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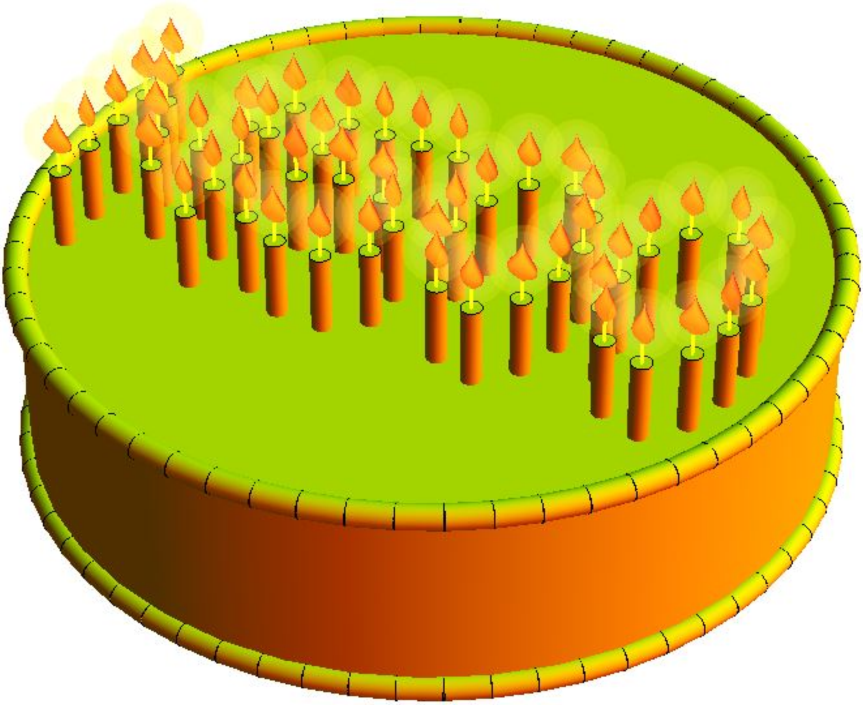
M500 310



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: `m500.org.uk`.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The Revision Weekend is a residential Friday to Sunday event providing revision and examination preparation for both undergraduate and postgraduate students. For details, please go to the Society's website and see below.

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Advice for authors We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to the Editor, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. For more information, go to `m500.org.uk/magazine/`.

M500 Revision Weekend 2023 We are planning to run the Revision Weekend in 2023 over 12th-14th May at Kents Hill Park, Milton Keynes. For further details and an application form please check the website, or e-mail the Revision Weekend Organizer, weekend@m500.org.uk.

Please let me know if you require any more information.

Moments of inertia without integration

Alan Davies

1 Introduction

We are concerned here with a *uniform* lamina defined by a closed plane curve $\mathcal{C}$ with area A. The lamina has surface density (mass per unit area) σ and mass m $(= \sigma A)$.

The mass centre of the lamina corresponds to the centroid, G, of the area enclosed by $\mathcal{C}$. We wish to find the moment of inertia, I, of the lamina, about the principal axis, $\mathcal{G}$, perpendicular to the lamina; see Figure 1.

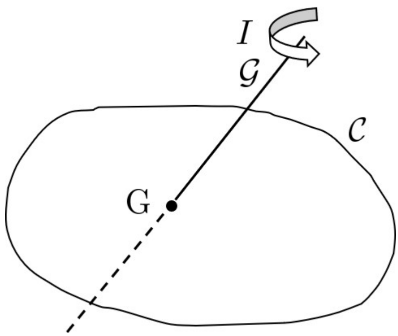


Figure 1: General lamina with moment of inertia I.

We may calculate I using the integral

$$I \;=\; \sigma \int_A D^2 \, dA,$$

where D is the distance in the plane from G and the integral is taken over the area A.

Usually the integral would be performed using either Cartesian or plane polar coordinates:

(i) *Cartesian coordinates*, (x, y): $D^2 = x^2 + y^2$ and $dA = dx\,dy$.

(ii) *Plane polar coordinates*, (r, θ): $D^2 = r^2$ and $dA = r\,dr\,d\theta$.

In certain special situations I can be obtained without integration, using only a simple geometric property of the shape and the *Parallel axes theorem* which states:

The moment of inertia, $I_{\mathcal{Z}}$, about an axis $\mathcal{Z}$, parallel to $\mathcal{G}$, is given by

$$I_{\mathcal{Z}} \;=\; I + md^2$$

where d is the distance between the two axes; see Figure 2.

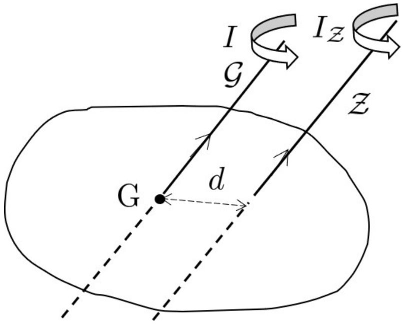


Figure 2: Principal moment of inertia I and moment of inertia $I_{\mathcal{Z}}$, about a parallel axis, $\mathcal{Z}$.

The interested reader may wish to show that the moment of inertia I is the largest of the three principal moments of inertia and we shall refer to it as the *first* moment of inertia.

2 Equiangular triangular lamina

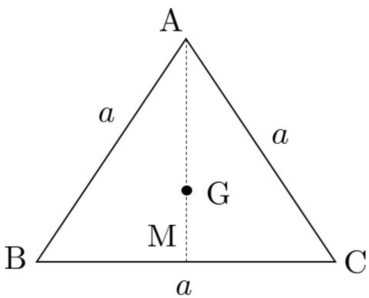


Figure 3: Equiangular triangle ABC.

Consider the equiangular triangle ABC, side a, whose centroid lies on the median AM, with AG:AM = 2:3; see Figure 3.

Since we have a uniform equiangular lamina it is defined by the two parameters m and a. Hence, we can suppose that the first moment of inertia, I, is given by

$$I = \lambda m a^2 \tag{1}$$

where λ is a dimensionless parameter to be found.

Now consider three identical triangles based on the sides AB, BC and CA to give a larger equiangular triangle, DEF. The centroid of $\triangle$DEF coincides with G, that of $\triangle$ABC, as shown in Figure 4.

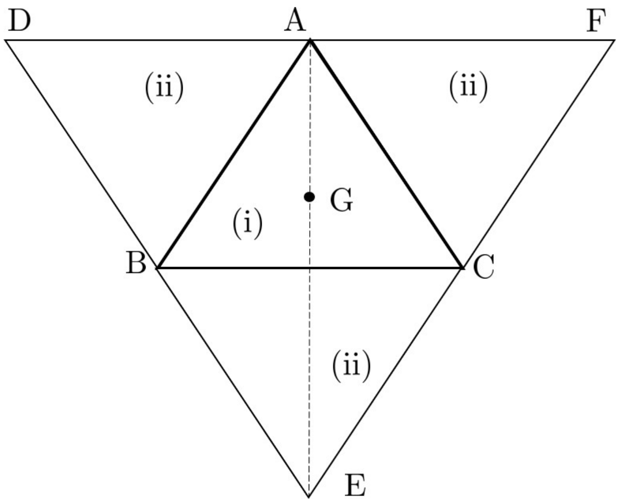


Figure 4: Equiangular triangle DEF, with centroid G and four smaller similar triangles.

Using equation (1), since $\triangle$DEF is an equilateral triangle, side $2a$ and mass $4m$, its first moment of inertia is given by

$$\begin{aligned} I_{\text{big}} &= \lambda(4m)(2a)^2 \\ &= 16\lambda m a^2. \end{aligned} \tag{2}$$

The moment of inertia, I_{big}, of the lamina DEF may be obtained using the original triangle ABC, (i) in Figure 4, together with three identical triangles, (ii).

Consider the moment of inertia of a typical type (ii) triangle, say $\triangle$ACF, about the axis through G; see Figure 5.

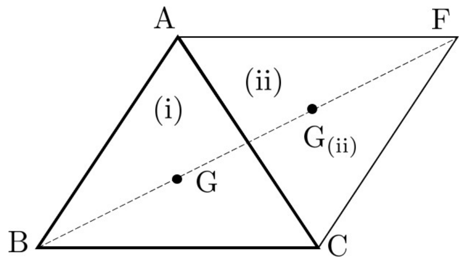


Figure 5: Triangles ABC and ACF; $\mathrm{G}_{\text{(ii)}}$ is the centroid of △ACF.

Since the distance $\mathrm{GG}_{\text{(ii)}}$ is $a\sqrt{3}/3$, it follows that, using the parallel axes theorem and equation (1), the moment of inertia of lamina ACF about the axis through G is given by

$$I_{\text{(ii)}} = \lambda m a^2 + \frac{1}{3} m a^2.$$

Now △DEF comprises three such triangles together with △ABC, as shown in Figure 4, so that

$$I_{\text{big}} = 3\left(\lambda m a^2 + \frac{1}{3} m a^2\right) + \lambda m a^2. \tag{3}$$

It follows, from equations (2) and (3), that

$$\lambda = \frac{1}{12}$$

and hence, that the moment of inertia is given by

$$I = \frac{1}{12} m a^2.$$

As an aside, it is very easy to use this result to find the moment of inertia of a uniform regular hexagonal lamina of side a. The hexagon comprises six equilateral triangles of the type shown in Figure 3. The parallel axis theorem gives the moment of inertia of the triangular lamina about a perpendicular axis through A as

$$I_{\mathrm{A}} = \frac{1}{12} m a^2 + m\left(\frac{2}{3}\frac{\sqrt{3}}{2}a\right)^2 = \frac{5}{12} m a^2.$$

Hence, if the hexagonal lamina has mass m then its first moment of inertia, I_{hex}, is given by

$$\begin{aligned} I_{\text{hex}} &= 6\left(\frac{5}{12}\left(\frac{m}{6}\right)a^2\right) \\ &= \frac{5}{12}ma^2. \end{aligned}$$

3 Square lamina

A square lamina may be treated in a similar fashion. The square lamina ABCD, see Figure 6, is defined by the two parameters a and m, the side and mass respectively. Its moment of inertia, I, about the perpendicular axis through G is given by

$$I = \lambda m a^2 \tag{4}$$

where, as before, λ is a dimensionless parameter to be found.

Construct eight new squares on the original square as shown in Figure 6. The squares PQRS and ABCD have the same centroid, G.

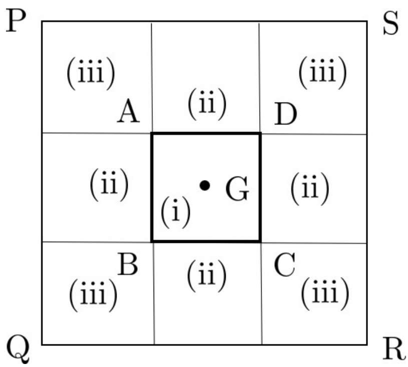


Figure 6: Square ABCD and eight adjacent squares.

Using equation (4), the lamina defined by the square PQRS has first moment of inertia given by

$$\begin{aligned} I_{\text{big}} &= \lambda(9m)(3a)^2 \\ &= 81\lambda m a^2. \end{aligned} \tag{5}$$

For squares in position (ii) the distance from G to the centroids is a and for squares in position (iii) the distance from G to the centroids is $a\sqrt{2}$.

Hence, using the parallel axes theorem,

$$\begin{aligned} I_{\text{big}} &= \lambda ma^2 + 4\left(\lambda ma^2 + ma^2\right) + 4\left(\lambda ma^2 + m\left(\sqrt{2}a\right)^2\right) \\ &= (9\lambda + 12)\, ma^2. \qquad (6) \end{aligned}$$

Using equations (5) and (6) it follows that

$$\lambda = \frac{1}{6}$$

and hence, using equation (4), that

$$I = \frac{1}{6}ma^2.$$

4 Rectangular lamina

Suppose we have a uniform rectangular lamina defined by three parameters a, b and m, the two sides and mass respectively. It is convenient to introduce the *aspect ratio*, r, given by

$$r = \frac{b}{a}.$$

The symmetry suggests that its moment of inertia, I, about the perpendicular axis through G is given by

$$I = \lambda(r)ma^2 \qquad (7)$$

where, in this case, the dimensionless parameter, $\lambda(r)$, is a function of r.

Proceeding in a manner similar to that for the square we construct eight similar rectangles as shown in Figure 7 and we have

$$\begin{aligned} I_{\text{big}} &= \lambda(r)(9m)(3a)^2 \\ &= 81\lambda(r)ma^2. \qquad (8) \end{aligned}$$

The distances from G to the centroids of the rectangles are as follows:

position (ii), $\sqrt{a^2 + b^2}$ $\left(= a\sqrt{1 + r^2}\right)$;
position (iii), a;
position (iv), b $(= ra)$.

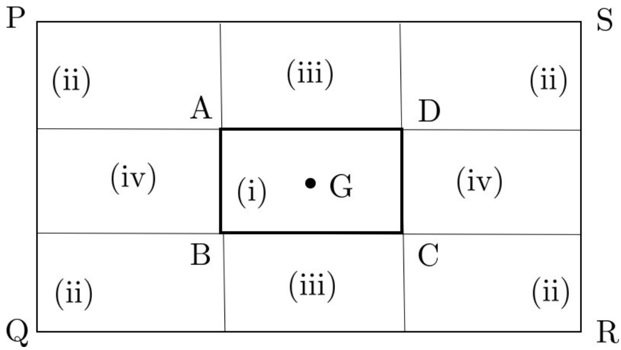


Figure 7: Rectangle ABCD and eight adjacent rectangles; AB and BC are of lengths a and b respectively.

Hence, using the parallel axes theorem

$$\begin{aligned} I_{\text{big}} &= \lambda(r)ma^2 + 4\left(\lambda(r)ma^2 + ma^2\left(1+r^2\right)\right) \\ &\quad +2\left(\lambda(r)ma^2 + ma^2\right) + 2\left(\lambda(r)ma^2 + m\left(ra\right)^2\right) \\ &= \left(9\lambda(r) + 6\left(1+r^2\right)\right)ma^2. \end{aligned} \tag{9}$$

From equations (8) and (9)

$$\lambda(r) = \frac{1}{12}\left(1+r^2\right)$$

and hence

$$I = \frac{1}{12}m\left(a^2+b^2\right).$$

COLIN GRAYSON

We are sorry to have to tell you that M500 member Colin Grayson died in hospital on 15 December 2022. He had been unwell for some time. Sad news - we have good memories of Colin from his time on the M500 Society Committee. And Pauline was great with the hospitality whenever we had meetings at Scarborough.

Our sympathy goes to his family and friends, especially to Pauline and daughter Wendy.

Solution 305.8 – Integral

Show that if t is an odd positive integer, then

$$\int_0^1 \frac{\sqrt{1-x^{2/t}}}{\sqrt{1+x^{-2/t}}} \;=\; A+B\pi,$$

where A and B are rational. For example,

$$\int_0^1 \frac{\sqrt{1-x^2}}{\sqrt{1+x^{-2}}} = \frac{\pi}{4}-\frac{1}{2}, \qquad \int_0^1 \frac{\sqrt{1-x^{2/3}}}{\sqrt{1+x^{-2/3}}} = \frac{6}{4}-\frac{3\pi}{8},$$

$$\int_0^1 \frac{\sqrt{1-x^{2/5}}}{\sqrt{1+x^{-2/5}}} = \frac{5\pi}{8}-\frac{5}{3}, \;\ldots, \quad \int_0^1 \frac{\sqrt{1-x^{2/27}}}{\sqrt{1+x^{-2/27}}} = \frac{4608}{1001}-\frac{11583\pi}{8192}, \;\ldots.$$

Tommy Moorhouse

I have considered two possible solutions to this problem. One allows verification that the given integrals are of the form $A+B\pi$ with A and B rational, while the other enables us to calculate A and B. The first step is a substitution.

Let

$$I_k \;=\; \int_0^1 \frac{\sqrt{1-x^{2/(2k+1)}}}{\sqrt{1+x^{-2/(2k+1)}}}dx \;=\; \int_0^1 x^{1/(2k+1)}\frac{\sqrt{1-x^{2/(2k+1)}}}{\sqrt{1+x^{2/(2k+1)}}}dx.$$

The substitution $x^{1/(2k+1)}=y$ gives

$$I_k \;=\; (2k+1)\int_0^1 y^{2k}\frac{\sqrt{1-y^2}}{\sqrt{1+y^2}}ydy.$$

A further substitution of $y^2=z$ leads to

$$I_k \;=\; \frac{2k+1}{2}\int_0^1 z^k\Big(\frac{1-z}{1+z}\Big)^{1/2}dz \;=\; \frac{2k+1}{2}\int_0^1 \frac{z^k(1-z)}{\sqrt{1-z^2}}dz.$$

First method

The last expression for I_k involves integration of two expressions of the form $z^m/\sqrt{1-z^2}$. We define J_k as follows:

$$I_k \;=\; \frac{2k+1}{2}\int_0^1 \frac{z^k(1-z)}{\sqrt{1-z^2}}dz \;\equiv\; \frac{2k+1}{2}(J_k-J_{k+1}).$$

This type of integral can be expressed in terms of integrals of $\arcsin z$ and $\sqrt{1-z^2}$ by partial integration. The (repeated) partial integration involves multiplication and addition of a finite number of rational expressions (and of π times such terms), leading to the stated result. The details are straightforward but a little messy so I won't go through them here. Although it is possible to find recurrence relations for the coefficients it does not seem to be a simple matter to find A and B this way.

Second method

Our old favourite the beta function comes to the rescue when we need to find the values of A and B. In J_k substitute $z^2 = s$ to find

$$J_k \;=\; \frac{1}{2}\int_0^1 s^{(k-1)/2}\,(1-s)^{-1/2}\,ds \;=\; \frac{1}{2}B\left(\frac{1}{2}(k+1), \frac{1}{2}\right).$$

This is just

$$\frac{\Gamma((k+1)/2)\Gamma(1/2)}{2\Gamma(1+k/2)}.$$

We have $\Gamma(1/2) = \sqrt{\pi}$ and the recurrence relation $\Gamma(1+z) = z\Gamma(z)$, allowing us to do the calculations. If k is odd then factors of $\sqrt{\pi}$ cancel between the numerator and denominator, leaving a rational expression. If k is even there is a factor of π multiplying a rational expression. We can calculate the exact expressions. For example, if $k = 13$ we have

$$\frac{27}{2}J_{13} \;=\; \frac{27}{2}\frac{\Gamma(7)\Gamma(1/2)}{2\Gamma(15/2)} \;=\; \frac{4508}{1001}$$

and

$$\frac{27}{2}J_{14} \;=\; \frac{27}{2}\frac{\Gamma(15/2)\Gamma(1/2)}{2\Gamma(8)} \;=\; \frac{11583}{8192}\pi.$$

Here we have used, for example,

$$\Gamma(15/2) \;=\; \frac{13}{2}\Gamma(13/2) \;=\; \frac{13}{2}\frac{11}{2}\Gamma(11/2) \;=\; \frac{13}{2}\frac{11}{2}\cdots\frac{1}{2}\sqrt{\pi}.$$

Putting it together we have

$$I_{13} \;=\; \int_0^1 \frac{\sqrt{1-x^{2/27}}}{\sqrt{1+x^{-2/27}}}dx \;=\; \frac{4508}{1001} - \frac{11583}{8192}\pi.$$

A hyperbola and the 9-point circle

Tony Forbes

Consider the hyperbola $\mathcal{H}$ defined by $t \mapsto 1/t$. Let $X = (x, y)$ be a point not on $\mathcal{H}$ such that the quartic

$$t^4 - xt^3 + yt - 1 \;=\; 0$$

has four distinct real roots, p, q, r, s. Let P, Q, R, S be points on $\mathcal{H}$ determined by $t = p$, q, r, s, respectively. Then PX, QX, RX and SX are normals to $\mathcal{H}$, S is the orthocentre of $\triangle PQR$, and the origin $O = (0, 0)$ is on the 9-point circle of $\triangle PQR$, [J. Margetson and R. Buckingham, The ten-point circle, *Math. Gazette* **73** (1989)].

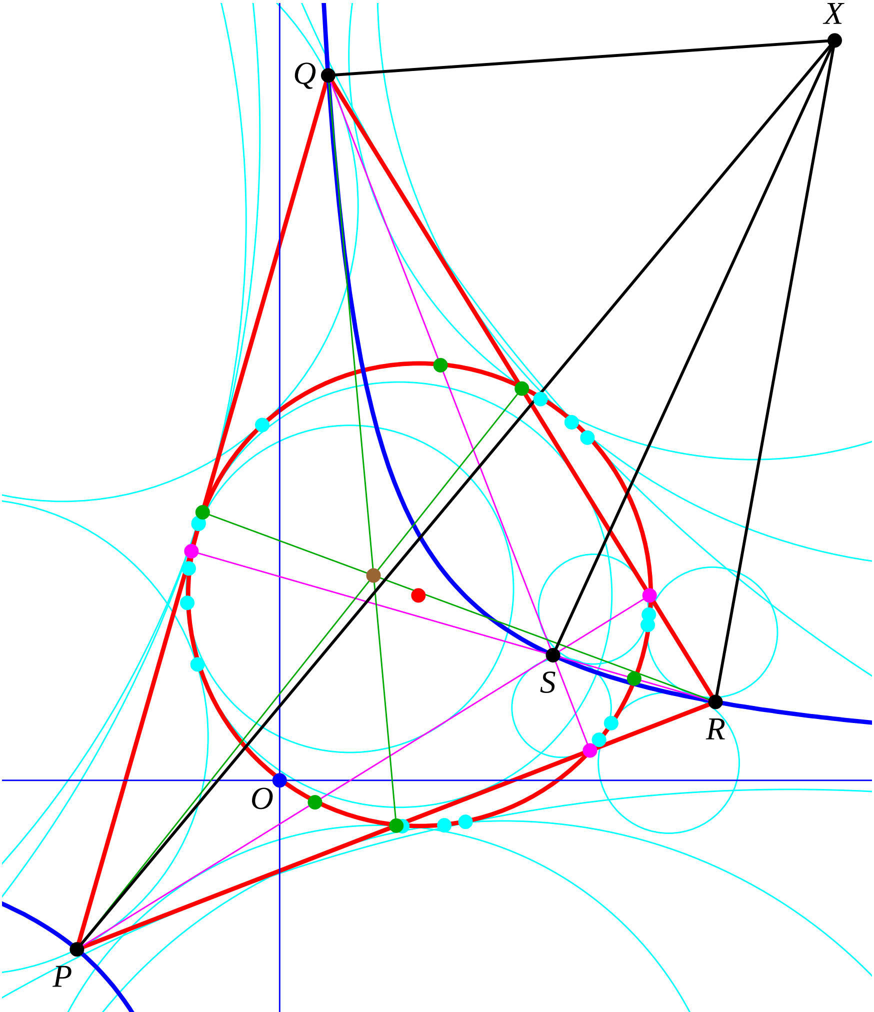
I do not know if this process reversible. That is, given $\triangle PQR$ with orthocentre S, construct its 9-point circle, $\mathcal{C}$, and find a rectangular (asymptotes meet at 90 degrees) hyperbola $\mathcal{H}$ centred on $\mathcal{C}$ such that for some point X not on $\mathcal{H}$, PX, QX, RX and SX are normals to $\mathcal{H}$. If you can provide a proof, please send it to us.

Recall that the orthocentre is where the three altitudes of the triangle meet. It seems to me that instead of studying one particular triangle, it is sensible to consider $\mathcal{T} = \{P, Q, R, S\}$ as a set of four distinct points with the property that any one of them is the orthocentre of the triangle whose vertices are the other three. Then the 9-point circle has a nice construction, which is the same for each choice of triangle from $\mathcal{T}$. Its centre is $(P + Q + R + S)/4$ and the nine points on it fall into two types.

- (i) The three intersections $PQ \cap RS$, $PR \cap QS$, $PS \cap QR$. These are the bases of altitudes of the chosen triangle, whatever it may be, and therefore we have $PQ \perp RS$, $PR \perp QS$, $PS \perp QR$.
- (ii) The six midpoints of pairs of points of $\mathcal{T}$.

See the diagram on the next page, where $X = (3, 4)$ and you can confirm that the centre of the circle is at $(3/4, 1)$, which lies on OX.

If 10 is not enough, there's another collection of interestingly defined points to add. These are where the in-circle and the three ex-circles of each triangle touch the 9-point circle, which indeed they do by Feuerbach's theorem, [Ralph Beauclerk, Geometry of the triangle – Feuerbach's Theorem, M500 **136**]. That's a further 16 points, which in a sufficiently general situation will be distinct. They are coloured cyan in the picture. The 9-point circle has become the 26-point circle and the diagram is in now danger of becoming rather cluttered.



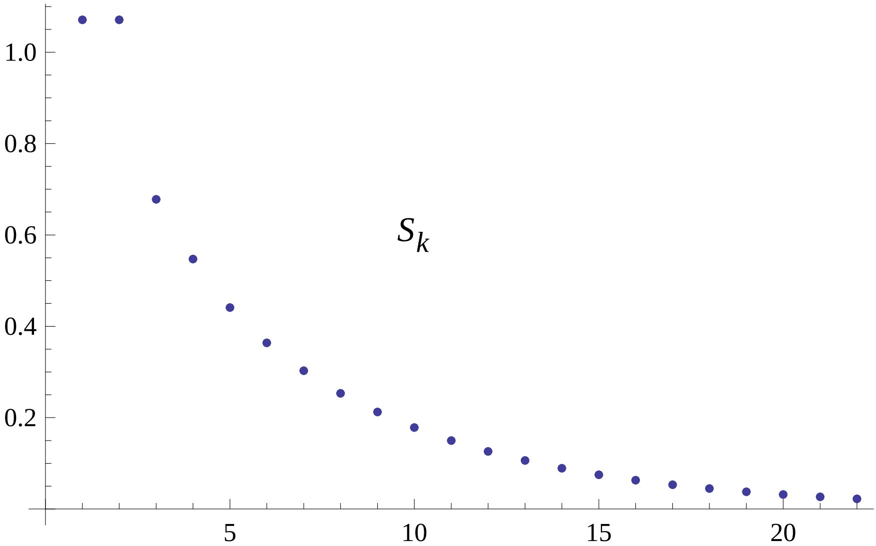
Triangle PQR has orthocentre S. The prominent round thing is the 9-point circle, which clearly goes through O, the origin of the hyperbola, as well as the bases of the altitudes of $\triangle PQR$ and the midpoints of PQ, PR, PS, QR, QS and RS, 10 points altogether. Its centre is at $(P+Q+R+S)/4$. The normals to the hyperbola at P, Q, R and S go through X. By permuting (P, Q, R, S), we obtain three more triangles with the same nine-point circle.

A trigonometric series

Tony Forbes

For positive integer k, define

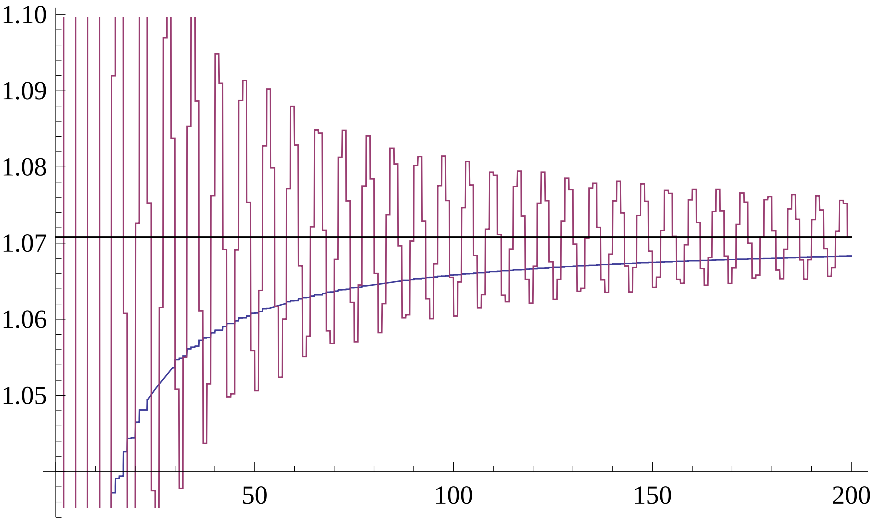
$$S_k = \sum_{n=1}^{\infty} \frac{\sin^k n}{n^k}.$$



For not too large k the sum can be evaluated exactly by MATHEMATICA, and the results are somewhat surprising (at least to me). When $1 \le k \le 6$ we get simple expressions of the form $a_k\,\pi - 1/2$ for certain positive rational numbers a_k.

k	1	2	3	4	5	6
S_k	$\frac{\pi}{2} - \frac{1}{2}$	$\frac{\pi}{2} - \frac{1}{2}$	$\frac{3\pi}{8} - \frac{1}{2}$	$\frac{\pi}{3} - \frac{1}{2}$	$\frac{115\pi}{384} - \frac{1}{2}$	$\frac{11\pi}{40} - \frac{1}{2}$

Immediately we observe that $S_1 = S_2 = (\pi - 1)/2$, resulting in a definite kink occurring at the beginning of the otherwise smooth plot of S_k against k, above. The reader is invited to prove this interesting equality, which we formally state as Problem 310.1, page 16. Be aware that the series for S_1 does not converge absolutely. On the next page we have plotted the partial sums for S_1 (the wildly oscillating curve) and S_2 (the other one). You are welcome to draw any conclusions that might occur to you.



Beyond $k = 6$ things get rather more complicated. The next three values are these monstrous expressions,

$$
\begin{aligned}
S_7 &= \frac{1}{46080}\big(-23040 + 129423\pi - 201684\pi^2 + 144060\pi^3 \\
&\qquad - 54880\pi^4 + 11760\pi^5 - 1344\pi^6 + 64\pi^7\big), \\
S_8 &= \frac{1}{5040}\big(-2520 + 17592\pi - 28672\pi^2 + 21504\pi^3 \\
&\qquad - 8960\pi^4 + 2240\pi^5 - 336\pi^6 + 28\pi^7 - \pi^8\big), \\
S_9 &= \frac{1}{10321920}\big(-5160960 - 6498981\pi + 42062688\pi^2 - 59068800\pi^3 \\
&\qquad + 41311872\pi^4 - 16853760\pi^5 + 4225536\pi^6 \\
&\qquad - 645120\pi^7 + 55296\pi^8 - 2048\pi^9\big).
\end{aligned}
$$

Thereafter the pattern is similar, at least for $k = 10, 11, \ldots, 24$, the limit of my evaluations. The formula for S_k is $S_k(\pi)$, where $S_k(x)$ is a polynomial in x of degree k with rational coefficients. Also we have $S_k(0) = -1/2$ and, apart from the constant term, the coefficients of $S_k(x)$ do not have small denominators.

Needless to say, I am mystified by the sudden fundamental change in the nature of S_k when k goes from 6 to 7.

Solution 307.4 – Ant

An elastic rope has length 1 m at time $t = t_0$. It is being stretched in such a manner that the velocity of one end relative to the other end is $v(t)$ m/s. At time $t = t_0$ an ant starts at one end of the rope and walks towards the other end at a constant velocity u m/s relative to the rope.

It is well known that when $v(t)$ is constant the ant will eventually reach the other end. However, if u is small, say $u = 0.01$, and $v(t)$ is not small, say $v(t) = 100$, it will take quite a long time. You might like to determine how long.

On the other hand, if the stretching is accelerating sufficiently rapidly, the ant will never reach the other end. So what we are asking for is a simple function $v(t)$ where the said insect only just manages to complete its journey.

Reinhardt Messerschmidt

Let $f(t)$ be the distance at time t between the ant and the end that it started from. Let $g(t)$ be the length of the rope at time t; then

$$g(t) \;=\; 1 + \int_{t_0}^{t} v(s)\,ds.$$

There are two effects on $f(t)$:

(i) it increases at a rate of u due to the walking of the ant;

(ii) it increases at a rate of $(f(t)/g(t))v(t)$ due to the stretching of the rope.

It follows that

$$f'(t) \;=\; u + \frac{f(t)}{g(t)}v(t) \;=\; u + \frac{f(t)g'(t)}{g(t)}.$$

Dividing both sides by $g(t)$ and rearranging,

$$\frac{f'(t)g(t) - f(t)g'(t)}{g^2(t)} \;=\; \frac{u}{g(t)};$$

therefore

$$\frac{d}{dt}\left(\frac{f(t)}{g(t)}\right) \;=\; \frac{u}{g(t)};$$

therefore

$$\frac{f(t)}{g(t)} = uG(t) + c,$$

where c is a constant and

$$G(t) = \int_{t_0}^{t} \frac{1}{g(s)}\, ds.$$

The initial conditions

$$f(t_0) = G(t_0) = 0 \quad \text{and} \quad g(t_0) = 1$$

imply that $c = 0$; therefore

$$\frac{f(t)}{g(t)} = uG(t).$$

In other words, $uG(t)$ is the proportion of the rope that the ant has traversed at time t. The ant reaches the other end at time t if and only if $uG(t) = 1$.

An example with nonconstant velocity

Suppose t_0 and α are strictly positive real numbers. Let

$$v(t) = \frac{\alpha + 1}{t_0}(t/t_0)^{\alpha}.$$

If $\alpha \geq ut_0 - 1$, then for every $t \geq t_0$ we have

$$v(t) \geq \frac{\alpha + 1}{t_0} \geq u,$$

as required for there to be a paradox. Note that

$$g(t) = 1 + \int_{t_0}^{t} \frac{\alpha + 1}{t_0}(s/t_0)^{\alpha}\, ds = 1 + \frac{s^{\alpha+1}}{{t_0}^{\alpha+1}}\Bigg|_{s=t_0}^{s=t} = (t/t_0)^{\alpha+1},$$

and

$$uG(t) = u\int_{t_0}^{t} {t_0}^{\alpha+1} s^{-\alpha-1}\, ds = ut_0 \frac{{t_0}^{\alpha} s^{-\alpha}}{-\alpha}\Bigg|_{s=t_0}^{s=t} = \frac{ut_0}{\alpha}\big(1 - (t/t_0)^{-\alpha}\big).$$

The function $uG(t)$ is strictly increasing with respect to t, and

$$\lim_{t\to\infty} uG(t) = \frac{ut_0}{\alpha}.$$

It follows that

(i) if $\alpha \geq ut_0$, then the ant asymptotically traverses the proportion ut_0/α of the rope, so it never reaches the other end;

(ii) if $\alpha < ut_0$, then the ant reaches the other end at

$$t = t_0\left(\frac{ut_0}{ut_0-\alpha}\right)^{1/\alpha}.$$

Problem 310.1 – A trigonometric series

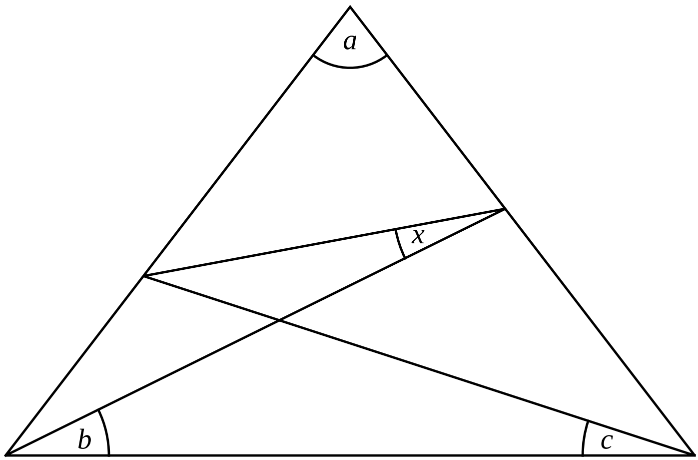
Tony Forbes

Show that

$$\sum_{n=1}^{\infty}\frac{\sin n}{n} = \sum_{n=1}^{\infty}\frac{\sin^2 n}{n^2} = \frac{\pi-1}{2}.$$

Problem 310.2 – Isosceles triangle

The two non-horizontal sides of the big triangle have the same length. Angles a, b and c are given. What's x?



Solution 231.4 – Four tans

Prove that

$$\tan 70^\circ \;=\; \tan 20^\circ + 2\tan 40^\circ + 4\tan 10^\circ.$$

Tony Forbes

Here is a possible attempt at solving this problem. This is in addition to more credible solutions presented in M500 **296** by Peter Fletcher and by someone else. I must admit that I don't like what follows. It seems too easy, especially the last sentence. Comments are invited.

Use your 12-digit pocket calculator to prove that

$$-10^{-11} \;<\; \tan 70^\circ - \tan 20^\circ - 2\tan 40^\circ - 4\tan 10^\circ \;<\; 10^{-11}. \qquad (1)$$

Now let $z = e^{\pi i/18}$ and

$$Z \;=\; \frac{z^{14}-1}{z^{14}+1} - \frac{z^4-1}{z^4+1} - 2\,\frac{z^8-1}{z^8+1} - 4\,\frac{z^2-1}{z^2+1} \;=\; \frac{A}{B},$$

where

$$B \;=\; (z^{14}+1)(z^4+1)(z^8+1).$$

Then A and B are algebraic integers and $B \neq 0$ since $z^n = -1$ only if n is an odd multiple of 18. Also

$$\tan 70^\circ - \tan 20^\circ - 2\tan 40^\circ - 4\tan 10^\circ \;=\; \frac{Z}{i}.$$

Hence, by (1), since $|i| = 1$ and $|B| \le 8$,

$$-10^{-10} \;<\; |A| \;=\; |B||Z/i| \;<\; 10^{-10}.$$

But A is an algebraic integer, and therefore $|A|$ must be exactly zero.

Problem 310.3 – Unit sum

Show that

$$\sum_{n=1}^{\infty} \frac{3 + 10n + 10n^2 - 6n^4}{(n^2+n)^4} \;=\; 1.$$

Problem 310.4 – Shuffle

Let $(X_1, X_2, \dots)$ be a sequence of positive integers. Let k be a positive integer. Construct a sequence $S_k(X) = (Y_1, Y_2, \dots)$ as follows.

$$Y_i = \begin{cases} X_{i/2} & \text{if } i \le 2k \text{ and } i \text{ is even,} \\ X_{k+(i+1)/2} & \text{if } i \le 2k \text{ and } i \text{ is odd,} \\ X_i & \text{if } i > 2k. \end{cases}$$

Alternatively, think of interleaving the first k elements with the next k elements. Start with $N_0 = (1, 2, \dots)$, the sequence of positive integers. Compute $N_1 = S_1(N_0)$, $N_2 = S_2(N_1)$, $N_3 = S_3(N_2)$, and so on, as illustrated below.

$$\begin{aligned}
N_1 &= (2,1,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20) \\
N_2 &= (3,2,4,1,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20) \\
N_3 &= (1,3,5,2,6,4,7,8,9,10,11,12,13,14,15,16,17,18,19,20) \\
N_4 &= (6,1,4,3,7,5,8,2,9,10,11,12,13,14,15,16,17,18,19,20) \\
N_5 &= (5,6,8,1,2,4,9,3,10,7,11,12,13,14,15,16,17,18,19,20) \\
N_6 &= (9,5,3,6,10,8,7,1,11,2,12,4,13,14,15,16,17,18,19,20) \\
N_7 &= (1,9,11,5,2,3,12,6,4,10,13,8,14,7,15,16,17,18,19,20) \\
N_8 &= (4,1,10,9,13,11,8,5,14,2,7,3,15,12,16,6,17,18,19,20) \\
N_9 &= (2,4,7,1,3,10,15,9,12,13,16,11,6,8,17,5,18,14,19,20) \\
N_{10} &= (16,2,11,4,6,7,8,1,17,3,5,10,18,15,14,9,19,12,20,13)
\end{aligned}$$

Now for the problem. Either prove that each positive integer eventually occurs as the first element of N_m for some integer m, or find one that doesn't.

For example, if we track the location of 42 as the positive integers are shuffled, we find that it occurs in positions 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 41, 37, 27, 5, 10, 20, 40, 23, 46, 31, 62, 59, 51, 33, 66, 59, 43, 9, 18, 36, 72, 59, 31, 62, 33, 66, 37, 74, 49, 98, 93, 81, 55, 1, with the move to position 1 occurring after the 54th shuffle. And we would be very interested if you can resolve the question mark in the following table.

n	First move to position 1									
1–10	3	1	2	8	5	4	78	37	6	11
11–20	28	12	349	13	383	10	18	16	29	17
21–30	33	210	14	133	32	60	19	106	57	20
31–40	48	26	21	35	97	217	25	22	13932	863
41–50	205	54	30452	306	2591	40	44	39	49	38
51–60	51	47	30	?	2253	101	112	246	402	119

To create the entries in the table on the previous page, I found it convenient to define the function $P(n,k)$, the position occupied by the positive integer n after the k-th shuffle, starting with the sequence $(1,2,\dots)$. You can verify that

$$P(n,k) \;=\; \begin{cases} n & \text{if } 2k < n, \\ 2P(n,k-1) & \text{if } 2k \ge n \text{ and } P(n,k-1) \le k, \\ 2P(n,k-1) - 2k - 1 & \text{if } 2k \ge n \text{ and } P(n,k-1) > k. \end{cases}$$

Problem 310.5 – 7NT

Tony Forbes

As any bridge player knows, it is good to have a lot of high-card points—perhaps as many as 37—between declarer and dummy in order to have a reasonable chance of making 7NT. But how many points are necessary?

The answer is actually zero. I tried this out on some bridge players I know. I asked, "Can you make seven no trumps with a combined holding containing no card higher than a 9?" "No!" was the usual instant response. But of course they were assuming hostile defence. Here's one possible situation. Dealer South.

North: ♠ 3 2 ♡ 9 8 7 6 5 4 3 2 ♢ 9 7 ♣ 2

West: ♠ A K Q J 10 8 6 4 ♡ A K ♢ – ♣ A K Q

East: ♠ – ♡ Q J 10 ♢ A K Q J 10 8 6 ♣ J 10 9

South: ♠ 9 7 5 ♡ – ♢ 5 4 3 2 ♣ 8 7 6 5 4 3

The bidding (in case it's relevant): South pass, 7♠, 7NT, all pass.

The play: West ♠8, ♠2, ♡Q, ♠9; South ♠7, ♠6, ♠3, ♡J; South ♠5, ♠4, ♣2, ♡10; South ♢3, ♡A, ♢9, ♢8; North ♢7, ♢6, ♢2, ♡K; North ♡9 and claims the rest.

Now for the problem. Either (i) make the North–South hands weaker, or (ii) prove that (i) cannot be done.

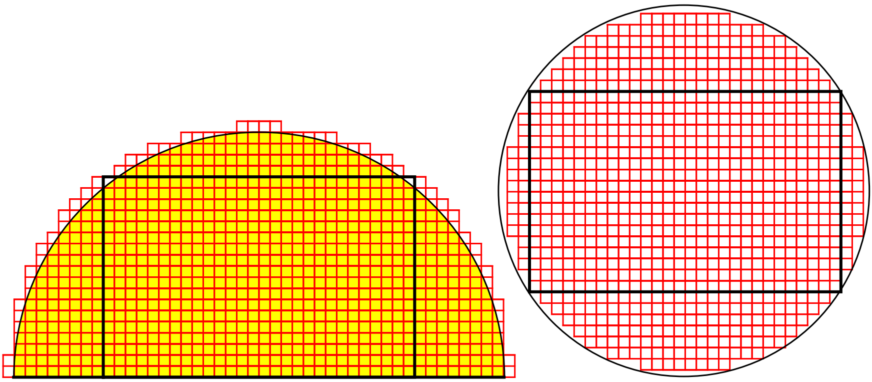
Alternatively, and this is indeed how I arrived at the problem, find a deal with the strongest North–South hands that will fail to make 7NT under every possible defence. However, it might be that the two problems are not equivalent.

Problem 310.6 – Semicircle dissection

Tony Forbes

This is like Problem 264.6 – Semicircle dissection. You have a circular plate, P, of radius 1 and half a pizza, D, which we model by a semicircular disc of area $7\pi/8$. You must dissect D into pieces using a finite number of horizontal and vertical straight-line cuts (with a ruler and a knife, say) so that the pieces will fit on P without any overlapping. What is the smallest number of pieces that must be created? In case an exact solution is too difficult a good upper bound will suffice.

To give some idea of what can be done, the left-hand diagram shows how to dissect D into 812 pieces along grid-lines spaced $d = 0.06$ apart.



Evidently they can be rearranged to fit comfortably on P (right-hand diagram) with room to spare. So we have proved that 812 is an upper bound, although it can be very much reduced by grouping the little squares together in some imaginative way. For example, the central $28d \times 18d$ rectangle would count as one piece. If the general problem is too difficult, we would be interested in the solution to this particular example. Obtain the grouping of the 812 little squares into the minimum number of rectangles.

Problem 310.7 – Approximate trigonometry

Show that $\cos\left(\dfrac{\pi}{2\sin 1}\sin(\cos x)\right)$ is non-negative and that its integral from 0 to π is 4/3, very nearly. What is the significance of this?

Fifty years of M500

You might have noticed the picture on the front cover. We are 50 years old! M500 began life as the *Solent M202 Newsletter*. Issue 1 was published in 1973; according to founding editor Marion Stubbs: At 2300 hrs., precisely, 16 February, 1973, Southampton, England, 51° N, 1°25′ W (born out of despair)—24 copies, together with an application form to join the Solent OU Mathematics Self-Help Scheme, dashed off in four hours flat for an M202 tutorial. It was an instant success. M202, *Topics in Pure Mathematics*, was perhaps the most difficult course the Faculty had to offer in the early years of the OU, and the *Newsletter* was just the kind of thing that geographically isolated students needed.

When issue 6 went out in July 1973, readers were invited to supply a new title as it was no longer restricted to the Solent, nor to M202. Peter Weir suggested M500 for various reasons: (1) Why not? (2) A top-level course in communications. Full credit. (3) It's an overview of OU maths. (4) Why not? [*sic*] (6) [*sic*] I thought of it. By Issue 7, the first to bear the name M500, readership had risen to about 200, and later it rose to over 800 when the Faculty allowed M500 to be advertised in the Mathematics stop presses. The familiar M500-in-a-circle-in-a-circle thing first appeared on M500 **59**. Incidentally, if you care to make some accurate measurements, you will discover a connection between our current logo and the Golden Ratio.

The editorship has been remarkably stable throughout the last 50 years. Marion did everything for the first few issues. Eddie Kent joined her from Number 25 and thereafter Eddie edited while Marion published. Jeremy Humphries was recruited as Problems Editor and later took over from Eddie at M500 **68**. Seventeen years later Jeremy handed the job to me (TF) at M500 **161**. As you can see, I am still here, and a simple calculation confirms that M500 **310** marks my 20th anniversary.

"Out of all the [undergraduate mathematics] magazines I've seen, you're the best," was the enthusiastic comment of an eminent mathematician. It's because you the readers are the contributors. If you look at other similar publications, you will often notice an obvious division: authors are vastly superior, omni-cognate beings while readers are mere mortals. However, there is, we hope, no such class distinction in M500. We like to encourage the notion that readers and writers operate on equal terms.

We will continue to flourish if you keep up the good work. You have done very well to keep M500 going for such a long time against fierce competition from the internet—many, many thanks. But do keep the contributions coming. As usual, be as formal or informal as you like and write to us about anything to do with mathematics. Remember, *no mathematics is too trivial for* M500.

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04.02

Problem 310.8 – Sum

Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n^3 + n^4} = \frac{\pi^2}{3} - 3.$$

Front cover Cake with 50 candles on top.