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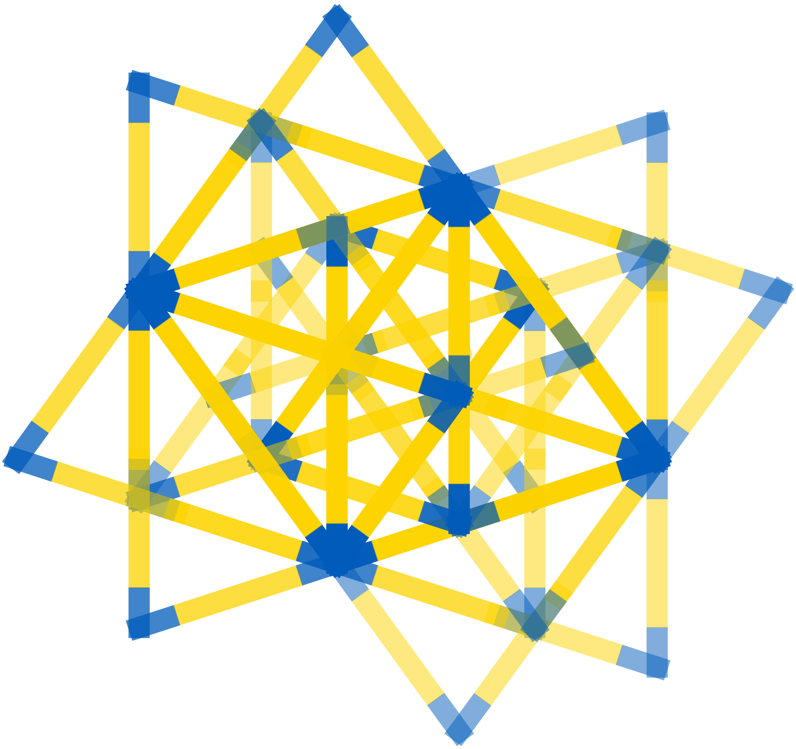

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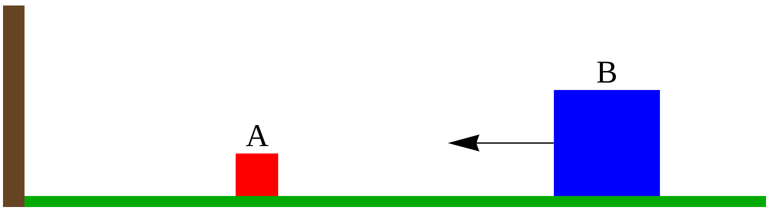
m500.org.uk/winter-weekend/

Solution 305.4 – Collisions

There is a wall on the left. To the right of the wall there is a stationary mass, A, of 1 kg, as in the picture. Mass B of m kg approaches from the right at non-zero speed and eventually collides with A. Mass A now moves towards the wall, rebounds from it and then heads towards B for another encounter. And so on.

The experiment stops when B is travelling to the right with sufficient speed to avoid a further collision with A. Assume all this is taking place on level ground, there is no friction, and collisions are perfectly elastic. Also you might want to ignore mutual gravitational attraction between A, B and the wall.



Show that for large m, the total number of collisions is $\sqrt{m}\,\pi$, at least approximately.

Tommy Moorhouse

Notation

We will use a slightly modified notation to that used in the problem, denoting the mass of the small mass by m, that of the large mass by M. We will also denote the respective speeds after k mass-mass collisions by $v_{(k)}$ and $V_{(k)}$. We take $V_{(k)}$ to be positive (moving rightward) rather than negative here. We will also write x for m/M.

Strategy

First we will demonstrate that the changes in velocities of the masses at each collision can be represented by the action of a constant matrix $\mathcal{M}$ on a vector. Next we will find a recurrence relation for the elements of the matrix powers. This will lead us to identifying a particular element of $\mathcal{M}^k$ as a well known polynomial. Finally we will use the approximation $M \gg m$ to deduce the result that the number of collisions $N \approx \pi\sqrt{M/m}$. The references for properties of the polynomials are in Chapter 22 of [Abramowitz & Stegun].

The collision matrix

The large mass has initial speed $V_{(0)}$, say. We will express all speeds in units of $V_{(0)}$, effectively setting it to 1. Consider the situation after k collisions between the masses ($2k-1$ collisions including those between the small mass and the wall), when the speeds of the masses are $v_{(k)}$ and $V_{(k)}$. The small mass collides with the wall and again hits the large mass. In the mass-mass collision energy and momentum are conserved:

$$\begin{aligned} mv_{(k+1)} + MV_{(k+1)} &= mv_{(k)} + MV_{(k)}, \\ mv^2_{(k+1)} + MV^2_{(k+1)} &= mv^2_{(k)} + MV^2_{(k)}. \end{aligned}$$

We can solve these two equations for the vector $(v_{(k+1)}, V_{(k+1)})$:

$$\begin{aligned} v_{(k+1)} &= \frac{2}{1+x} V_{(k)} - \frac{1-x}{1+x} v_{(k)}, \\ V_{(k+1)} &= \frac{1-x}{1+x} V_{(k)} + \frac{2x}{1+x} v_{(k)}. \end{aligned}$$

Now note that $v_{(k)}$ is reversed before the $(k+1)$st collision. This means that

$$\vec{v}_{(k+1)} \equiv (v_{(k+1)}, V_{(k+1)}) = \mathcal{M}^k \vec{v}_{(0)},$$

where

$$\mathcal{M} = \begin{pmatrix} \alpha & \alpha+1 \\ \alpha-1 & \alpha \end{pmatrix}$$

and $\alpha = (1-x)/(1+x)$. Observe that $\det \mathcal{M} = 1$, which is a consequence of the conservation of (kinetic) energy. Momentum is not conserved overall.

Recurrence relations and generating function

We will write the powers of $\mathcal{M}$ as

$$\mathcal{M}^n = \begin{pmatrix} p_n & q_n \\ r_n & s_n \end{pmatrix}.$$

We will only need the recurrence relations for q_n and s_n. Multiplying out the elements we arrive at the familiar relations (i.e. the same as those satisfied by $|v_{(k)}|$ and $V_{(k)}$):

$$\begin{aligned} q_{n+1} &= \alpha q_n + (\alpha+1) s_n, \\ s_{n+1} &= (\alpha-1) q_n + \alpha s_n. \end{aligned}$$

Let

$$Q(t) \;=\; \sum_{n=0}^{\infty} q_n t^n, \qquad S(t) \;=\; \sum_{n=0}^{\infty} s_n t^n.$$

We can write out $S(t)$ and $Q(t)$ and use the recurrence relations to eliminate $Q(t)$ and show that

$$S(t) \;=\; \frac{1 - t\alpha}{1 - 2t\alpha + t^2}.$$

This is proportional to the generating function for the Chebyshev polynomials $T_n(\alpha)$. In detail we have (with initial values $q_0 = 0, s_0 = 1$)

$$\begin{aligned} Q(t) &= q_0 + q_1 t + q_2 t^2 + \cdots + q_m t^m + \cdots \\ &= t(\alpha + 1)s_0 + t^2(\alpha q_1 + (\alpha + 1)s_1) + \cdots \\ &= t\alpha Q(t) + (\alpha + 1)tS(t), \\ Q(t)(1 - \alpha t) &= (\alpha + 1)tS(t). \end{aligned}$$

Similarly

$$S(t)(1 - \alpha t) \;=\; 1 + (\alpha - 1)tQ(t).$$

Eliminating $Q(t)$ gives the desired result. Now suppose that after the kth mass-mass collision the large mass is at rest. Then we must have $T_k(\alpha) \approx 0$.

Now we use the approximation $x \ll 1$, so that $(1 - x)/(1 + x) \approx 1 - 2x$ and the result that the zeros of $T_n(\alpha)$ are at

$$\alpha \;=\; \cos((2m - 1)\pi/2n).$$

We take $m = 1$ for the smallest zero. Thus, using $\arccos(1 - 2\epsilon) \approx 2\sqrt{\epsilon}$ for small ϵ,

$$\frac{\pi}{2k} \;\approx\; \arccos(1 - 2x) \;\approx\; 2\sqrt{x}.$$

Thus the number, N, of collisions, including collisions between the small mass and the wall, when the interaction is over, is approximately given, when $M \gg m$, by

$$N \;=\; 4k \;\approx\; \pi\sqrt{\frac{M}{m}}.$$

Reference

Abramowitz and Stegun, *Handbook of Mathematical Functions*, Dover, 1964 (reprinted 1970).

Alternative Solution

Notation

We denote the mass of the small mass by m, that of the large mass by M. We consider the evolution of a one-dimensional analogue of a frictionless piston of mass M forming one wall of a cylinder in which a gas molecule of mass m is moving. The large mass has speed $V(t)$, the small mass has speed $v(t)$. This method does not involve recurrence relations and orthogonal polynomials, relying rather on elementary calculus. We will work out the momentum transfer from the particle in the 'container' and hence the acceleration of the large mass. We will then use conservation of kinetic energy to find a differential equation for the distance moved by the mass, and finally use integration to find the number of collisions. A dot over a variable indicates differentiation w.r.t. time.

Momentum transfer

The large mass is initially at rest at a distance x_0 along the x-axis from the wall. The small mass is moving back and forth colliding alternately with the wall and the large mass. During each collision the momentum of the small mass changes by approximately $2mv$. If there are δn collisions with the large mass in a time δt then

$$M\delta V \;=\; 2mv\delta n.$$

By thinking of the x-axis with marks at intervals of length $x(t)$ at the instant t we can see that the number of collisions in time δt is

$$\delta n \;=\; \frac{v\delta t}{x(t)}.$$

We must remember that only half of these collisions transfer momentum to the large mass (the other half are with the wall). Consequently

$$M\dot{V} \;=\; \frac{mv(t)^2}{x(t)}.$$

Note that $V = \dot{x}$.

Energy conservation and an equation for $x(t)$

The only energy involved is kinetic. Suppose the initial speed of the small mass is v_0. Then

$$mv(t)^2 \;=\; mv_0^2 - MV(t)^2.$$

Combining this with the momentum equation we find

$$\begin{aligned} Mx\ddot{x} &= mv^2 \\ &= mv_0^2 - M\dot{x}^2, \text{ so that} \\ x\ddot{x} + \dot{x}^2 &= \frac{m}{M}v_0^2. \end{aligned}$$

This can be rewritten using

$$\frac{d^2(x^2)}{dt^2} = 2(x\ddot{x} + \dot{x}^2),$$

and some elementary integrations give

$$x^2 = \frac{m}{M}v_0^2 t^2 + 2Kt + L.$$

The initial conditions $x(0) = x_0$ and $V(0) = 0$ lead to

$$x(t) = \sqrt{\frac{m}{M}v_0^2 t^2 + x_0^2}.$$

This in turn determines

$$V = \frac{\frac{m}{M}v_0^2 t}{\sqrt{\frac{m}{M}v_0^2 t^2 + x_0^2}} \quad \text{and} \quad v = \frac{\sqrt{M}v_0 x_0}{\sqrt{mv_0^2 t^2 + Mx_0^2}}.$$

Final step

The expression used above for δn can be integrated. We use the limits $t = 0$ and $t \to \infty$ since there will be no further collisions once the speeds of the masses match. We have

$$\dot{n} = \frac{v}{x}$$

so that

$$\begin{aligned} n &= v_0 x_0 M \int_0^\infty \frac{dt}{mv_0^2 t^2 + Mx_0^2} \\ &= \frac{x_0 M}{v_0 m} \int_0^\infty \frac{dt}{t^2 + Mx_0^2/(mv_0^2)} \\ &= \frac{x_0 M}{v_0 m} \left(\frac{v_0\sqrt{m}}{x_0\sqrt{M}} \right) \left[\arctan\left(\frac{tv_0\sqrt{m}}{x_0\sqrt{M}} \right) \right]_{t=0}^\infty \\ &= \frac{\pi}{2}\sqrt{\frac{M}{m}}. \end{aligned}$$

The final result is just twice this.

Remembering Professor Uwe Grimm

Martin Hansen

It was Uwe who introduced me to that most fascinating of mathematical toys, the infinite Fibonacci word. It's a deceptively simple substitution, θ, on an alphabet of only two letters, $\mathcal{A}(a, b)$, defined by $a \to ab$ and $b \to a$. It gives us the finite Fibonacci words, $\mathcal{F}_n = \theta^n(a)$. The first few are

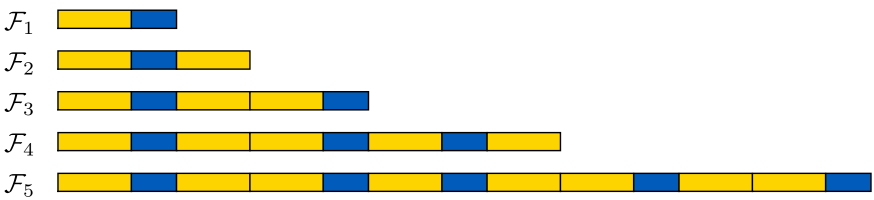
$$\mathcal{F}_0 = a, \quad \mathcal{F}_1 = ab, \quad \mathcal{F}_2 = aba, \quad \mathcal{F}_3 = abaab$$

and so on. Throw away the last couple of letters on any given word and what's left is a palindrome. As example, $\mathcal{F}_4 = abaababa$ which, without its rightmost two letters, is $abaaba$. This palindromic nature, along with the remarkable concatenation property that

$$\mathcal{F}_n \;=\; \mathcal{F}_{n-1}\mathcal{F}_{n-2} \quad \text{for } n \geq 3,$$

guarantees that the Fibonacci words abound with symmetries. As $n \to \infty$ the infinite Fibonacci word emerges as a fixed point of the iteration.

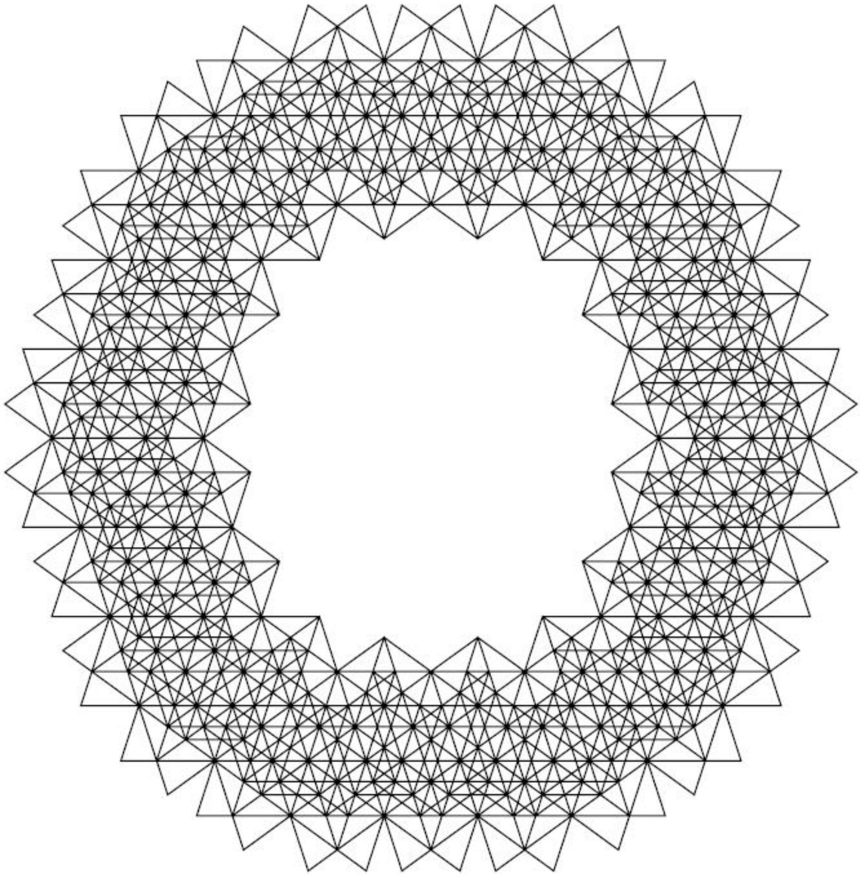
Uwe took pleasure in finding geometric visualisations to complement his algebraic researches. These were often stunningly beautiful creations that non-mathematicians could marvel over. When he died, I had just begun studying the Open University's M840 graduate course, *Aperiodic Tilings and Symbolic Dynamics*. In the course topic guide (co-authored with Reem Yassawi), he showed how, via the Fibonacci word's substitution incidence matrix, the left eigenvector gave rise to an aperiodic tiling of $\mathbb{R}^+$.



For my dissertation I ended up running with this idea, under the watchful eye of Dan Rust, who kindly stepped in to supervise Uwe's orphaned students and keep the course going. I took Uwe's tiled path and twisted it back and forth, often with it tiling over itself, and looked at the properties of the resulting figures. The twisting was via a drawing rule that took each letter of the Fibonacci word in turn and used it as an instruction to say how the next tile should be placed. Some attractive images resulted. To give a flavour of what can occur the adjacent image is for $\mathcal{F}_{22}$ under the following drawing rule.

Symbol	Action
a	forward ϕ (The golden ratio, about 1.618)
b	forward 0.5, turn 108°, forward 0.5.

I like to think that my visualisation of $\mathcal{F}_{22}$ is in the spirit of the mathematics that inspired it; Uwe's mathematics. And that he would approve.



Solution 301.5 – Matrix powers

For integer $n \geq 3$, let A be an $n \times n$ matrix whose r-th row is $[0\,1\,0\,0\cdots0\,1]$ rotated right by $r-1$ places. Show that for $h = 1, 2, \ldots, n-1$ and $i = 1, 2, \ldots, n$, we have

$$\left[A^h\right]_{i,i} \;=\; \begin{cases} 0, & \text{if } h \text{ is odd,} \\ \dbinom{h}{h/2}, & \text{if } h \text{ is even,} \end{cases}$$

Can you extend this formula to $n \leq h \leq 2n-1$?

J. M. Selig

Let P be the $n \times n$ permutation matrix corresponding to the n-cycle,

$$P = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}; \text{ thus } P^{n-1} = P^{-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

This can also be written:

$$[P]_{i,j} \;=\; \begin{cases} 1 & \text{if } i - j \equiv 1 \pmod{n}, \\ 0 & \text{otherwise.} \end{cases}$$

We also have

$$\left[P^h\right]_{i,j} \;=\; \begin{cases} 1 & \text{if } i - j \equiv h \pmod{n}, \\ 0 & \text{otherwise,} \end{cases}$$

The point of this is that clearly $A = P + P^{-1}$; so the powers of A can be computed as

$$A^h \;=\; \left(P + P^{-1}\right)^h.$$

By the binomial theorem we have

$$A^h \;=\; \sum_{k=0}^{h} \binom{h}{k} P^{h-k} P^{-k} \;=\; \sum_{k=0}^{h} \binom{h}{k} P^{h-2k}.$$

For $h = 1, 2, \ldots, n-1$, the only term that has non-zero diagonal elements is the term where the exponent of P is zero. This can only happen if $k = h/2$

is an integer. If h is odd $P^0 = I$ does not appear in the expansion above and hence the diagonal elements of A^h are 0. If h is even, the coefficient of $P^0 = I$ in the expansion is $\binom{h}{k} = \binom{h}{h/2}$. This answers the first part of the question but also gives a result for the other elements of A^h. It is not too difficult to see that

$$\left[A^h\right]_{i,j} = \sum_k \binom{h}{k}, \quad \text{where the index satisfies } h - 2k \equiv i - j \pmod{n}.$$

For example, when $n = 6$ we have

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

For $h = 5$, we get

$$\begin{aligned} A^5 &= P^5 + 5P^3 + 10P + 10P^{-1} + 5P^{-3} + P^{-5} \\ &= P^5 + 5P^3 + 10P + 10P^5 + 5P^3 + P \\ &= 11P^5 + 10P^3 + 11P. \end{aligned}$$

That is,

$$A^5 = \begin{pmatrix} 0 & 11 & 0 & 10 & 0 & 11 \\ 11 & 0 & 11 & 0 & 10 & 0 \\ 0 & 11 & 0 & 11 & 0 & 10 \\ 10 & 0 & 11 & 0 & 11 & 0 \\ 0 & 10 & 0 & 11 & 0 & 11 \\ 11 & 0 & 10 & 0 & 11 & 0 \end{pmatrix}.$$

The second part of the question concerns the range $n \le h \le 2n - 1$. The general result above applies in this range, but if we are only interested in the diagonal elements and only this range we can see that the only ways to get $h - 2k \equiv 0 \pmod{n}$ are when $k = h/2$ or $k = (h \pm n)/2$. There are four cases to consider here, depending on whether h and n are odd or even. Alternatively, we could just assume that the binomial coefficient is zero if one of the arguments is not an integer. Then we get

$$\left[A^h\right]_{i,i} = \binom{h}{h/2} + 2\binom{h}{(h-n)/2}, \quad n \le h \le 2n - 1.$$

Time travel and the theory of everything

The world of mathematics is an integral part in establishing that goal

Colin P. George

According to Stephen Hawking there is one area where it's unlikely you're going to get a research grant, and that is the subject of time travel. For mathematicians and theoretical physicists who have brilliant ideas this is quite daunting. However, there is nothing to stop one from performing a theoretical thought experiment just to see how things might play out. Mathematicians and scientists are very fond of thought experiments because they don't cost any money. After all, Einstein's *General Relativity* (1915) began as a thought experiment and much of its contents over the years have been proven to be correct.

In the past, mathematicians, physicists, philosophers, authors and artists have come up with some novel ideas. Although displaying different approaches many appear to be based on special relativity, general relativity, quantum theory, string, M or some other theory. However, this is hardly surprising as many appear to have elements about them that indicates that this subject can be treated scientifically

Fig 1: Mathematics plays an integral role in many articles and papers about time travel.

We know that time is part of Einstein's four dimensional universe. Multiplying t with c gives it parity with the other three. We call this the space–time continuum (x, y, z and ct), but there is the question we keep asking. Is time travel possible? The world of mathematics is an integral part of establishing that goal. Those studying advanced mathematics will be familiar with the equation

$$t = \frac{t_0}{\sqrt{(1 - v^2/c^2)}}$$

in Fig 1(a). It is a means of calculating time dilation effects of elementary particles, such as muons, which travel at incredible speeds. The muons' clocks run slow, meaning that their half-lives appear to an Earth based observer to be longer than in their rest frame. The consequence is that more muons survive to reach the ground than their rest frame half-lives would suggest. After many weeks of flight, passenger airliners experience the same phenomenon but less profound. It appears to be in line with Special Relativity (1905), which says moving clocks run slow.

Time is affected by heavy mass objects too. This was described in Einstein's field equations (Fig 2 (a)). However, it is easier to imagine space as a flat surface that becomes deformed by heavy objects that form cup-like depressions (see Fig 2 (a)). The Sun, the Earth and other planets are heavy enough to cause measurable distortions in the geometry of space. Since time is an integral part of that geometry it too will be affected. A clock placed on Earth will be telling a different time from a clock placed far away from Earth. This is in line with general relativity.

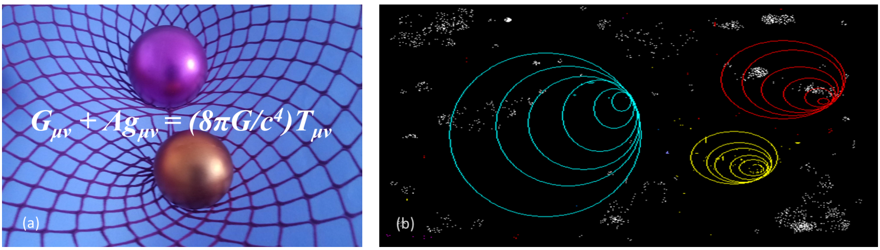


Fig 2: (a) Einstein's equation used to describe gravity and the shape of space-time. (b) It offers solutions to many other theoretical ideas regarding our universe, wormholes etc.

So it seems that by altering space geometry in the right way we can do something with time. It requires a certain amount of mass or the energy equivalent ($E = mc^2$). There is no known technology that can compress something so small that it creates even a modest distortion. Physicists subscribe to what they call a 'negative energy density' ($-\rho E = -\rho c^2$), a theoretical tool used to bend and manipulate space–time geometry. One is talking of a hypothetical 'exotic' substance that displays negative mass/energy. Alternatively, maybe it is something that emulates the equivalent (e.g. a negative pressure density, like the Casimir effect between two parallel plates). There are other proposals on how it could be done but many are theoretical. We still have a long way to go. On the subject of wormholes (Fig 2 (b)): if they exist, we may have to consider the possibility of additional dimensions especially if one wishes to entertain the idea of connecting to a remote region far away (e.g. if a sheet of paper is curved through the third dimension then an ant who tunnels through can get from A to B a lot quicker). Mathematicians and physicists believe wormholes may offer an opportunity for time travel.

Other time travel schemes include the Tipler cylinder (Kornel Lanczos, 1924; Willem Jacob van Stockum, 1936; Frank Tipler, 1974), a theoretical time machine based on the notion of a cylinder that spins so incredibly fast that it warps space–time. Kurt Gödel (mathematician and logician) came up with his famous Gödel metric (1949) based on the idea of a rotating universe which lacks Hubble expansion.

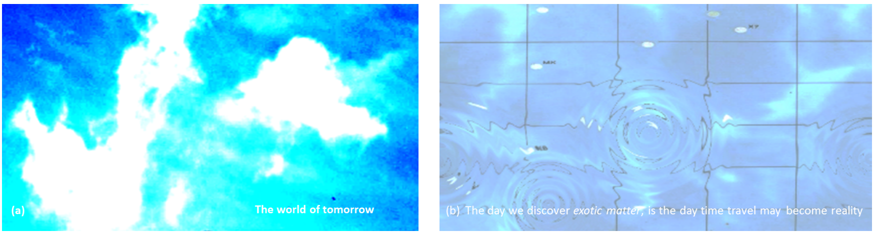


Fig 3: (a) An 'exotic' form of matter may one day be discovered giving rise to negative energy density conditions; (b) thus altering space-time geometry.

According to Stephen Hawking the possibility of time travel cannot be ruled out. The wormhole is a topic that is covered in general relativity, but is yet to be proven. Our universe does not quite emulate Gödel's universe. The Tipler cylinder has mixed opinions. We still do not know much about this 'exotic' material required to create a negative energy density. Yet it is recognised in quantum mechanics. There are things that are possible in relativity that do not agree with quantum mechanics. This is where many papers fall flat on their face (violations included) and lose credibility. Could it be possible before we build a time machine we need some kind of unifying theory that encompasses all of them in order to iron out all those holes and kinks. *The theory of everything?*

Whether or not you want to discover the theory of everything or discover the secrets of time travel, one thing we can be sure of: mathematics will be an integral part in establishing that goal.

Solution 306.7 – Quintic roots

Let a and b be positive numbers. Show that the quintic

$$2x^5 - 5a^3x^2 + 3b^5 \; = \; 0$$

has $2 + \operatorname{sign}(a - b)$ real roots.

Reinhardt Messerschmidt

Let

$$f(x) \; = \; 2x^5 - 5a^3x^2 + 3b^5;$$

therefore

$$f'(x) \; = \; 10x^4 - 10a^3x \; = \; 10x(x^3 - a^3) \; = \; 10x(x - a)(x^2 + ax + a^2).$$

The discriminant of $x^2 + ax + a^2$ is $a^2 - 4a^2 = -3a^2 < 0$; therefore $x^2 + ax + a^2 > 0$ for every real x. It follows that

$$\begin{aligned} x \in (-\infty, 0) \quad &\Longrightarrow \quad f'(x) > 0, \\ x \in (0, a) \quad &\Longrightarrow \quad f'(x) < 0, \\ x \in (a, \infty) \quad &\Longrightarrow \quad f'(x) > 0; \end{aligned}$$

therefore f is strictly increasing on $(-\infty, 0) \cup (a, \infty)$ and strictly decreasing on $(0, a)$. Let

$$u \; = \; -(3/2)^{1/5}b, \qquad v \; = \; (5/2)^{1/3}a;$$

therefore $u < 0$, $v > a$, and

$$\begin{aligned} f(u) \; &= \; 2u^5 - 5a^3u^2 + 3b^5 \; < \; 2u^5 + 3b^5 \; = \; 0, \\ f(0) \; &= \; 3b^5 \; > \; 0, \\ f(a) \; &= \; 3b^5 - 3a^5 \; = \; 3(b - a)(b^4 + b^3a + b^2a^2 + ba^3 + a^4), \\ f(v) \; &= \; v^2(2v^3 - 5a^3) + 3b^5 \; = \; 3b^5 \; > \; 0. \end{aligned}$$

It follows that

(i) f has no roots in $(-\infty, u] \cup \{0\} \cup [v, \infty)$;
(ii) f has exactly one root in $(u, 0)$;
(iii) if $b - a < 0$, then f has exactly one root in $(0, a)$ and exactly one root in (a, v);
(iv) if $b - a > 0$, then f has no roots in $(0, v)$;
(v) if $b - a = 0$, then f has a root at a, and it is of multiplicity 2, because f' has a root at a of multiplicity 1.

Solution 306.2 – Approximate roots

> Let n be an integer greater than 1, let x be positive number and let
> $$r = \lfloor \sqrt[n]{x} \rfloor.$$
> Show that if x is not too small, then
> $$\frac{(n+1)x + (n-1)r^n}{(n-1)x + (n+1)r^n}\, r \approx \sqrt[n]{x}.$$
> Clearly, any x in the interval $(0, 1)$ is too small, but when x is sufficiently large this approximation can be quite effective. For example, with $n = 3$ and $x = 1100$, so that $r = 10$, it gives 10.3226 whereas $\sqrt[3]{1100} = 10.3228$.

Ted Gore

Let $p = \sqrt[n]{x}$ so that
$$x = p^n = (r + \varepsilon)^n,$$
where ε is an element of $[0, 1)$. Then $x = r^n + \alpha$, where α is a function of r and ε. (Alternatively, $\alpha = p^n - r^n$.) We can therefore say that
$$\begin{aligned} q &= r\,\frac{(n+1)(r^n + \alpha) + (n-1)r^n}{(n-1)(r^n + \alpha) + (n+1)r^n} \\ &= r\,\frac{2nr^n + (n+1)\alpha}{2nr^n + (n-1)\alpha} = r + \frac{2r\alpha}{2nr^n + (n-1)\alpha}. \end{aligned}$$
Now
$$(r + \varepsilon)^n = r^n + nr^{n-1}\varepsilon + \ldots;$$
so that we can take $nr^{n-1}\varepsilon$ as an approximation to α.

Dividing both top and bottom of the second term by this approximation and rearranging gives us
$$q^* = r + \frac{2r}{2r/\varepsilon + (n-1)} = r + \frac{\varepsilon}{1 + (n-1)\varepsilon/2r}.$$
Now
$$\frac{\varepsilon}{1 + (n-1)\varepsilon/2r}$$
is zero if ε is zero and is less than ε if $n > 1$ and $\varepsilon > 0$ so that q^* is an approximation to $\sqrt[n]{x}$.

Using the example in the question, $q^* = 10.3127$, which is the cube root of 1096.774.

The term $(n-1)\varepsilon/2r$ generates the error in taking q^* as an approximation to $\sqrt[n]{x}$. The value of this error is

$$\frac{\varepsilon^2(n-1)}{2r+(n-1)\varepsilon},$$

and for the example we get 0.0101, as we would expect.

We can retrieve q by introducing a factor β, where $\alpha = nr^{n-1}\varepsilon\beta$ so that

$$q \;=\; r + \frac{\varepsilon}{1/\beta + (n-1)\varepsilon/2r} \;=\; r + \frac{\varepsilon}{1 + (1-\beta)/\beta + (n-1)\varepsilon/2r}.$$

The error in q is now

$$\frac{\varepsilon\,[2r(1-\beta) + \beta(n-1)\varepsilon]}{2r + \beta(n-1)\varepsilon}.$$

For the example, this is 0.0002.

For $n > 1$ and $\varepsilon > 0$, β is greater than 1 so that $(1-\beta)/\beta$ is negative, which makes q a more accurate approximation than q^*.

Problem 309.1 – Three squares suffice

Dave Wild

Most positive numbers can be expressed as the sum of three squares. The exceptions are numbers of the form $4^m(8n+7)$. If we look at a few numbers of this form such as 7, 368, and 999 then they can also be expressed using three squares. $7 = 2^2+2^2-1^2$, $368 = 9^2+24^2-17^2$ and $999 = 14^2+42^2-31^2$.

If a, b and c are integers, which integers can be expressed in the form $a^2 + b^2 \pm c^2$.

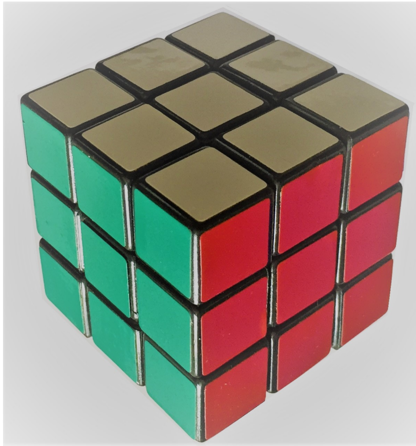
Problem 309.2 – A marginal problem

Dave Wild

Let x, y, z and N be positive integers. If $n = 2N+1$ and $x+y = pq$, where p and q are different primes, then show that the equation $x^n + y^n = z^n$ has no solutions.

Problem 309.3 – Rubik's cube colourings

Tony Forbes



The picture on the right shows a familiar object. The cube has 6 faces each divided into 9 squares, 54 squares in all. How many of the 6^{54} ways to colour the little squares (from a palette of 6 colours) are realizable by Rubik cube movements from a cube in its identity state, where each face has all of its squares coloured according to the scheme: (front, green), (back, blue), (left, orange), (right, red), (up, white), (down, yellow)?

Problem 309.4 – Primes

Tony Forbes

Let $\mathcal{P}$ be a set of integers such that

$$n \in \mathcal{P} \quad \text{iff} \quad n > 1 \text{ and } \gcd((\sqrt{n})\#, n) = 1,$$

where $x\#$ is the product of all the elements of $\mathcal{P}$ that lie in the interval $[2, x]$, i.e.

$$x\# \;=\; \prod_{2 \le n \le x,\ n \in \mathcal{P}} n.$$

What is $\mathcal{P}$?

Problem 309.5 – Regular graphs

A look at Markus Meringer's website

`http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html#CRG`

will reveal that of those (unlabelled) graphs which have 14 vertices, 21609300 are connected and 6-regular whereas 21609301 are 7-regular. Show that this is not a coincidence.

The same site indicates that there are 1470293675 connected 15-vertex 6-regular graphs. How many 15-vertex graphs are 8-regular?

Problem 309.6 – Square and reverse

We are interested in examples of integers $m, n \geq 1$, n not divisible by 10, such that m^2 is obtained from n^2 by reversing the order of its decimal digits. Thus for $n \leq 1000$, we have the following (n, m) pairs:

(1, 1), (2, 2), (3, 3), (11, 11), (12, 21), (13, 31), (21, 12), (22, 22),
(26, 26), (31, 13), (33, 99), (99, 33), (101, 101), (102, 201), (103, 301),
(111, 111), (112, 211), (113, 311), (121, 121), (122, 221), (201, 102),
(202, 202), (211, 112), (212, 212), (221, 122), (264, 264), (301, 103),
(307, 307), (311, 113), (836, 836).

You can check, for instance, that $311^2 = 96721$, the reverse of $113^2 = 12769$. As you can see, in most cases either $m = n$, in which case n^2 happens to be palindromic, or m is the reverse of n. In fact, the only exceptions in this list are (33, 99) and of course (99, 33). Further exceptions do occur, but they are rare. Here is a complete list with $n < m$ and $n \leq 10000000$:

(33, 99), (3168, 6501), (20508, 21468), (110922, 219111),
(303577, 304877), (1100922, 2191011), (1109211, 1119111),
(1110922, 2191111), (3080367, 3140793).

Now for the problem. Characterize these exceptional pairs.

Problem 309.7 – Limit

Show that

$$\frac{x \sin y - y \sin x}{x \cos y - y \cos x} \quad \to \quad \tan(x - \arctan x) \qquad \text{as } y \to x.$$

M500 308 Puzzle solution

^{14}8	^{11}1	5	^{16}9	7	^{5}3	2	^{11}4	^{7}6
6	3	2	^{19}4	8	5	^{17}9	7	1
^{29}9	4	7	^{11}6	2	^{11}1	8	^{15}5	3
^{8}1	7	9	5	3	6	^{18}4	8	2
^{5}3	2	^{10}4	^{14}8	1	^{23}9	7	6	5
^{13}5	8	6	2	4	7	^{14}3	1	9
^{7}2	5	^{13}3	7	^{14}6	8	1	^{13}9	4
^{16}7	9	^{20}1	3	^{20}5	4	6	^{5}2	^{15}8
4	6	8	1	^{11}9	2	5	3	7

Some people complained that the puzzle has no solution. Possibly they failed to read the small print, or rather the lack of small print. Unlike all published 'Killer Sudoku' puzzles I (TF) have seen, there was no rule saying that the numbers in the grey-bordered areas must be distinct. In my opinion, that rule is artificial and detracts from the otherwise simplicity of the instructions. Moreover, in the vast majority of published examples it is not actually required! Indeed, one can get quite excited on those rare occasions where the distinct-numbers rule must be invoked in order to obtain a unique solution.

04.00

Problem 309.8 – Pythagorean triples

Find all positive integers n such that there exist positive integers a and b such that

$$n^2 \;=\; (n-1)^2 + a^2 \;=\; (n-2)^2 + b^2.$$

Is it possible to find positive integers a, b and c such that

$$n^2 \;=\; (n-1)^2 + a^2 \;=\; (n-2)^2 + b^2 \;=\; (n-3)^2 + c^2?$$

Front cover Fibonacci word $\mathcal{F}_{12}$; see page 6.