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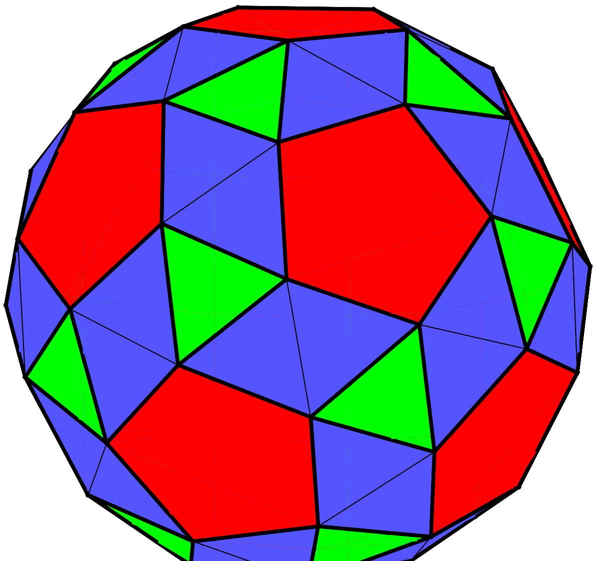

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The Revision Weekend is a residential Friday to Sunday event providing revision and examination preparation for both undergraduate and postgraduate students. For details, please go to the Society's website.

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Please let me know if you require any more information.

A remark about limit cycles and French rivers

Michael Grinfeld

The Poincaré–Bendixson Theorem [3, Theorem 11.1] summarises all the possibilities for asymptotic behaviour in time of bounded orbits of autonomous ordinary differential equations (ODEs) in the plane $\mathbb{R}^2$. As a consequence of that theorem, for example, one can immediately conclude that chaos in such equations can never occur. This theorem can also be used constructively: if one manages to find a domain Ω, which does not contain any rest points, such that orbits can only enter Ω and cannot leave (or only leave and cannot enter), then the domain Ω will contain periodic orbits. See [3, Section 11.1] for details of the technique used.

Consider the system of equations (primes stand for derivatives with respect to time)

$$\begin{cases} x' &= -y - x(x^2 + y^2 - 2y - 3), \\ y' &= x - y(x^2 + y^2 - 2y - 3). \end{cases} \tag{1}$$

Then by the above technique, it is possible to show that there are periodic orbits in the annulus (in polar coordinates) $r_0 < r < r_1$, where $r_0 < 1$ and $r_1 > 3$, which is our domain Ω in this case. In polar coordinates, equations (1) become

$$\begin{cases} r' &= -r(r^2 - 2r\sin\theta - 3), \\ \theta' &= 1. \end{cases} \tag{2}$$

In this case, using, for example, the results of Gasull and Giacomini [2, Section 4], one can prove that the periodic orbit is unique, thus a limit cycle.

Last May, during the M500 revision weekend, I was asked by a student whether the locus

$$r^2 - 2r\sin\theta - 3 \;=\; 0$$

in the polar plane defines the limit cycle, that is, whether the equation of the limit cycle in Cartesian coordinates is $x^2 + (y-1)^2 \;=\; 4$. Of course the answer to this question is negative, because if the limit cycle were defined by $r^2 - 2r + 2\sin\theta - 3 \;=\; 0$, we would have $r' \equiv 0$ but the locus is the circle of radius 2 with centre at $(0, 1)$, for which r is not constant.

On the other hand, numerically $x^2 + (y-1)^2 \;=\; 4$ gives an excellent approximation to the limit cycle as Figure 1 shows. Here the approximation is in a thick dashed line, and the 'true' limit cycle is the solid closed curve. The match is not perfect, of course, as for example the limit cycle does not pass though the point $(-2, 1)$.

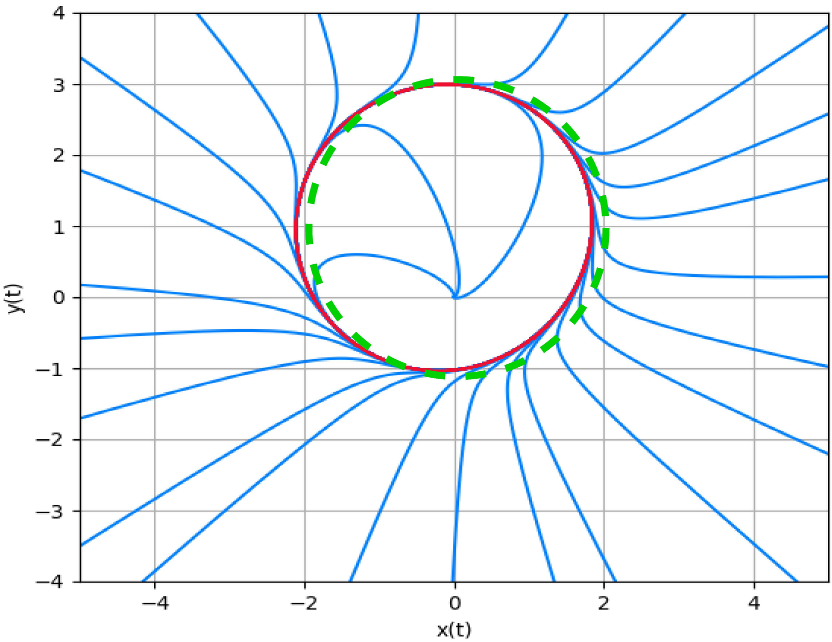


Figure 1: The limit cycle and its approximation.

The question is why a simple expression, immediately derivable from the equations, gives such a good approximation.

In brief, the key is the concept of a river, or a *fleuve* in French, as this area of asymptotics of ordinary differential equations has been largely developed in francophone countries as well as being largely overlooked in English-speaking ones; the most directly relevant paper is by Michel [4].

For a precise definition of a river, please see [1, Definitions 1 and 2]. Roughly speaking, rivers are orbits in systems of ODEs in the plane that have very strong attraction or repulsion properties; please see Figures 2 and 3 (on which all the analysis below will be done for simplicity), where the location of the rivers, both attracting and repelling, is obvious as is the reason these orbits are called rivers.

If we divide the r equation by the θ equation, we get

$$\frac{dr}{d\theta} \;=\; -r(r^2 - 2r\sin\theta - 3).$$

and Michel's theory of oscillating rivers applies to such equations.

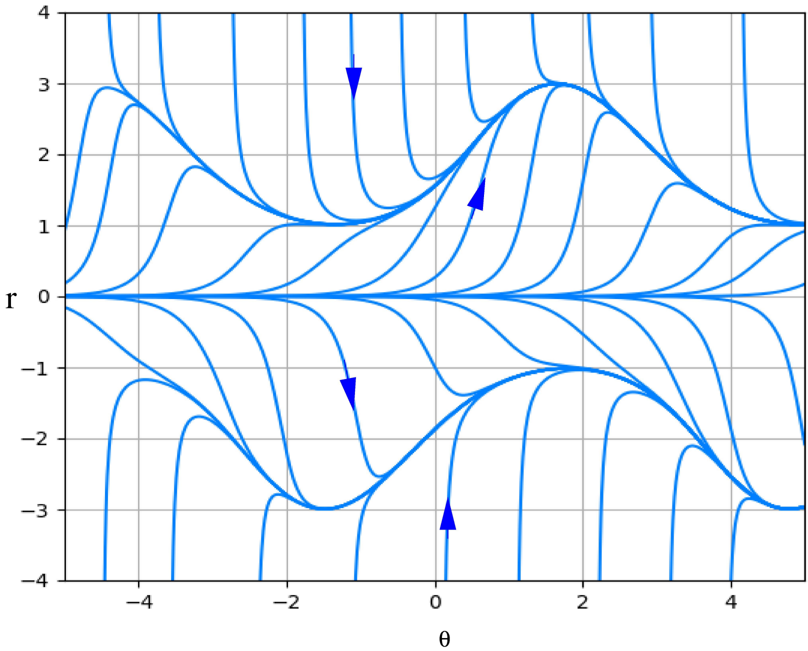


Figure 2: The limit cycle unfolded as a river.

From the Michel theory, we immediately have that

Claim 1: The limit cycle $r(\theta)$ is a river.

Actually, the number of limit cycles is precisely the number of positive rivers, which is nice. But more is true.

Claim 2: $r_0(\theta) \;=\; \sin\theta + \sqrt{\sin^2\theta + 3}$ is the leading term of the river expansion

$$r(\theta) \;\sim\; r_0(\theta) + r_1(\theta) + \dots.$$

The algorithm for the expansion is explained in [4]. To find $r_1(\theta)$ in our case is possible but rather computationally involved. Let me show how it is done in a simpler case (in this case there are two rivers). I will compute two terms of one of them, and will indicate the properties of the exact solution.

So let us consider the much simpler equation,

$$x' \;=\; x^2 - t^2. \tag{3}$$

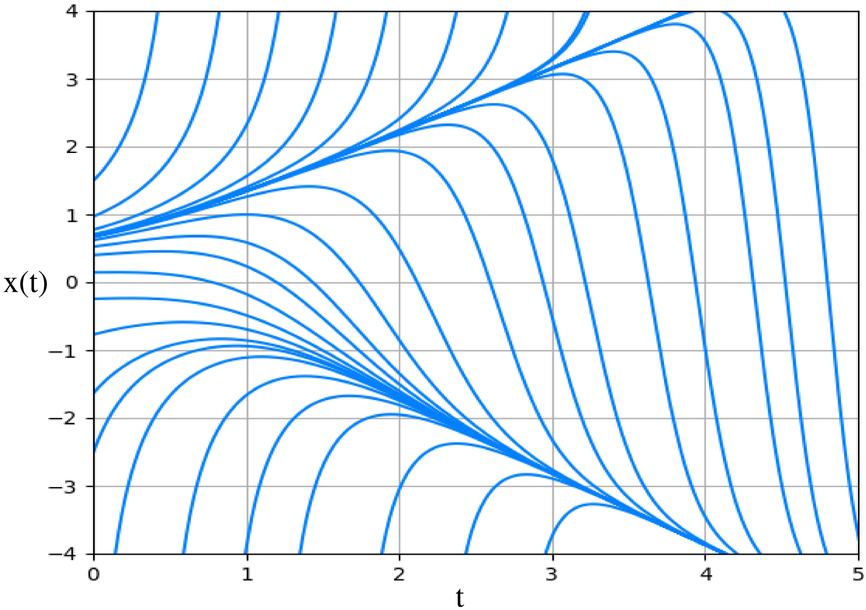


Figure 3: Some more rivers.

To compute the asymptotics of the positive (repelling) river, put the right-hand side of the equation equal to zero. Choose the positive root of $x^2 = t^2$, call it $R_0(t)$. So $R_0(t) = t$ is the leading term of the (positive) river expansion. Now let us compute the next term. Consider $x - R_0$:

$$(x-t)' \;=\; x' - 1 \;=\; (x - t + t)^2 - t^2 - 1 \;=\; (x-t)^2 + 2t(x-t) - 1.$$

Now rename $x - t$ to x:

$$x' \;=\; x^2 + 2tx - 1.$$

Solve the rhs for x and call the positive solution $R_1(t)$, i.e.

$$R_1(t) \;=\; \sqrt{t^2+1} - t.$$

So our more sophisticated approximation to the positive river is:

$$R(t) \;=\; R_0(t) + R_1(t) \;=\; t + \sqrt{t^2+1} - t \;=\; \sqrt{t^2+1}. \tag{4}$$

I hope it is clear how to find the next term (renaming is the key).

Note that in this way we get an asymptotic expansion as $t \to \infty$. In fact, (4) gives

$$R(t) \;\sim\; t + \frac{1}{2t} + O\left(\frac{1}{t^3}\right) \quad \text{as } t \to \infty.$$

To work out the $O(1/t^3)$ term one needs to compute $R_2(t)$.

In this case, one can use computer algebra software to find the exact solution for the river; this is left as an excellent exercise. The general solution of (3) can be found in terms of Bessel functions, and a judicious choice of the constant allows one to pick the (unique) solution that grows like t as $t \to \infty$. Its asymptotic behaviour turns out to be

$$R(t) \;\sim\; t + \frac{1}{2t} - \frac{8}{t^3} + O\left(\frac{1}{t^5}\right)$$

and, gratifyingly,

$$R(0) \;=\; \frac{[\Gamma(3/4)]^2\sqrt{2}}{\pi} \;\approx\; 0.67597824.$$

References

[1] F. Blais, Asymptotic expansions of rivers, in: *Dynamic Bifurcations*, ed. E. Benoît, Springer, Berlin, Heidelberg 1991, 181–189.

[2] A. Gasull and H. Giacomini, Effectiveness of the Bendixson–Dulac theorem, *J. Diff. Eq.* **305** (2021), 347–367.

[3] D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations*, 4th Edition, OUP, Oxford, 2007.

[4] F. Michel, Fleuves oscillants, *Bull. Belgian Math. Soc. – Simon Stevin* **2** (1995), 127–141.

The point in the concept of geometric shapes

Mako Sawin

1 Introduction

In geometry, the point, line and the curve are essential for measuring plane shapes and volumes of solid figures. The current definition for the point is that it has only a position. It has no length, width or dimension and it has no size in a geometric figure.

The shortest line between two points is called a straight line. A line only has one dimension, it has length but no thickness or width or size. The point and the line have no size because they are not considered as objects, but only show the positions or directions of objects. From the definitions of the point and the line, we conclude that they have no area and no shape such as a circle or a square. However, mathematics deals with fact and logic. Therefore, the point somehow gives a geometric shape area. It might be associated with a nano of length and width.

We can give an example. Splitting a large area into two parts or more, we might give the point size 1 cm, while for splitting a small area, the Planck length would be appropriate. Also, we can conclude that somehow a point gives shape. Here, the question arises, what is the shape of a point in our real life? Mathematicians do not know the shape of points. If a point has a square shape, that means it has two dimensions. But if the point is a circle, it has only one dimension, which contradicts the current definition of the point.

However, in real life any object has a shape, and its position is determined by a point; that means the point should have a shape. But what kind of geometric shape is suitable for the point? If we choose a circular shape, it might look temporarily perfect; but if we join two points, there will be a tiny gap between them and they will not give us a perfect line. However, if we join two points with a square shape, we get a perfectly straight line. For drawing a perfect shape like a square, the points might take a square shape rather than a circle shape, whereas for drawing the circle shape, we can choose the circle as the shape of the points.

The concept of the point in geometry is not only the position of the object but it is the reflection too. For example, if the plane bends, then the points will give the exact position of the bending. Therefore the shape of the point either becomes smaller or its shape is changed. To give an example, if the line consists of at least three points, then if it bends, the number of its points either decreases, or the shape of the points changes. Therefore we can conclude that the point has a shape and it depends on the shape of the objects.

2 Point in the concept of the Area

One of the most important subjects in mathematics is measuring the areas of plane shapes and the volumes of solid figures. The geometric shape can be defined as the space occupied by a flat shape, with the points lying on its surface. A plane shape only has two dimensions and we can only measure the area of its surface, like a sheet of paper. For some examples of plane figure shapes, there are squares, rectangles, etc. The area of the shape is the measure of the set of the points that cover the figure. To prove that the area of the square is x^2 we illustrate in Figure 1a that each side of the square is composed of a set of points. Let X_1 be made from $x_{1,1}$, $x_{1,2}$, …,

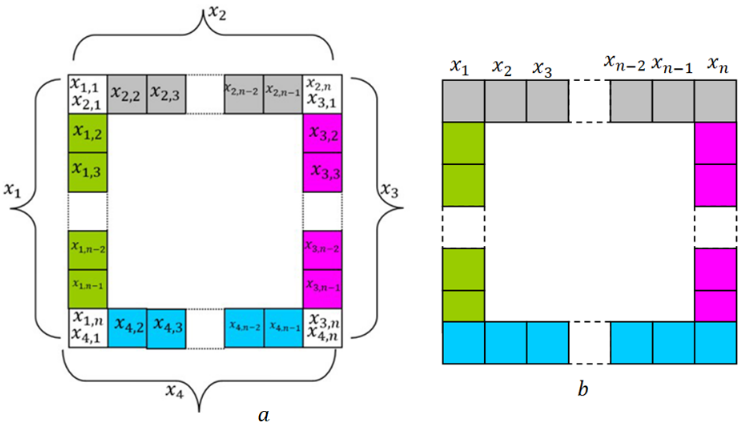


Figure 1

$x_{1,n}$, and X_2 from $x_{2,1}$, $x_{2,2}$, …, $x_{2,n}$. In each point of X_1 we can draw a straight line to a point in X_3 such as $x_{1,2}$ to $x_{3,2}$. All vertical lines are equal to X_1 and all horizontal lines are equal to X_2. To calculate the sum of all points,

$$A = X_2 \times \sum_{i=1}^{n} X_{1,i} \quad \text{and} \quad x_{1,1} + x_{1,2} + \cdots + x_{1,n} = \sum_{i=1}^{n} x_{1,i} = X_1.$$

Thus

$$A = X_1 \times X_2 \quad \text{and} \quad X_1 = X_2 = X_3 = X_4.$$

Therefore the area is

$$A = X^2. \tag{1}$$

The area of a square is a collection of all straight lines either vertically or horizontally; see Figure 1b. Therefore,

$$A \;=\; \sum_{i=1}^{n} x_{1,i} + \sum_{i=1}^{n} x_{2,i} + \cdots + \sum_{i=1}^{n} x_{n,i} \;=\; \sum_{i=1}^{n}\sum_{k=1}^{n} x_{k,i}. \qquad (2)$$

The same principle will apply to rectangles and triangles and other shapes.

Points can change their size according to the shape of the geometric plane. To give an example, when the line bends the points on the line will change their size. Also, if two lines or more are partially overlapped, then their points are changed too. By partially overlapping two or more points the sizes of the points change and they share a tiny area.

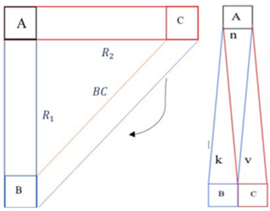


Figure 2

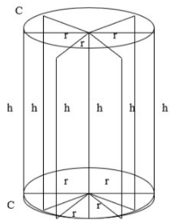


Figure 3

To illustrate, draw two equal lines R_1 and R_2 at the point A to the line BC, as shown in Figure 2. Then make the straight line BC shorter. We can see the points of both R_1 and R_2 overlapping partially. The area is divided into three equal parts. Thus k and v represent half of the points of R_1 and R_2. Let n be the number of points in the intersection between two lines with half of the points overlapping. If we find the area by considering these points, then

$$A \;=\; \frac{1}{2}AB + \frac{1}{2}AC + \frac{1}{2}(AB \cap AC)$$

In fact $n = AB \cap AC$, which has half the points of both R_1 and R_2. Thus n makes half of the points of each line, and we can see that half of the points have vanished. Therefore, the size of the points has changed and a number of the points vanished, even when the points are infinitely tiny. We can conclude that the points changed the shape and the size of each of lines R_1 and R_2.

We can give another example to prove that some points vanish if the lines are partially overlapping. The circle is a geometric shape which is a set of points enclosed by a curved line. The area of the circle is

$$A = \frac{1}{2}rc. \tag{3}$$

To find the area of the circle as a collection of points, we draw from the centre the line, which is the radius r_i, to the circumference. The number of radii is

$$N = \sum_{i=1}^{K} i, \quad \text{where} \quad K \to \infty.$$

Then the area is the sum of the points,

$$A = c \times \sum_{i=1}^{K} r_i \Rightarrow A = rc,$$

where $K \to \infty$ and r_i is a set of points, and we see that half of the radial points have vanished from the area of the circle, $rc/2$.

Thus, however tiny the points are, by partial overlapping of the radii always some points vanish. The points will reduce the size which it had with the original straight line. We can conclude for a circle shape that half of the points vanish.

To conclude this section, if two lines are drawn from one point and there is a partial overlapping, then each line has a portion of the area. The shape and the size of the points are changed accordingly, contrary to the axiom which says a point has no size. Also we see that the size and the shape of the points are changed; otherwise, adding all together would make the area of the circle rc.

3 Point in the concept of the Volume of a Solid Figure

The solid figure is three-dimensional, having length, width and height, and its volume is measured by a cube. The volume of a cube is a collection of all layers of squares, where each layer is a set of points. This kind of measurement is applied to all kinds of prisms, such as polygons, rhombuses, parallelograms, triangles and stars.

Here we examine the point based on some solid figures and how its shape and size will change.

3.1 The Volume of the Cylinder

The usual way to find the volume of the cylinder is to multiply its height by the area of the circle, since it consists of layers of circles,

$$V = \frac{1}{2}r \times c \times h. \tag{4}$$

However, if we consider the shape of a cylinder, we can see that it consists of a set of either rectangles or squares, depending on the height. See Figure 3 on page 8. The area of any rectangle is

$$\text{Area} \;=\; h_i r_i.$$

Thus, summing all the rectangles,

$$S_r \;=\; \sum_{i=1}^{n} h_i r_i.$$

Also, the number of rectangles is the same as the circumference of the circle. Therefore the volume is

$$V \;=\; hrc,$$

which contradicts equation (4). We can see half the points of the volume of a cylinder vanish in the same way as half the points vanish from the area of the circles. If we have a cylinder and we make it flat, it would have a rectangular or square shape, depending on the height, and the area of the surface is not equal to $c \times h$; again, half of the points vanish.

The same method will apply to the volume of pyramids. If the some of the points have not vanished by overlapping, then the volume of the pyramid is $V = h/2 \times \text{base}$, which contradicts the actual formula.

3.2 Bending Points in the Sphere

If the point cannot be bent or changed in its size, we arrive at another new formula for finding the volume of a sphere and its surface area. The surface area of the sphere is

$$A \;=\; 4\pi r^2. \tag{5}$$

The volume of the sphere is

$$V \;=\; \frac{4}{3}\pi r^3. \tag{6}$$

Suppose we have a sphere as shown in Figure 4. Then if we take any quarter of the circle in the sphere, it is equal to the others. Then the quarter of the circumference of the circle is

$$a \;=\; b \;=\; \ldots \;=\; n \;=\; \frac{1}{4}c.$$

Therefore, in half of a sphere, the base is a circle and $c/4$ goes round the

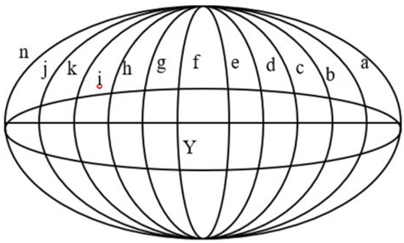


Figure 4

circumference as shown in Figure 4. This means that the surface area of the half sphere is

$$A \;=\; \frac{1}{4}c \times c \;=\; r^2\pi^2$$

and the surface area of the sphere is

$$A \;=\; 2r^2\pi^2, \tag{7}$$

which contradicts the actual formula, (5).

In the same way we see that a quarter of the area of the circle is

$$A \;=\; \frac{1}{4}\left(\frac{1}{2}rc\right).$$

Then the volume of the half-sphere is

$$V \;=\; \left(\frac{1}{8}rc\right) \times c \;=\; \frac{1}{4}\pi^2 r^3.$$

Thus the volume of the sphere is

$$V \;=\; \frac{1}{2}\pi^2 r^3, \tag{8}$$

which is not equal to the formula (6). We can see a contradiction in the formula for the volume of a sphere and its surface area if the points cannot bend or change from their size. We see that inside the sphere the points are massively intersecting. The points of the line cannot be the same points on the same lines if the line is changed to a curve. The points will bend and their shapes will transform. If the point is a location of the object, then, however tiny the points are, their shapes will change according to the transformations of the objects.

4 Conclusion

We conclude with the observation that however small the point is, it has a size that can be changed according to the size of the area. When two points are partially overlapping the size also changes and a measurable area as a portion of the point vanishes. If the point has no size, shape, length and width as defined, therefore, we come to conclude the following.

The area of the circle is	$r \times c$,
the volume of the cylinder is	$r \times c \times h$,
the volume of the cone is	$hrc/2$,
the volume of the pyramid is	$h \times \text{base}/2$,
the area of the surface of the sphere is	$2r^2\pi^2$,
and the volume of the sphere is	$\pi^2 r^3/2$.

Problem 311.1 – Binary binomial coefficients

Tony Forbes

Let $R(n)$ be the number obtained by writing the elements of row n of Pascal's triangle in binary one after the other with no spacing in between. Thus for $n = 0, 1, \ldots,$ we have

$$\begin{aligned} R(n) \;=\; & 1,\ 3,\ 13,\ 63,\ 1641,\ 55979,\ 1963261,\ 1051838303,\ 427823653777, \\ & 899765549835411,\ 962612860717614517,\ \ldots. \end{aligned}$$

For example, when $n = 4$ we have

$$(1, 4, 6, 4, 1) \to (1, 100, 110, 100, 1) \to 11001101001_2 = 1641 = R(4).$$

One is tempted to conjecture that

$$R(n) \;<\; 2^{n^2/(\log 4)} \qquad \text{for all sufficiently large } n.$$

Either prove that this is true, or find infinitely many n for which $R(n) \geq 2^{n^2/(\log 4)}$.

Problem 311.2 – Triangle-free regular graphs

Tony Forbes

Prove that there exists a k-regular graph with $2k + 1$ vertices and girth at least 4 only when k is 0 or 2. Or find a counter-example.

Problem 311.3 – Circle construction

Tommy Moorhouse

Take a circle of radius 1 centred on the origin in $\mathbb{R}^2$, and a vertical line L parallel to the y-axis, passing through the point $(1, 0)$. Given a point P on the right hand half of the circle we define the height t to be given by the intersection of the production of $\overrightarrow{OP}$ with L. Let ρ be the distance between the origin and the intersection of the line $\overrightarrow{NP}$ with the x-axis.

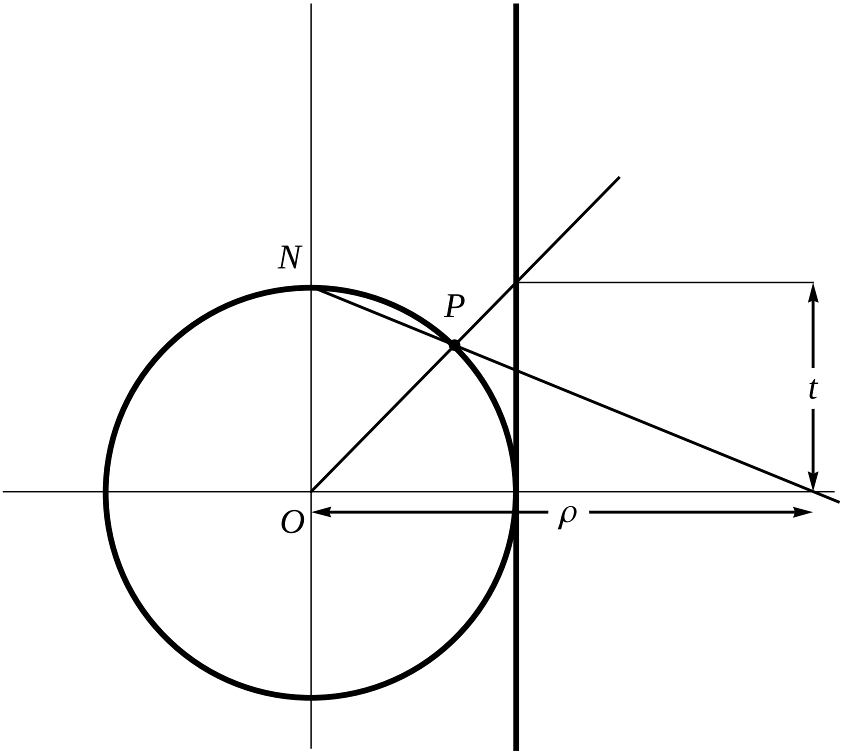


Figure 1: Construction of t and ρ.

Show that

$$\rho^2 - 2t\rho - 1 \;=\; 0.$$

Solution 308.3 – Arithmetic progression

Show that
$$x + y = -\cos(\pi/9)$$
if $x + y < 0$ and $\dfrac{1}{1+x}, \dfrac{1}{1-y}, \dfrac{1}{x}, \dfrac{1}{y}$ are in arithmetic progression.

J. M. Selig

Since we are trying to find $x + y$ let's change variables so that $z = x + y$ and $w = x - y$. The inverses of these relations are

$$\begin{aligned} x &= \frac{1}{2}(z + w), \\ y &= \frac{1}{2}(z - w). \end{aligned}$$

Using these relations to substitute into the arithmetic progression gives

$$\frac{2}{2+w+z}, \frac{2}{2+w-z}, \frac{2}{w+z}, \frac{2}{z-w}.$$

As these expressions form an arithmetic progressions their second differences must vanish, that is

$$\nabla^2 a_i = a_i - 2a_{i+1} + a_{i+2} = 0.$$

This will give us two equations,

$$\frac{2}{2+w+z} - \frac{4}{2+w-z} + \frac{2}{w+z} = 0$$

and

$$\frac{2}{2+w-z} - \frac{4}{w+z} + \frac{2}{z-w} = 0.$$

Next, we can put everything over a common denominator and simplify. In particular, we set the numerator of each to zero. This results in a pair of equations,

$$(2z - 1)w + (2z^2 + z - 2) = 0$$

and

$$w^2 + (3 - 2z)w + (z^2 - z) = 0.$$

Now, we can use the first of these equations to eliminate w from the second equation. Again, we can just set the numerator of the resulting expression to zero. The result is a quartic in z, which factorises as

$$(8z^3 - 6z + 1)(z - 1) \;=\; 0.$$

The question tells us that $z = x+y < 0$, so $z = 1$ is not the solution sought. We need to solve the cubic factor in the above. There is an intriguing method to solve cubic equations using trigonometry. We can compare the cubic to the cosine triple angle formula, this is

$$\cos(3\theta) \;=\; 4\cos^3\theta - 3\cos\theta.$$

Rearranging the cubic gives

$$-\frac{1}{2} \;=\; 4z^3 - 3z;$$

so, if we set $\cos(3\theta) = -1/2$, then $3\theta = 2\pi/3$. Actually we get an infinite number of solutions,

$$3\theta \;=\; \frac{2\pi}{3} + 2n\pi,$$

where n is any integer. Now, $z = x + y = \cos\theta$; this gives three different real solutions,

$$x + y \;=\; \cos\left(\frac{2\pi}{9}\right), \cos\left(\frac{2\pi}{9} + \frac{2\pi}{3}\right), \cos\left(\frac{2\pi}{9} + \frac{4\pi}{3}\right).$$

(The other values of n just repeat these three solutions.) The first solution is positive, as is the third,

$$\cos\left(\frac{2\pi}{9} + \frac{4\pi}{3}\right) \;=\; \cos\left(\frac{14\pi}{9}\right) \;=\; \sin\left(\frac{\pi}{18}\right).$$

The only negative solutions is

$$\cos\left(\frac{2\pi}{9} + \frac{2\pi}{3}\right) \;=\; \cos\left(\frac{8\pi}{9}\right) \;=\; -\cos\left(\frac{\pi}{9}\right).$$

TF writes: A similar problem appears as LXXXVIII.7 in *Mathematical Problem Papers* by E. M. Radford (CUP, 1923). I was so intrigued by the process by which one goes from a simple 4-term arithmetic progression to an even simpler trigonometric equality that I thought it deserved the attention of M500 readers.

Problem 311.4 – Absolute differences

Let n be a power of 2.

(1) Write down n positive integers in a column.

(2) Form a new column thus: for $k = 1, 2, \ldots, n$, next to the k-th integer write the absolute difference between it and the $((k \bmod n) + 1)$-th integer.

(3) If the result is n zeros, STOP; otherwise go to (2).

Must the process stop? If that's too difficult, we would be interested in examples for small powers of 2 that take a long time to finish. Here is a sequence for $n = 4$.

82	77	401	183	84	40	198	88	36	12	104	48	8	16	16	16	0
5	478	218	99	44	238	110	52	24	116	56	40	24	32	0	16	0
483	260	119	55	282	128	58	28	140	60	16	16	56	32	16	16	0
223	141	64	337	154	70	30	168	80	44	32	72	24	16	0	16	0

Numbers n that are not powers of 2 behave in a different manner. For example, starting with $(6, 40, 15, 7, 26)$ the algorithm soon goes into a 15-cycle beginning with $(1, 1, 0, 1, 1)$.

Problem 311.5 – Colours and shapes

There are 9 objects. They could be a child's bricks, but their exact nature need not concern us. Each has one of 3 colours, {red, blue, green} say, and one of 3 shapes, {triangle, square, pentagon}. All 9 distinct combinations are represented.

They are to be arranged in a circle such that two adjacent objects differ in colour or in shape but not both and not neither. For example,

(R3, R4, R5, B5, B3, B4, G4, G5, G3),

where to save printing space the circle is a straight line and we have employed some hopefully obvious abbreviations.

How many ways?

What about c colours and s shapes?

Snub cube and snub dodecahedron

Tony Forbes

A snub cube consists of 6 squares and 32 equilateral triangles. It is known (see, for example, `https://mathworld.wolfram.com/SnubCube.html`) that the triangle–triangle dihedral angle is given by

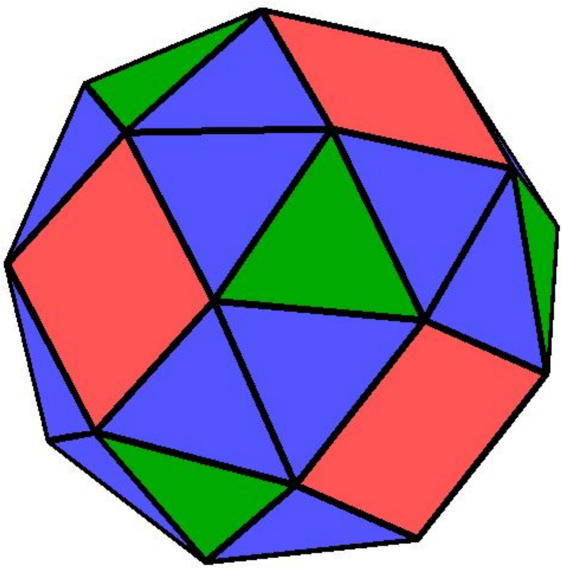
$$\pi - \arccos \frac{2t-1}{3} \;=\; 2.67445 \;=\; 153.235^\circ,$$

where $t = 1.83929$ is the real root of the cubic $x^3 - x^2 - x - 1$. The picture shows a clockwise snub cube—in the sense that the squares are rotated $\arccos \sqrt{t/2}$ clockwise from their original orientations. View it in a mirror for the anticlockwise version. The triangles are divided into two classes.

(i) Octahedral triangles (green in the picture): those where the vertices are coincident with the vertices of three distinct squares. There are 8, and one can see that if you discard the other faces and contract, they might form the eight faces of a regular octahedron.

(ii) Snub triangles (blue): the other 24. They each share a common edge with a square.

Moreover, John Smith asks in M500 **200**: Why is the dihedral angle between two snub triangles the same as the dihedral angle between a snub triangle and an octahedral triangle? We still don't really know. As far as we can see (and admittedly this might not be exceedingly far), there is nothing about the structure of the snub cube that obviously implies equality. Is there a simple explanation?



One can also ask the same question about the snub dodecahedron. This fascinating Archimedean solid has 12 pentagons and a lot of triangles which, as with the snub cube, are of two types.

(i) Twenty icosahedral triangles, coloured green in the picture on the front cover, which clearly shows that the vertices are coincident with the vertices of three pentagons. One can check that the angle between nearest pairs is the same as the dihedral angle of a regular icosahedron, $2\arctan((3+\sqrt{5})/2) = 138.190°$.
(ii) Sixty snub triangles, blue, each sharing an edge with a pentagon.

Here is something interesting. Start with a rhombicosidodecahedron with edge length 1, top-left in the sequence illustrated on page 20. There are 12 red pentagons, 20 green icosahedral triangles and 30 blue squares, each of which one can consider as being split into two $(1, 1, \sqrt{2})$ blue snub triangles. Angles between various pairs of faces are tabulated below.

face colours	symbol	dihedral angle	degrees
blue–blue	s33	π	180°
blue-green	i33	$\arccos\left(-\frac{\sqrt{3}+\sqrt{15}}{6}\right)$	159.095°
red–blue	s35	$\arccos\left(-\sqrt{\frac{5+\sqrt{5}}{10}}\right)$	148.283°
red–green	i35	$\arccos\left(-\sqrt{\frac{5+2\sqrt{5}}{15}}\right)$	142.623°

Now rotate all the pentagons about their centres of 5-fold symmetry by θ anticlockwise (when looking at them from outside). At the same time adjust their distances from the centre so that the icosahedral triangles retain the same edge length, 1. Observe what happens to s33, i33, s35, i35 and the shapes of the blue triangles.

As θ increases from 0 to 36°, s33 decreases while i33 and s35 increase, but i35 remains constant throughout—it is the icosidodecahedron triangle–pentagon dihedral angle. The green triangles remain equilateral with side 1. Three more examples are illustrated on page 20.

At $\theta = 8°$ the snub triangles are still short and fat, with sides $(1, 1, c)$ where $c > 1$, and i33 is still less than s33. However, at $\theta = 20°$ the snub triangles have become tall and thin, with $c < 1$, and i33 is greater than s33.

By continuity, therefore, two things happen somewhere between $\theta = 8°$ and $\theta = 20°$.

(A) At some point $\theta = \theta_{\rm A} \in (8°, 20°)$ the snub triangles become equilateral, and
(B) at some point $\theta = \theta_{\rm B} \in (8°, 20°)$ dihedral angles i33 and s33 are equal.

Unless I have missed something, I can see no obvious reason why (A) and (B) should occur at the same value of θ. But the amazing truth is that they do. Let $\sigma = 0.142850164945$ be the smallest positive root of

$$64x^6 + 64x^5 + 800x^4 + 240x^3 - 800x^2 - 306x + 59 \tag{1}$$

and let

$$\alpha \;=\; \frac{3\pi}{10} - \frac{\arccos(\sigma)}{2} \;=\; 13.1064033769°. \tag{2}$$

Then $\theta_{\rm A} = \theta_{\rm B} = \alpha$, and when $\theta = \alpha$ the solid is a snub dodecahedron—an anticlockwise one since α is positive. This is illustrated on the front cover. To get the clockwise version, rotate by $\theta = -\alpha$.

When $\theta = 36°$ the snub triangles are extremely thin and the solid has become an icosidodecahedron. Angle s33 is exactly 144°, which is sensible because two adjacent snub triangles have degenerated into lines that form two sides of a regular decagon that bisects the icosidodecahedron. But you will also note that i33 and s35 attain limiting values of 169.188° and 153.435° respectively. Unfortunately I do not know what significance this has other than that $\text{i33} + \text{s35} \equiv \text{i35} \pmod{180°}$.

I am not entirely happy with the polynomial (1), which I got from *Wolfram Mathworld*. As you can verify from *Wikipedia*, the two dihedral angles, the pentagon in-radius, the triangle in-radius, the mid-radius, the circumradius, the surface area and the volume can be expressed as elementary functions of the Golden Ratio, $\phi = (\sqrt{5} + 1)/2$, and

$$\xi = 0.943151259244, \quad \text{the real root of } \; x^3 + 2x^2 - \phi^2.$$

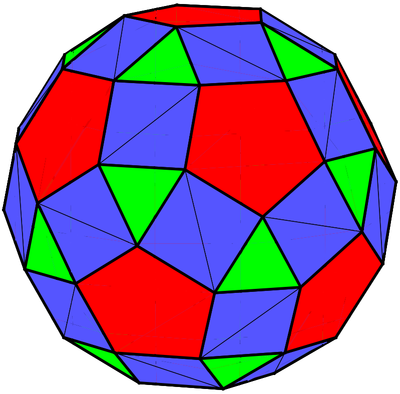
So what about σ and α? It turns out that (1) factorizes over the field $\mathbb{Q}(\phi)$, and σ is the root of one of its factors,

$$8x^3 + 4x^2 + (50\phi + 24)x - (5\phi + 7),$$

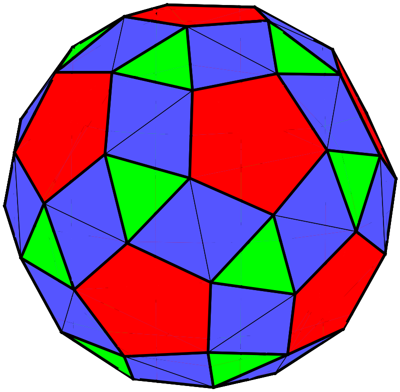
which can be solved to get

$$\sigma = \frac{\Delta}{12} - \frac{5(7 + 15\phi)}{3\Delta} - \frac{1}{6}, \quad \Delta = \left(20\big(95 + 36\sqrt{5} + 3\sqrt{15(1315 + 2128\phi)}\big)\right)^{1/3}$$

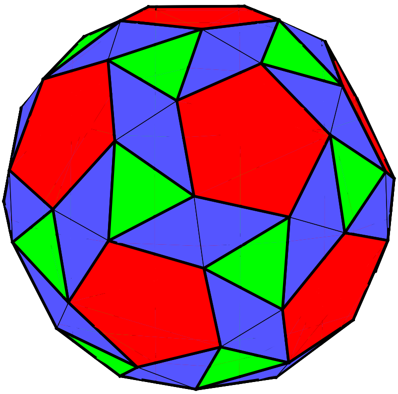
from which we can obtain an exact expression for α via (2). Admittedly this formula is a bit of a mess—and it doesn't involve the number ξ in a nice way if at all—but it will have to do for now.



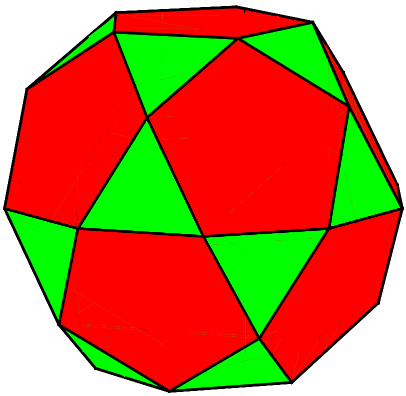
θ=0
s33=180. i33=159.095
s35=148.283 i35=142.623



θ=8
s33=169.63 i33=162.551
s35=151.582 i35=142.623



θ=20
s33=157.679 i33=165.918
s35=154.03 i35=142.623



θ=36
s33=144. i33=169.188
s35=153.435 i35=142.623

The *sphericity* of an object is defined as

$$36\pi\,\frac{(\text{volume})^2}{(\text{surface area})^3}.$$

For example, it's about 0.32973 for an $8 \times 4 \times 2$ brick. The 36π is obviously there to make the expression 1 for a sphere. For a snub dodecahedron, we have from *Wikipedia*:

$$\begin{aligned}\text{surface area} &= 20\sqrt{3} + 3\sqrt{20\phi + 15} && = 55.2867,\\ \text{volume} &= \frac{(3\phi+1)\xi^2 + (3\phi+1)\xi - \phi/6 - 2}{\sqrt{3\xi^2 - \phi^2}} && = 37.6166,\end{aligned}$$

and a simple calculation gives a sphericity of 0.94700. Therefore it would seem that a snub dodecahedron would make a significantly better football than the Archimedean object that is currently employed for that purpose, the truncated icosahedron, which has sphericity only 0.90317. However, the choice of two chiral options and consideration of the difference in their aerodynamic behaviour on the football pitch would risk adding an undesirable level of complexity to the game.

Problem 311.6 – Square and dodecagon

Tony Forbes

Show that the square and the dodecagon are the only regular polygons that have rational areas when inscribed in a unit circle.

In fact the areas are integers, 2 and 3. Obviously this was the inspiration for the introduction of the 12-sided 3d piece in 1937 to replace the minute 'silver' coin of the same denomination. However, we do not believe there were any plans at the time for the minting of a square 2d coin.

Problem 311.7 – Grand slam

Tony Forbes

As any bridge player knows, it is possible to make all 13 tricks with a combined holding of only 5 high-card points (A and J of trumps).

Devise a plausible situation where it is possible to make a grand slam in a suit with no card higher than a 10.

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Problem 311.8 – Factorial ratio

Tony Forbes

For non-negative integers a and b, define

$$F(a,b) \;=\; \frac{(2a)!\,(2b)!}{4\,a!\,b!\,(a+b)!}.$$

Show that $F(a,b)$ is an integer if and only if $a+b$ is positive and not a power of 2. For something similar, see Problem 295.3 – Integers.

Front cover Snub dodecahedron; see page 17.