results. Specifically we set up a pair of sequences,

$$a_{i+1} \;=\; (a_i+g_i)/2, \qquad g_{i+1} \;=\; \sqrt{a_i g_i},$$

with starting values a_0 and g_0. It is not too difficult to show that

$$a_0 \ge a_1 \ge a_2 \ge a_3 \ge \cdots \ge g_3 \ge g_2 \ge g_1 \ge g_0.$$

In fact, both sequences converge to the same limit and this limit is defined as the value of $\operatorname{agm}(a_0,\, g_0)$.

Moreover, it is clear from the definition that $\operatorname{agm}(ca,\, cg) = c \operatorname{agm}(a,\, g)$ for any constant c. The area of the bubble is thus

$$A \;=\; \frac{4\sqrt{2}\operatorname{agm}(1,\, 1/\sqrt{2})}{5} a^{5/2} \;=\; \frac{4}{5}\operatorname{agm}(\sqrt{2},\, 1)\, a^{5/2}.$$

The sequences for the arithmetic geometric mean converge very quickly; below is a table of the first few values of the sequences converging to $\operatorname{agm}(\sqrt{2}, 1)$.

i	a_i	g_i
0	1.414213562373095048801689	1.000000000000000000000000
1	1.207106781186547524400844	1.189207115002721066717500
2	1.198156948094634295559172	1.198123521493120122606586
3	1.198140234793877209082879	1.198140234677307205798384
4	1.198140234735592207440631	1.198140234735592207439214
5	1.198140234735592207439922	1.198140234735592207439922
6	1.198140234735592207439922	1.198140234735592207439922

Notice, we get 24 decimal places after just 5 iterations. So, finally we have our result for the area of the bubble,

$$A \;=\; \frac{4}{5}\operatorname{agm}(\sqrt{2},\, 1)\, a^{5/2} \;\approx\; 0.958512187788473765951938 0\, a^{5/2},$$

which agrees with my computer's evaluation using gamma functions.

Problem 308.1 – Powers of 2 and 3

Write down the powers of 2 in a long line. Underneath them write down the powers of 3 according to the rule: 3^n is written under 2^j, where j is chosen such that $2^j < 3^n < 2^{j+1}$.

4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768	65536	131072
	9	27		81	243		729		2187	6561		19683	59049		177147

Is this always possible?

Irregularity of the calendar

Terry S. Griggs

Methodist churches are organised into circuits. In the one to which I belong it was decided that on every fifth Sunday of a month, there would not be services in each church but two joint circuit services. The aim of this is so that people from different congregations would be able to worship together approximately quarterly. If a year consisted of 364 days and was divided into four quarters of 91 days with months of 30, 30 and 31 days always in the same order in each quarter, this is what would occur; a *perfect schedule.* But an ordinary year contains 365 days and a leap year 366 days and the distribution of the number of days in each month is irregular.

January 1st	Alternate months with five Sundays
Sunday	October and December
Monday	July and September
Tuesday	
Wednesday	June and August
Thursday	March and May
Friday	August and October
Saturday	May and July

Table 1: Alternate months with five Sundays, ordinary year

January 1st	Alternate months with five Sundays
Sunday	July and September
Monday	
Tuesday	June and August
Wednesday	March and May
Thursday	August and October
Friday	May and July
Saturday	October and December

Table 2: Alternate months with five Sundays, leap year

It is the aim of this short note to explore to what extent this causes the schedule to deviate from the ideal. In particular it was a surprise to some that recently there were five Sundays in two months separated only by a single month (May and July). Call these *alternate months.* Note that it is impossible for two consecutive months both to contain five Sundays. For this to happen the total number of days in the two months would have to be at least $64 = 9 \times 7 + 1$. So it is relevant to ask whether the occurrence of five Sundays in alternate months will be a common feature or one that is comparatively rare?

There are either 52 or 53 Sundays in a year. In the former case eight months contain 4 Sundays and four months contain 5 Sundays, and in the latter case seven months contain 4 Sundays and five months contain 5 Sundays. Consider first the former case. Constructing a sequence of eight 4s and four 5s in which all occurrences of the 5s are at least two apart can only be achieved if the sequence is regular, i.e. 544544544544 or 454454454454 or 445445445445. In the latter case the construction of such a sequence is impossible. This suggests that the phenomenon of five Sundays in alternate months might be more common than originally thought.

January 1st	Four months between five Sundays
Sunday	December, and April of the next year
Monday	
Tuesday	
Wednesday	November, and March of the next year
Thursday	
Friday	January and May *
Saturday	January and May *

Table 3: Four months between five Sundays, ordinary year

January 1st	Four months between five Sundays
Sunday	
Monday	
Tuesday	November, and March of the next year
Wednesday	
Thursday	
Friday	January and May
Saturday	December, and April of the next year

Table 4: Four months between five Sundays, leap year

A complete analysis can be done by drawing up two tables, one for an ordinary year and one for a leap year, showing the number of Sundays in each month when January 1st is each day of the week. The reader is invited to do this for herself but in Tables 1 and 2 on page 6 we present the results, which can easily be verified.

In addition, five Sundays occur in months November and January of the next year when January 1st is on a Thursday in an ordinary year and a Wednesday in a leap year. Thus the phenomenon that there will be alternate months both containing five Sundays is common, not occurring only when an ordinary year begins on a Tuesday and a leap year on a Monday. In both of those years five Sundays occur regularly every three months. In fact in an ordinary year which is both preceded and followed by ordinary years, this regularity extends from the September of the previous year to the June of the next year, a period of 22 months, giving a false impression of the actual situation.

The analysis also allows us to identify when there are four months between those with five Sundays; see Tables 3 and 4 on page 7. It is impossible for there to be more than four months. Table 3 assumes that an ordinary year is followed by an ordinary year. When it is followed by a leap year only the two asterisked entries occur.

Our calendar is indeed irregular.

Problem 308.2 – Minifigures

Jeremy Humphries

Jonathan told me about a problem he and Felix had been discussing, concerning LEGO Minifigures. There are 12 Minifigures in a set, and you can buy them in packs of six. The packs consist of six different Minifigures, apparently randomly selected from the 12 available. If we assume that is true, how many six-packs would you expect to buy in order to get a complete set of all 12 Minifigures?

Problem 308.3 – Arithmetic progression

Show that

$$x + y \;=\; -\cos(\pi/9)$$

if $x + y < 0$ and $\dfrac{1}{1+x}, \dfrac{1}{1-y}, \dfrac{1}{x}, \dfrac{1}{y}$ are in arithmetic progression.

Solution 303.2 – Regular graphs with girth 6

Given integer $n \geq 2$, show that an $(n+1)$-regular graph with $2(n^2+n+1)$ vertices and girth 6 must be the incidence graph of a projective plane of order n. Or find a counter-example.

Recall that in a projective plane of order n, there is a set P of n^2+n+1 points and a set L of n^2+n+1 lines such that:

(i) each line is incident with $n+1$ points,

(ii) each point is incident with $n+1$ lines,

(iii) for any two distinct points, there is a unique line incident with both points, and

(iv) for any two distinct lines, there is a unique point incident with both lines.

Its incidence graph has vertices $P \cup L$ and there is an edge $p \sim \ell$ whenever point p is incident with line ℓ. Clearly the graph is $(n+1)$-regular and has $2(n^2+n+1)$ vertices. Moreover, it is not too difficult to show that it has girth 6. Therefore it is sensible to pose the stated problem.

Tommy Moorhouse

We will describe a construction of the $(n+1)$-regular graph G of girth six having $2(n^2+n+1)$ vertices and show that it is isomorphic to the incidence graph $I(\mathbb{F}_n\mathbb{P}^2)$ of the projective plane of order n, $\mathbb{F}_n\mathbb{P}^2$. Although the construction goes through for any integer n only prime power n will be considered here, because the idea of projective space of finite order makes sense over fields.

First we will get a better idea of the structure of the incidence graph of $\mathbb{F}_n\mathbb{P}^2$. The points of $\mathbb{F}_n\mathbb{P}^2$ are triples $[X,Y,Z]$, with each element in $\mathbb{F}_n$ and the equivalence relation $[X,Y,Z] \sim [kX,kY,kZ]$ where $k \in \mathbb{F}_n^*$, the multiplicative subgroup of $\mathbb{F}_n$. We do not include the triple $[0,0,0]$ so naively there are n^3-1 points. The equivalence relation for prime power n reduces this to $(n^3-1)/(n-1) = n^2+n+1$ distinct points.

The lines in $\mathbb{F}_n\mathbb{P}^2$ are again given by triples $[a,b,c]$ and a point $[X,Y,Z]$ is coincident with a line if $aX+bY+cZ \equiv 0 \pmod{n}$. A little thought shows that the set of lines L has the same structure as the set of points P, and there is a duality between the two, sending points to lines and vice versa.

The graph $I(\mathbb{F}_n\mathbb{P}^2)$ is bipartite, meaning that the vertices fall into two disjoint sets (here L and P) where elements of one set are linked by an edge to elements of the other set only. Thus the vertex-type sequence starting from a point will be $\cdots - p - l - p - l \cdots$. The girth of a graph is the length of its shortest cycle, a closed path using a subset of the edges of the graph once. In this case the girth is 6 and there are many cycles of length 6. An important point to keep in mind is that all the vertices of G are on the same footing: there are no distinguished vertices or groups of vertices.

Stage 1 – core and clusters We are given an $(n+1)$-regular graph G of girth six having $2(n^2+n+1)$ vertices, and we want to relate it to the incidence graph of $\mathbb{F}_n\mathbb{P}^2$. We will do this by construction. To start our construction select any vertex from G. For later convenience we call this the core P-vertex p_0. We can imagine drawing a point p_0 on paper (which is a nice way to check the construction for small values of n). Join p_0 by edges to $n+1$ vertices from G, say l_0, l_1, l_2, $\cdots$, l_n, calling these 'L-vertices'. Join each of the L-vertices by an edge to n unused vertices from G, which we also call P vertices. We have now placed $n(n+1)+1$ P-vertices and $n+1$ L-vertices. This part of the construction is clearly universal to all $(n+1)$-regular graphs with no cycles of length less than 4 and we naturally obtain two different types of vertex. Figure 1 shows this stage for $n=2$.

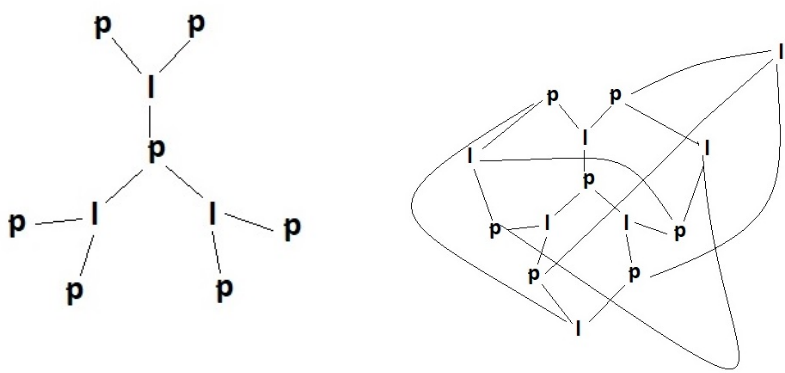


Figure 1: First and final stages of construction for $n = 2$.

We call the n new P-vertices joined to l_k 'cluster k', so that the different clusters are connected by a '$l - p - l$' bridge passing through the core. If we join any non-core P-type vertex from cluster k to any non-core vertex of cluster $m \neq k$ through an intermediate L-type vertex we find that we have created a 6-cycle. This suggests that we can complete the construction by placing n^2 L-vertices and connecting every vertex from each cluster to every other vertex from the other clusters, so that each of the new L-vertices has a single connection to each of the $n+1$ clusters. This part of the construction has an essentially unique outcome.

Stage 2 – linking clusters Label the non-core P vertices in each cluster $k \in \{0, 1, \cdots, n\}$, $p_{k,1}$ to $p_{k,n}$. Join vertex $p_{0,1}$ to $p_{k,1}$ of each other cluster $k > 0$ through a new intermediate vertex $l_{1,1}$. This vertex has degree $n+1$ and no cycles of length less than 6 are produced because the cycle must pass through the core vertex. Note that joining two vertices on the same cluster produces a cycle of length 3 and is not allowed.

Stage 3 – completing the construction, uniqueness Now join vertex $p_{0,1}$ to a new L-vertex $l_{1,2}$ and connect this to vertex $p_{k,2}$ of each other cluster $k > 0$. Again no short cycles are produced. Connecting to vertex $p_{k,1}$ of any of the clusters would produce a 4-cycle, which is not allowed. We have therefore essentially no choice but to connect to a different P-vertex, which we have labelled vertex $p_{k,2}$: recall that all the vertices in a cluster are equivalent so the labels are just to help us keep track of things. We continue the construction by linking vertex $p_{0,m}$ via the vertex $l_{m,q}$ to each vertex $p_{k,q}$, $k > 0$, until all the distinct P-vertices of each cluster have degree $n+1$. In this way we get n^2 degree $n+1$ vertices $l_{k,m}$, and every vertex has degree $n+1$. The construction is unique up to the equivalence of the vertices, which naturally fall into two disjoint sets. It is easily checked that the resulting graph is isomorphic to $I(\mathbb{F}_n\mathbb{P}^2)$ for prime power n.

Conclusion We have set out the graph G in such a way as to map it isomorphically to $I(\mathbb{F}_n\mathbb{P}^2)$ for prime power n. Thus the $(n+1)$-regular graph G of girth six having $2(n^2 + n + 1)$ vertices is unique up to isomorphism.

Problem 308.4 – Group construction

Tony Forbes

Let G be a finite group of even order. Let H be a subgroup of G of order $|G|/2$. Let $A = G \setminus H$ and assume there exists an element a of A that commutes with every element of A. Show that $(A, \circ)$ is a group, where the operation $\circ$ is defined by

$$x \circ y \ = \ xya.$$

The value of the zeta function at zero

Mako Sawin

The zeta function is

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where s is a complex variable, and when $s = 0$ we have,

$$\zeta(0) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^0} \;=\; \sum_{n=1}^{\infty} 1.$$

Consider two identical sums:

$$\sum_{n=0}^{\infty}(2n+1) \;=\; 1+3+5+7+9+11+13+\ldots$$

and

$$\sum_{n=1}^{\infty}(2n-1) \;=\; 1+3+5+7+9+11+13+\ldots.$$

Then

$$\begin{aligned}\sum_{n=1}^{\infty}(2n-1) \;&=\; \sum_{n=0}^{\infty}(2n+1),\\ \sum_{n=1}^{\infty} 2n - \sum_{n=1}^{\infty} 1 \;&=\; 1 + \sum_{n=1}^{\infty} 2n + \sum_{n=1}^{\infty} 1,\\ -2\sum_{n=1}^{\infty} 1 \;=\; 1, \qquad \sum_{n=1}^{\infty} 1 \;&=\; -\,\frac{1}{2}.\end{aligned}$$

Therefore

$$\zeta(0) \;=\; \sum_{n=1}^{\infty} 1 \;=\; -\,\frac{1}{2}.$$

By taking another two identical series such as

$$\sum_{n=2}^{\infty}(n-1) \;=\; \sum_{n=0}^{\infty}(n+1),$$

we obtain

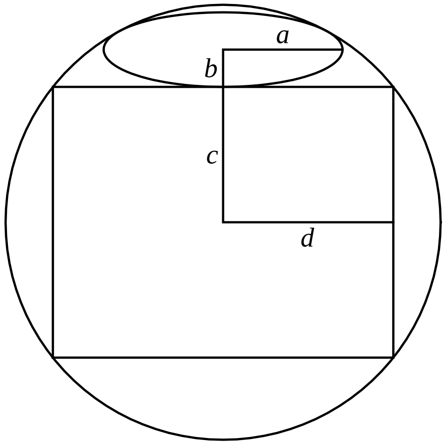
$$\sum_{n=2}^{\infty}(n-1) \;=\; 1+2+\sum_{n=2}^{\infty}(n+1).$$

Then
$$-2\sum_{n=2}^{\infty} 1 \;=\; 1+2,$$
which leads to
$$\sum_{n=2}^{\infty} 1 = -\frac{3}{2}, \quad \sum_{n=3}^{\infty} 1 = -\frac{5}{2}, \quad \ldots, \quad \sum_{n=k}^{\infty} 1 = -\frac{2k-1}{2}, \tag{1}$$
where k is a positive integer. Using (1), we deduce from
$$\sum_{n=2}^{\infty}(\zeta(n)-1) \;=\; 1$$
that
$$\sum_{n=2}^{\infty}\zeta(n) - \sum_{n=2}^{\infty} 1 \;=\; 1, \quad \sum_{n=2}^{\infty}\zeta(n) - \left(-\frac{3}{2}\right) \;=\; 1.$$
Thus
$$\sum_{n=2}^{\infty}\zeta(n) \;=\; -\frac{1}{2} \;=\; \zeta(0).$$
Similarly from
$$\sum_{n=1}^{\infty}(\zeta(2n)-1) \;=\; \frac{3}{4}$$
and (1) we obtain
$$\sum_{n=1}^{\infty}\zeta(2n) \;=\; \frac{1}{4},$$
and from
$$\sum_{n=1}^{\infty}(\zeta(4n)-1) \;=\; \frac{7}{8} - \frac{\pi}{4}\,\frac{e^{2\pi}+1}{e^{2\pi}-1}$$
we deduce that
$$\sum_{n=1}^{\infty}\zeta(4n) \;=\; \frac{3}{8} - \frac{\pi}{4}\,\frac{e^{2\pi}+1}{e^{2\pi}-1}.$$

Received 1/4/22 — TF

Solution 304.5 – A rectangle and an ellipse

This is like Problem 269.2 – Two rectangles, except that one of the rectangles is not a rectangle. A rectangle and an ellipse are packed inside a circle of radius 1, possibly but not necessarily according to the pattern indicated on the right. What's the largest area they can occupy?



When you have solved the problem as stated, try again but this time with the extra condition that the rectangle must be larger than the ellipse.

Ted Gore

Let the height of the rectangle be $2c$ and the width $2d$, where $d = \sqrt{1-c^2}$. Let the vertical semi diameter of the ellipse be b and the horizontal a.

The area of the rectangle is $A_r = 4cd$, the area of the ellipse is $A_e = \pi ab$, and the total area, $A = A_r + A_e$. There are certain constraints on a, b and c:

$$\begin{array}{rclrcl} c & > & 0, & c & < & 1; \\ b & > & 0, & b & \le & (1-c)/2; \\ a & > & 0, & a & \le & \sqrt{1-f^2}, \end{array}$$

where $f = b + c$. A computer program was used to calculate the area of configurations using a range of values conforming to these constraints.

There was one further constraint. It was necessary to exclude configurations where part of the ellipse fell outside the circle. These were identified as follows. The equation for the ellipse is

$$\frac{x^2}{a^2} + \frac{(y-f)^2}{b^2} = 1.$$

The ellipse touches the circle where $x^2 = 1 - y^2$ so that

$$y^2(a^2 - b^2) - 2fa^2y + b^2 + a^2f^2 - a^2b^2 = 0$$

and after some manipulation,

$$y = \frac{a^2f \pm \sqrt{b^2(a^2 - b^2)(a^2 - 1) + a^2b^2f^2}}{(a^2 - b^2)}.$$

Now the ellipse may touch the circle in 0, 1 or 2 points (considering only the upper right quadrant of the figure).

For 0 points, the ellipse is completely inside the circle. For 2 points, part of the ellipse falls outside the circle.

For 1 point, the ellipse touches the circle but does not continue outside it. To ensure there is just one touching point we require that the discriminant in the above equation for y is zero. That is,

$$b^2(a^2 - b^2)(a^2 - 1) + a^2b^2f^2 = 0.$$

From this we calculate

$$a^2 = \frac{1 + b^2 - f^2 \pm \sqrt{(f^2 - 1 - b^2)^2 - 4b^2}}{2},$$

which gives two possible values of a. Let these be a^+ and a^-.

The following table shows the maximum area of various configurations according to the value of a, whether the ellipse touches the point $(0, 1)$ and whether the area of the rectangle is greater than that of the ellipse.

c	b	a	$+$ or $-$	$(0,1)$	A_r	A_e	A
0.639	0.1805	0.424853	$+$ & $-$	y	1.966092	0.240916	2.207008
0.622	0.171612	0.549696	$+$	n	1.948147	0.296360	2.244507
0.201	0.3995	0.632060	$+$ & $-$	y	0.787591	0.793277	1.580869
0.218	0.36363	0.745789	$+$	n	0.851027	0.851973	1.703000
0.635	0.1825	0.427000	$-$	n	1.962181	0.244931	2.207112
0.199	0.4005	0.632851	$-$	n	0.780080	0.796258	1.576337

In addition, there are solutions where there is no point of contact between the ellipse and the circle. These occur when the discriminant in the equation for y is negative. In these cases, a is the x coordinate of the ellipse when the y coordinate is f. These ellipses lie completely within the circle. The table below shows the maximum area for solutions of this type.

c	b	a	$(0,1)$	A_r	A_e	A
0.62	0.1729	0.548166	n	1.945812	0.297890	2.243702
0.21	0.3713	0.740573	n	0.821269	0.863724	1.684993

We can make the following observations:

(a) Using this approach, the greatest area found is approximately 2.244507 and the greatest area where $A_e > A_r$ is approximately 1.703, and these can be thought of as limits to the cases in the second table.

(b) The eight rows in the two tables can be put into four pairs that have similar parameters and areas.

(c) Whenever the ellipse touches the point $(0,1)$ we know that $f + b = c + 2b = 1$. We can substitute into the equation for a^2 to show that $a^+ = a^-$ and $b = a^2$.

All the above results are for a configuration in which the centre of the rectangle is at the centre of the circle.

Alternatively, we could place the centre of the ellipse at the centre of the circle.

Let the horizontal semi-axis of the ellipse be 1 and the vertical semi-axis be $1-\varepsilon$, $\varepsilon > 0$. In this case the area of the ellipse is $\pi(1-\varepsilon)$. We could fit a rectangle with area less than $\pi\varepsilon/2$ above it. As ε approaches 0 the combined area of ellipse and rectangle approaches π.

Problem 308.5 – Integral

Let a be a positive integer. Show that

$$\int_0^1 \frac{\sqrt{1-x^a}}{\sqrt{1+x^a}}\,dx \;=\; \sqrt{\pi}\left(\frac{\Gamma\left(\dfrac{2a+1}{2a}\right)}{\Gamma\left(\dfrac{a+1}{2a}\right)} - \frac{1}{a+1}\,\frac{\Gamma\left(\dfrac{3a+1}{2a}\right)}{\Gamma\left(\dfrac{2a+1}{2a}\right)}\right).$$

Did the chap in the lions' den ever crack the question of doubling the cube? No, ⋆⋆⋆⋆⋆⋆ failed to resolve the ⋆⋆⋆⋆⋆⋆ problem. (The missing words are anagrams.)

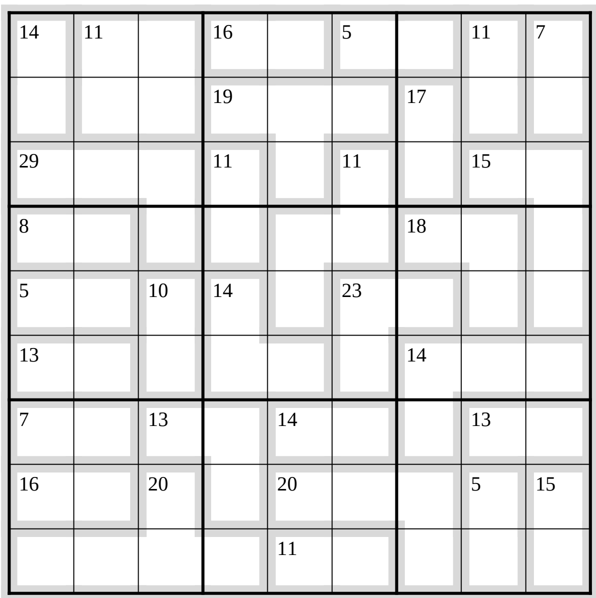
— Jeremy Humphries

Things you can't buy in shops – V

Tony Forbes

Following on from the lists that we printed in M500s **278**, **289**, **293** and **296**, here are some more useful items for you to enquire about if your browsing in a shop gets interrupted by the words "Can I help you?"

1. A kettle that turns the gas off when the water starts to boil.
2. A pencil eraser that comes with a 20-year guarantee.
3. A compact nuclear reactor suitable for providing power to a portable computer. It would save all that messing around with batteries and the charging thereof. (This probably exists but the Authorities would certainly frown upon using it without a licence.)
4. Software that makes one's computer or mobile telephone unusable for a period of 10 minutes in every hour.
5. A pesticide that actually kills pests but does not require military chemical warfare clothing to use safely.
6. A whiteboard that automatically fails to respond to permanent felt-tipped markers.



Puzzle

TF

Normal sudoku rules apply. Each row, column and 3 × 3 box must contain each of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once (not counting the little numbers already printed). In addition, the cells in an area enclosed by a thick grey border must sum to the total indicated.

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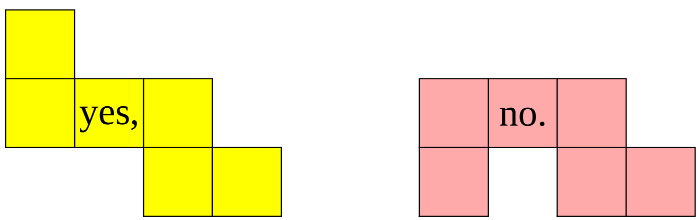
Problem 308.6 – Squares to cube

Find a simple test to determine whether six unit squares joined together in some way only along entire edges can be folded to form a unit cube. For instance,



Front cover Generators of a group divisible design with block size 5 and group type 6^{15}; https://arxiv.org/abs/2202.13911v2.