## M500 167



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## Sums of Powers - Again

## Barry Lewis

previous article (Hip, hip, Array, M500 162) started as an

Ainvestigation into the sums of the powers of the natural numbers, using matrices as its principal tool. This article continues the story, seeking explicit forms for the function $S$ of two natural number arguments, where

$$
S(n, r)=1^{r}+2^{r}+3^{r}+\ldots+n^{r} .
$$

Its principal tool is Calculus.
1 Formulae for $S(n, r)$ from Generating Functions. Consider the function

$$
f(x)=e^{x}+e^{2 x}+e^{3 x}+\ldots+e^{n x} .
$$

so that

$$
f(0)=e^{0}+e^{0}+\ldots+e^{0}=1+1+\ldots+1=n .
$$

But $f$ is differentiable and so we have

$$
\begin{aligned}
f^{\prime}(x) & =e^{x}+2 e^{2 x}+3 e^{3 x}+\ldots+n e^{n x} \\
\Rightarrow f^{\prime}(0) & =1+2.1+3.1+\ldots+n .1 \\
& =1+2+3+\ldots+n \\
& =S(n, 1)
\end{aligned}
$$

and if we differentiate again, we get

$$
\begin{aligned}
f^{\prime \prime}(x) & =e^{x}+2^{2} e^{2 x}+3^{2} e^{3 x}+\ldots+n^{2} e^{n x} \\
\Rightarrow \quad f^{\prime \prime}(0) & =1+2^{2}+3^{2}+\ldots+n^{2} \\
& =S(n, 2) .
\end{aligned}
$$

Continuing this process, we have
Lemma $\quad 1^{r}+2^{r}+3^{r}+\ldots+n^{r}=S(n, r)=f^{r}(0)$.

But there is another way of looking at $f$ :

$$
f(x)=e^{x}+e^{2 x}+e^{3 x}+\ldots+e^{n x}
$$

which is a geometric progression, and so

$$
f(x)=\frac{e^{(n+1) x}-1}{e^{x}-1}-1=\frac{e^{(n+1) x}-e^{x}}{e^{x}-1}
$$

So now we have our first result

## Theorem 1

$$
S(n, r)=1^{r}+2^{r}+3^{r}+\ldots+n^{r}=\operatorname{Lim}_{x \rightarrow 0} \frac{d^{r}}{d x^{r}}\left(\frac{e^{(n+1) x}-e^{x}}{e^{x}-1}\right) .
$$

This looks very impressive, but in this form it is not very practical. We have

$$
f^{\prime}(x)=\frac{\left(e^{x}-1\right)\left((n+1) e^{(n+1) x}-e^{x}\right)-\left(e^{(n+1) x}-e^{x}\right) e^{x}}{\left(e^{x}-1\right)^{2}}
$$

by the quotient rule, which simplifies to

$$
f^{\prime}(x)=\frac{n e^{(n+2) x}-(n+1) e^{(n+1) x}+e^{x}}{e^{2 x}-2 e^{x}+1}
$$

Now $f^{\prime}(0)$ is just $S(n, 1)$ but evaluating $f^{\prime}(0)$ poses a problem since the numerator and denominator both become zero. But L'Hôpital's rule is valid and if it is applied twice, we have:

$$
\begin{aligned}
f^{\prime}(0)=\operatorname{Lim}_{x \rightarrow 0} f^{\prime}(x) & =\operatorname{Lim}_{x \rightarrow 0} \frac{n(n+2) e^{(n+2) x}-(n+1)^{2} e^{(n+1) x}+e^{x}}{2 e^{2 x}-2 e^{x}} \\
& =\operatorname{Lim}_{x \rightarrow 0} \frac{n(n+2)^{2} e^{(n+2) x}-(n+1)^{3} e^{(n+1) x}+e^{x}}{4 e^{2 x}-2 e^{x}}
\end{aligned}
$$

$$
\text { i.e. } S(n, 1)=\frac{n(n+2)^{2}-(n+1)^{3}+1}{4-2}=\frac{n(n+1)}{2} \text {. }
$$

The prospect of differentiating and then evaluating the limit - with an increasing number of applications of L'Hôpital's Rule - is daunting. Can we automate the process? We can. We begin the process by writing $f$ in the form

$$
f(x)=\frac{e^{x}}{e^{x}-1} g(x) \quad \text { where } \quad g(x)=e^{n x}-1
$$

Then we have

$$
\begin{equation*}
e^{-x} f(x)=f(x)-g(x) . \tag{1}
\end{equation*}
$$

If we differentiate, then

$$
\begin{aligned}
-e^{-x} f(x)+e^{-x} f^{\prime}(x) & =f^{\prime}(x)-g^{\prime}(x) \\
e^{-x} f^{\prime}(x) & =f^{\prime}(x)-g^{\prime}(x)+e^{-x} f(x) \\
\Rightarrow \quad e^{-x} f^{\prime}(x) & =f^{\prime}(x)-g^{\prime}(x)+f(x)-g(x) \quad \text { using (1) again. }
\end{aligned}
$$

Now the trick. We denote the operation of differentiation not by $f^{\prime}, f^{\prime \prime}$ etc but by the operators

$$
D \equiv \frac{d}{d x}, \quad D^{2} \equiv D(D) \equiv \frac{d}{d x}\left(\frac{d}{d x}\right) \equiv \frac{d^{2}}{d x^{2}}, \quad \ldots \quad \text { etc. }
$$

So we can write this result in the form

$$
\begin{equation*}
e^{-x} D f(x)=(D+1)(f(x)-g(x)) \tag{2}
\end{equation*}
$$

Differentiating again

$$
\begin{aligned}
D\left(e^{-x} D f(x)\right) & =D(D+1)(f(x)-g(x)) \\
\Rightarrow \quad-e^{-x} D f(x)+e^{-x} D^{2} f(x) & =D(D+1)(f(x)-g(x)) \\
e^{-x} D^{2} f(x) & =D(D+1)(f(x)-g(x))+e^{-x} D f(x)
\end{aligned}
$$

and using (2) we now have

$$
\begin{aligned}
e^{-x} D^{2} f(x) & =D(D+1)(f(x)-g(x))(D+1)(f(x)-g(x)) \\
& =(D(D+1)+(D+1))(f(x)-g(x)) \\
& =(D+1)^{2}(f(x)-g(x)) .
\end{aligned}
$$

It is now an easy matter (by induction) to prove that

$$
e^{-x} D^{r} f(x)=(D+1)^{r}(f(x)-g(x))
$$

and rearranging

$$
\left((D+1)^{r}-e^{-x} D^{r}\right) f(x)=(D+1)^{r} g(x) .
$$

So when $x=0$ we have

$$
\left.\left((D+1)^{r}-e^{-x} D^{r}\right) f(x)\right|_{x=0}=\left.(D+1)^{r} g(x)\right|_{x=0} .
$$

But the function $g$ is easy to differentiate and evaluate at $x=0$ because

$$
\begin{array}{lll}
g(x)=e^{n x}-1 & \Rightarrow & g(0)=0 \\
g^{\prime}(x)=n e^{n x} & \Rightarrow & g^{\prime}(0)=n \\
g^{\prime \prime}(x)=n^{2} e^{n x} & \Rightarrow & g^{\prime \prime}(0)=n^{2}
\end{array}
$$

So we have

$$
\left.(D+1)^{r} g(x)\right|_{x=0}=(n+1)^{r}-1 \quad(\text { the }-1 \text { because } g(0)=0 \text { and not } 1)
$$

and then

| Theorem 2 | $\left.\left((D+1)^{\mathrm{r}}-D^{r}\right) f(x)\right\|_{x=0}=(n+1)^{r}-1$. |
| :--- | :--- |

This is the practical tool we need. We have already

$$
f(0)=n \text { and } f^{\prime}(0)=\frac{n(n+1)}{2} .
$$

Now consider Theorem 2 when $r=3$

$$
\begin{aligned}
\left.\left((D+1)^{3}-D^{3}\right) f(x)\right|_{x=0} & =(n+1)^{3}-1 \\
\left.\left(3 D^{2}+3 D+1\right) f(x)\right|_{x=0} & =(n+1)^{3}-1 \\
\left(3 f^{\prime \prime}(0)+3 f^{\prime}(0)+f(0)\right) & =(n+1)^{3}-1 \\
3 f^{\prime \prime}(0) & =(n+1)^{3}-1-3 f^{\prime}(0)-f(0) \\
& =n^{3}+3 n^{2}+3 n-\frac{3 n(n+1)}{2}-n \\
& =\frac{2 n^{3}+3 n^{2}+n}{2} \\
\Rightarrow \quad f^{\prime \prime}(0) & =\frac{n(n+1)(2 n+1)}{6}
\end{aligned}
$$

$$
\text { i.e. } \quad S(n, 2)=\frac{n(n+1)(2 n+1)}{6}
$$

We can continue the process and inductively we have

## Theorem $3 \quad S(n, r)$ is a polynomial in $n$ of degree $r+1$.

Exactly the same techniques lead to the related sums

$$
Q(n, r)=1^{r}-2^{r}+3^{r}-\ldots+(-1)^{n-1} n^{r} .
$$

Consider

$$
f(x)=e^{x}-e^{2 x}+e^{3 x}-\ldots+(-1)^{n-1} e^{n x} .
$$

Then we have, as before

$$
Q(n, 0)=f(0), \quad Q(n, 1)=f^{\prime}(0), \quad \ldots \text { etc } .
$$

But we have

$$
\begin{aligned}
Q(n, 0) & =1^{0}-2^{0}+3^{0}-\ldots(-1)^{n-1} n^{0}=1-1+1-\ldots(-1)^{n-1} \\
& =\left(\frac{(-1)^{n+1}+1}{2}\right) .
\end{aligned}
$$

Also, as before, we have a corresponding generating function $f(x)=\frac{e^{x}}{e^{x}+1}\left(1-(-1)^{n} e^{n x}\right)$ which we may write as $f(x)=\frac{e^{x}}{e^{x}+1} g(x)$ so that differentiating and putting $x=0$ we have

$$
\left.\left((D+1)^{r}+D^{r}\right) f(x)\right|_{x=0}=\left.(D+1)^{r} g(x)\right|_{x=0} .
$$

Again the function $g$ is easy to successively differentiate and evaluate at 0 :

$$
\begin{aligned}
g(x)=1-(-1)^{n} e^{n x} & \Rightarrow \quad g(0)=1-(-1)^{n}=1+(-1)^{n+1} \\
g^{\prime}(x)=-(-1)^{n} n e^{n x} & \Rightarrow \quad g^{\prime}(0)=n(-1)^{n+1} \\
g^{\prime \prime}(x)=-(-1)^{n} n^{2} e^{n x} & \Rightarrow \quad g^{\prime \prime}(0)=n^{2}(-1)^{n+1} . \\
\text { So }\left.\quad(D+1)^{r} g(x)\right|_{x=0} & =\left.D^{r} g(x)\right|_{x=0}+\left.{ }^{r} C_{1} D^{r-1} g(x)\right|_{x=0}+\ldots \\
& =(-1)^{n+1} n^{r}+C_{1}(-1)^{n+1} n^{r-1}+\ldots \\
& =(-1)^{n+1}(n+1)^{r}+1 .
\end{aligned}
$$

So when $r=1$

$$
\begin{array}{rlrl} 
& & \left.(2 D+1) f(x)\right|_{x=0} & =\left.(D+1) g(x)\right|_{x=0}=(-1)^{n+1}(n+1)+1 \\
\Rightarrow & 2 f^{\prime}(0)+f(0) & =(-1)^{n+1}(n+1)+1 \\
\Rightarrow & & f^{\prime}(0) & =(-1)^{n+1}(n+1)+1-\left(\frac{(-1)^{n+1}+1}{2}\right) .
\end{array}
$$

i.e. $\quad Q(n, 1)=\frac{(-1)^{n+1}(2 n+1)+1}{4}$.

When $r=2$

$$
\begin{aligned}
\left.\left(2 D^{2}+2 D+1\right) f(x)\right|_{x=0} & =\left.(D+1)^{2} g(x)\right|_{x=0}=(-1)^{n+1}(n+1)^{2}+1 \\
\Rightarrow 2 f^{\prime \prime}(0)+2 f^{\prime}(0)+f(0) & =(-1)^{n+1}(n+1)^{2}+1 \\
\Rightarrow \quad f^{\prime \prime}(0) & =(-1)^{n+1}(n+1)^{2}+1-\left(\frac{(-1)^{n+1}(2 n+1)+1}{4}\right)-\left(\frac{(-1)^{n+1}+1}{2}\right) .
\end{aligned}
$$

i.e. $\quad Q(n, 2)=\frac{(-1)^{n+1}\left(2 n^{2}+2 n\right)}{4}=\frac{(-1)^{n+1} n(n+1)}{2}$.

When $r=3$ it is easy to verify that

$$
Q(n, 3)=\frac{(-1)^{n+1}\left(4 n^{3}+6 n^{2}-1\right)-1}{8} .
$$

I will conclude this section with an elegant proof of the final result in the previous article. This was a set of identities, one for each natural number $n$ :

$$
\begin{aligned}
& -1.1^{0}+^{n} C_{1} 2^{0}-{ }^{n} C_{2} 3^{0}+\ldots+^{n} C_{n-1} n^{0}=0 \\
& -1.1^{1}+{ }^{n} C_{1} 2^{1}-{ }^{n} C_{2} 3^{1}+\ldots+{ }^{n} C_{n-1}^{n} 1^{1} \quad=0 \\
& -1.1^{2}+{ }^{n} C_{1} 2^{2}-{ }^{n} C_{2} 3^{2}+\ldots+{ }^{n} C_{n-1}^{n-1} n^{2}=0 \\
& -1.1^{n-2}+{ }^{n} C_{1} 2^{n-2}-^{n} C_{2} 3^{n-2}+\ldots+^{n} C_{n-1} n^{n-2}=0 .
\end{aligned}
$$

Now consider the generating function,

$$
f(x)=-e^{x}+{ }^{n} C_{1} e^{2 x}-{ }^{n} C_{2} e^{3 x}+\ldots++^{n} C_{n-1} e^{n x}
$$

By the same procedure

$$
\begin{array}{llrl} 
& & f^{\prime}(x) & =-1 e^{x}+{ }^{n} C_{1} 2 e^{2 x}-e^{n} C_{2} 3 e^{3 x}+\ldots++^{n} C_{n-1} n e^{n x} \\
\Rightarrow & f^{\prime}(0) & =-1.1+^{n} C_{1} 2 n^{n} C_{2} 3+\ldots+{ }^{n} C_{n-1} n \\
& & f^{\prime \prime}(x) & =-1 e^{x}++_{1}^{n} C_{1} 2^{2} e^{2 x}-C^{n} 3^{2} 3^{3 x}+\ldots+{ }^{n} C_{n-1} n^{2} e^{n x} \\
& f^{\prime \prime}(0) & =-1.1++^{n} C_{1} 2^{2}-{ }^{n} C_{2} 3^{2}+\ldots++^{n} C_{n-1} n^{2}
\end{array}
$$

and we can continue in this way to generate the left hand parts of the identities. All we now have to prove is that they take the zero value claimed on the right hand side. But

$$
\begin{aligned}
f(x) & =-1 e^{x}+{ }^{n} C_{1} e^{2 x}-{ }^{n} C_{2} e^{3 x}+\ldots+{ }^{n} C_{n-1} e^{n x} \\
& =-e^{x}\left(1-{ }^{n} C_{1} e^{x}-{ }^{n} C_{2} e^{2 x}+\ldots+{ }^{n} C_{n-1} e^{(n-1) x}\right) \\
& =-e^{x}\left(1-e^{x}\right)^{n-1}
\end{aligned}
$$

If we write this in the form

$$
f(x)=g(x)\left(1-e^{x}\right)^{n-1},
$$

then by differentiation we have

$$
\begin{aligned}
f^{\prime}(x) & =\left(1-e^{x}\right)^{n-1} g^{\prime}(x)+g(x)\left(-e^{x}\right)\left(1-e^{x}\right)^{n-2} \\
& =\left(1-e^{x}\right)^{n-2}\left(\left(1-e^{x}\right) g^{\prime}(x)-e^{x} g(x)\right) \\
& =h(x)\left(1-e^{x}\right)^{n-2} .
\end{aligned}
$$

So successive differentiations progressively reduce the power of the bracket until it reaches zero; moreover, when $x=0$ each of these brackets is zero as $\left(1-e^{0}\right)=1-1=0$, So

$$
\begin{array}{ll}
f(x)=g(x)\left(1-e^{x}\right)^{n} & \Rightarrow f(0)=0 \\
f^{\prime}(x)=h(x)\left(1-e^{x}\right)^{n-1} & \Rightarrow f^{\prime}(0)=0 \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
\end{array}
$$

which is precisely what we wanted for the right hand part of the identities.
2 General Properties of the Polynomial $S(n, r)$. We know from Theorem 3 that $S(n, r)$ is a polynomial, in $n$, of degree $r+1$. Consider a new polynomial $P_{r}(x)$ defined for all real numbers, with the property that,

$$
\begin{equation*}
P_{r}(x+1)=P_{r}(x)+(1+x)^{r} \text { and } P_{r}(1)=1 \tag{3}
\end{equation*}
$$

Then $S$ and $P$ are identical apart from their domains of definition. We aim now to derive results about $S$ from the properties of $P$. We can write $P_{r}(1)$ in two ways:

$$
P_{r}(1)=1
$$

and

$$
P_{r}(1)=P_{r}(0+1)=P_{r}(0)+(1+0)^{r} \text { by (3). }
$$

So

$$
1=P_{r}(0)+1
$$

$\Rightarrow \quad P_{r}(0)=0$.
Similarly, we can write $P_{r}(0)$ in two ways :
and

$$
P_{r}(0)=0
$$

So

$$
P_{r}(0)=P_{r}(-1+1)=P_{r}(-1)+(1+-1)^{r} \text { by }(3) .
$$

$\Rightarrow$

$$
P_{r}(-1)=0 .
$$

But each zero of a polynomial is linked to a factor of the polynomial, and as we have found two such zeros, we may deduce that

$$
P_{r}(x)=x(x+1) T_{r}(x)
$$

for some polynomial $T_{r}(x)$ of degree 2 less than $P_{r}(x)$. This gives us

| Theorem 4 | $S(n, r)$ has $n$ and $(n+1)$ as factors for all $r$. |
| :--- | :--- |
| Proof | We restrict $P_{r}(x)$ to the integers and the result follows. |

So by studying the real polynomial $P_{r}(x)$ we gain important information about the function $S(n, r)$ with natural number arguments. Continuing as before, we can write $P_{r}(-1)$ in two ways

$$
\begin{aligned}
& P_{r}(-1)=P_{r}(-2+1)=P_{r}(-2)+(1+-2)^{r} \\
& \text { so } \\
& 0=P_{r}(-2)+(-1)^{r} \\
& \Rightarrow \quad P_{r}(-2)=-(-1)^{r} \text {. }
\end{aligned}
$$

This descending process leads to the following result,

| Lemma | $P_{r}(x)=(-1)^{r-1} P_{r}(-x-1)$. |
| :---: | :---: |
| Proof | $\begin{aligned} & P_{r}(x)=P_{r}(x-1)+(1+(x-1))^{r} \quad \text { by }(1) \\ &=P_{r}(x-1)+x^{r} \\ &=P_{r}(x-2)+(1+(x-2))^{r}+x^{r} \quad \text { again by }(3) \\ &=P_{r}(x-2)+(x-1)^{r}+x^{r} \\ & \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \end{aligned}$ <br> When $r$ is odd we have: $P_{r}(x)=P_{r}(-x-1)-(x)^{r}-(x-1)^{r}+\ldots+(x-1)^{r}+x^{r}$ <br> and as $P_{r}(0)=0$ we have $P_{r}^{r}(x)=P_{r}(-x-1) .$ <br> When $r$ is even we have : $\begin{aligned} P_{r}(x) & =P_{r}(-x-1)+(x)^{r}+(x-1)^{r}+\ldots+(x-1)^{r}+x^{r} \\ & =P_{r}(-x-1)+2\left((x)^{r}+(x-1)^{r}+\ldots+0\right) \\ & =P_{r}(-x-1)+2 P_{r}(x) \\ \Rightarrow P_{r}(x) & =-P_{r}(-x-1) . \end{aligned}$ <br> Combining these two results we now have $P_{r}(x)=(-1)^{r-1} P_{r}(-x-1)$ as required . |


| Theorem 5 | If $r$ is even then $S(n, r)$ has $n, n+1$ and $2 n+1$ as factors. |
| :---: | :---: |
| Proof | We have already established the $n$ and ( $n+1$ ) parts; to prove the remaining part we use the lemma with $r$ even, so that $P_{r}(x)=-P_{r}(-x-1) .$ <br> When $x=-\frac{1}{2}$ we then have $\begin{aligned} P_{r}\left(-\frac{1}{2}\right) & =-P_{r}\left(\frac{1}{2}-1\right)=-P_{r}\left(-\frac{1}{2}\right) \\ \Rightarrow \quad 2 P_{r}\left(-\frac{1}{2}\right) & =0 \\ \Rightarrow \quad P_{r}\left(-\frac{1}{2}\right) & =0 . \end{aligned}$ <br> So $P_{r}(x)$ has $(2 x+1)$ as a factor: so that $S(n, r)$ has $(2 n+1)$ as a factor as required. |

Now $P_{r}(x)$ is a real polynomial and so differentiable. Can we use this? Recall that if a polynomial $f(x)$ is such that $f^{\prime}(x)$ has $(x-a)$ as a factor then $f(x)$ has $(x-a)^{2}$ as a factor. This leads us to differentiate the result of the lemma, which gives

$$
\begin{aligned}
P_{r}(x) & =(-1)^{r-1} P_{r_{r}}(-x-1) \\
\Rightarrow \quad P_{r}^{\prime}(x) & =(-1)^{r-1} P_{r}^{\prime}(-x-1)(-1) \text { by the chain rule }
\end{aligned}
$$

(Note that we regard $r$ as fixed - we are considering the sums of a particular power $r$.)

So

$$
\begin{aligned}
P_{r}^{\prime}(x) & =(-1)^{r} P_{r}^{\prime}(-x-1) \text { and when } x=0 \text { this yields } \\
P_{r}^{\prime}(0) & =(-1)^{r} P_{r}^{\prime}(-1) .
\end{aligned}
$$

But we can also differentiate (3) to get

$$
\begin{aligned}
P_{r}^{\prime}(x+1) & =P_{r}^{\prime}(x)+r(1+x)^{r-1} \text { and when } x=-1 \text { this gives } \\
P_{r}^{\prime}(0) & =P_{r}^{\prime}(-1) .
\end{aligned}
$$

Putting these results together we have

$$
P_{r}^{\prime}(-1)=(-1)^{r} P_{r}^{\prime}(-1) \quad \text { and } \quad P_{r}^{\prime}(0)=(-1)^{r} P_{r}^{\prime}(0)
$$

So when $r$ is odd:

$$
\begin{array}{lll}
P_{r_{r}^{\prime}}^{\prime}(-1)=-P_{r_{1}}^{\prime}(-1) & \Rightarrow & P_{r_{r}^{\prime}}^{\prime}(-1)=0 \\
P_{r}^{\prime}(0)=-P_{r}^{\prime}(0) & \Rightarrow & P_{r}^{\prime}(0)=0 .
\end{array}
$$

| Theorem 6 | If $r$ is odd and $r>2$ then $S(n, r)$ has as factors <br> $n^{2}$ and $(n+1)^{2}$. |
| :--- | :--- |
| Proof | $\mathrm{P}_{\mathrm{r}}^{\prime}(x)$ has 0 and -1 as zeros; so <br> $P_{r}(x)$ has $x^{2}$ and $(x+1)^{2}$ as factors as required. |

There is one more critical result obtained by differentiating (3):

$$
\begin{aligned}
P_{r}(x+1)-P_{r}(x) & =(x+1)^{r} \\
\Rightarrow \quad P_{r}^{\prime}(x+1)-P_{r}^{\prime}(x) & =r(x+1)^{r-1} .
\end{aligned}
$$

If we now replace $r$ by $r-1$ in (1) we get

$$
\begin{aligned}
& P_{r-1}(x+1)-P_{r-1}(x)=(x+1)^{r-1} \\
\Rightarrow \quad & r P_{r}(x+1)-r P_{r}(x)=r(x+1)^{r-1}
\end{aligned}
$$

and subtracting these two results, we have

$$
\begin{align*}
& P_{r}^{\prime}(x+1)-P_{r}^{\prime}(x)=r P_{r-1}(x+1)-r P_{r-1}(x) \\
\Rightarrow \quad & P_{r}^{\prime}(x+1)-r P_{r-1}(x+1)=P_{r}^{\prime}(x)-r P_{r-1}(x) \tag{4}
\end{align*}
$$

It's difficult at first sight to understand the significance of this equality; to simplify matters we define

$$
Z_{r}(x)=P_{r}^{\prime}(x+1)-r P_{r-1}(x+1)
$$

Then (4) becomes

$$
Z_{r}(x)=Z_{r}(x-1) .
$$

Now as $P_{r}$ is a polynomial in $x$ (of degree $r$ ) then $Z_{r}$ is also a polynomial of finite degree. But, and this is the important bit, it is also periodic. There is only one possibility.

| Lemma | $Z_{r}$ is constant |
| :--- | :--- |


| Proof | $Z_{r}(x)=Z_{r}(x-1)$ <br> As $Z_{r}$ is a polynomial it is differentiable and so <br> and again, $\begin{gathered} Z_{r}^{\prime}(x)=Z_{r}^{\prime}(x-1) \\ Z_{r}^{\prime \prime}(x)=Z_{r}^{\prime \prime}(x-1) \end{gathered}$ <br> and ultimately, as $Z_{r}$ has finite degree, <br> So <br> and then <br> But as $\begin{gathered} Z_{r}^{s}(x)=Z_{r}^{s}(x-1)=0 \\ Z_{r}^{s-1}(x)=Z_{r}^{s-1}(x-1)=c \quad \text { a constant } \\ Z_{r}^{s-2}(x)=Z_{r}^{s-2}(x-1)=c x+k . \\ Z_{r}(x)=Z_{r}(x-1) \text { this means } \end{gathered}$ $c x+k=c(x-1)+k$ <br> and so <br> $Z_{r}^{S-2}(x)=Z_{r}^{s-2}(x-1)=k$ a constant again. The same argument may be repeated in ascending order and so <br> $Z_{r}(x)=Z_{r}(x-1)=k$ as required. |
| :---: | :---: |

So from (4) we have $P_{r+1}^{\prime}(x)-(r+1) P_{r}(x)=k$ (a constant) and integration gives

$$
P_{r+1}(x)=\int\left((r+1) P_{r}(x)+k\right) d x
$$

so that, going from $P_{r+1}(x)$ to $S(n+1, r)$ and $P_{r}(x)$ to $S(n, r)$ in the usual way, we have

| Theorem 7 | $S(n, r+1)=\int\left((r+1) S(n, r)+k_{r+1}\right) d n$. |
| :--- | :--- |

This is a symbolic result: a function in $n$ cannot be integrated as $n$ takes only integer values not continuous real values; nonetheless the formula for $S(n, r)$ behaves as if it can be integrated in the natural way. It is a powerful tool.

We illustrate its use by extending the results we already have.

$$
\begin{aligned}
S(n, 3) & =\int\left(3 S(n, 2)+k_{3}\right) d n=\int\left(3\left(\frac{n^{3}}{3}+\frac{n^{2}}{2}+\frac{n}{6}\right)+k_{3}\right) d n \\
& =\frac{n^{4}}{4}+\frac{n^{3}}{2}+\frac{n^{2}}{4}+k_{3} n+C .
\end{aligned}
$$

However, $C=0$ as $S(n, r)$ always has $n$ as a factor - by Theorem 4. (This constant of the integration process will therefore always be zero.)

Also $\quad S(1,3)=1^{3}=1 \Rightarrow 1=\frac{1}{4}+\frac{1}{2}+\frac{1}{4}+k_{3} \Rightarrow k_{3}=0$ and so

$$
S(n, 3)=\frac{n^{4}}{4}+\frac{n^{3}}{2}+\frac{n^{2}}{4}=\frac{n^{2}(n+1)^{2}}{4} .
$$

Similarly

$$
\begin{aligned}
& S(n, 4)=\int\left(4 S(n, 3)+k_{4}\right) d n=\int\left(n^{4}+2 n^{3}+n^{2}+k_{4}\right) d n \\
& S(n, 4)=\frac{n^{5}}{5}+\frac{2 n^{4}}{4}+\frac{n^{3}}{3}-\frac{n}{30}=\frac{n(n+1)(2 n+1)\left(3 n^{2}+3 n-1\right)}{30}
\end{aligned}
$$

3 The Explicit Form for $S(n, r)$. In order to determine the explicit form of the function $S(n, r)$ we look at the results we have already and the tools that fashioned them. The results so far are:

| $S(n, r)$ | $n$ | $n^{2}$ | $n^{3}$ | $n^{4}$ | $n^{5}$ | $n^{6}$ | $\ldots$ |
| :--- | :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $r=0$ | $1 / 2$ | $1 / 2$ | 0 | 0 | 0 | 0 | $\ldots$ |
| $r=1$ | $1 / 6$ | $1 / 2$ | $1 / 3$ | 0 | 0 | 0 | $\ldots$ |
| $r=2$ | 0 | $1 / 4$ | $1 / 2$ | $1 / 4$ | 0 | 0 | $\ldots$ |
| $r=3$ | $-1 / 30$ | 0 | $1 / 3$ | $1 / 2$ | $1 / 5$ | 0 | $\ldots$ |
| $r=4$ | $\ldots$ | $\ldots . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . ~$ |  |  |  |  |  |

How does the table continue? Simple. Each entry generates the next entry on the downward diagonal by the integration technique of Theorem 7. This leads to a recurrence relation for the table. It works like this. Suppose that $k_{r}^{s}$ is the entry in the $r$ th row and $s$ th column. Then we have

$$
S(n, r+1)=\int\left((r+1) S(n, r)+k_{r+1}\right) d n
$$

and by comparing the corresponding $(s+1)$ th powers of $n$, we have

$$
\ldots+k_{r+1}^{s+1} n^{s+1}+\ldots=\int\left(\ldots(r+1) k_{r}^{s} n^{s}+\ldots\right) d n .
$$

But we can integrate this power of $n$ on the right hand side and if we do we obtain the following recurrence relation between consecutive diagonal entries in the table,

| Theorem 8 | $k_{r+1}^{s+1}=\frac{(r+1)}{(s+1)} k_{r}^{s}$. |
| :--- | :--- |

Notice that the 'constant of integration' $k_{r}$ that occurred in Theorem 7 is now called $k_{r}^{1}$ - it is the $r$ th entry in the first column, or alternatlitively, the first entry in the $r$ th row. Also, from Theorem 6, we know that $k_{2 r+1}^{1}=0$ (i.e. the odd entries in the first column after the first). This means that the diagonals generated (by the recurrence relation) from these starting values are all zero the zeros march onwards and downwards for ever. There are many other intriguing patterns to the diagonals.

The first diagonal is the harmonic series $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. We prove it inductively:
$\mathrm{k}_{0}^{1}=1 \Rightarrow k_{1}^{2}=\frac{1}{2} \cdot 1=\frac{1}{2}$ by the recurrence relation
$\Rightarrow \quad k_{2}^{3}=\frac{2}{3} \cdot \frac{1}{2}=\frac{1}{3}$

$$
\Rightarrow \quad k_{r}^{r+1}=\frac{r}{r+1} k_{r-1}^{r}=\frac{r}{r+1} \cdot \frac{1}{(r-1)+1}=\frac{1}{r+1} \text { as required. }
$$

The second diagonal is the constant series $\frac{1}{2}, \frac{1}{2}, \quad \frac{1}{2}, \quad \frac{1}{2}, \ldots$

$$
\begin{aligned}
k_{1}^{1}=1 / 2 & \Rightarrow k_{2}^{2}=\frac{2}{2} \cdot 1 / 2=1 / 2 \\
& \Rightarrow k_{3}^{3}=\frac{3}{3} \cdot 1 / 2=1 / 2 \\
& \Rightarrow k_{r}^{r}=1 / 2
\end{aligned}
$$

Every diagonal is a sequence of interest. To see this, take the row number $r$ and the denominator of the first entry in the row $r$ as reciprocal factors of each entry in the diagonal starting with the first entry in row $r$. So when $r=4$ and the first entry is $-1 / 30$ we obtain as the entries for this diagonal

$$
-\frac{1}{30}, \frac{5}{2}\left(-\frac{1}{30}\right), \frac{6}{3} \cdot \frac{5}{2}\left(-\frac{1}{30}\right), \ldots=-\frac{1}{30.4}(4,10,20,35,56, \ldots)
$$

which are the tetrahedral numbers. You can prove that they are such by using the recurrence relation. The pattern continues; the next non-zero diagonal contains the fifth order figurative numbers.

Every entry on a line - except the first - is the direct, diagonal, recursive result of an initial entry in the first column :
$k_{r}^{1}$

$$
\begin{aligned}
& k_{r+1}^{2}=\frac{(r+1)}{2} \cdot k_{r}^{1} \\
& k_{r+2}^{3}=\frac{(r+2)(r+1)}{3.2} \cdot k_{r}^{1}
\end{aligned}
$$

$$
k_{r+s}^{s+1}=\frac{(r+s)(r+s-1) \ldots(r+1)}{s(s-1) \ldots 3.2} \cdot k_{r}^{1}
$$

So by tracking forwards in this way, all the entries in the table may be produced. Equally, each entry of a line, except the first, may be written as the product of a polynomial in $r$ with a constant from the first column. Putting all this together gives

## Theorem 9

The polynomial $S(n, r)$ has the form
$=k_{r}^{1} \cdot n+\frac{r}{2} \cdot k_{r-1}^{1} \cdot n^{2}+\frac{r(r-1)}{3.2} \cdot k_{r-2}^{1} \cdot n^{3}+\ldots+\frac{r(r-1) \ldots 1}{(r+1) r(r-1) \ldots 2} \cdot k_{0}^{1} \cdot n^{r+1}$
$=k_{r}^{1} \cdot n+\frac{r}{2} \cdot k_{r-1}^{1} \cdot n^{2}+\frac{r(r-1)}{3.2} \cdot k_{r-2}^{1} \cdot n^{3}+\ldots+\frac{1}{2} \cdot k_{1}^{1} \cdot n^{r}+\frac{1}{r+1} \cdot k_{0}^{1} \cdot n^{r+1}$
and this means that $S(n, r)$ is completely determined by the numbers $k_{0}^{1}, k_{1}^{1}, k_{2}^{1}, \ldots k_{r}^{1}$ that appear in the first column. Our remaining task is to automate the determination of these numbers and the formula itself. First we define an operator denoted by $B$ (after Bernoulli) that transforms $k_{r-1}^{1}$ into $k_{r}^{1}$ so that $B\left(k_{r-1}^{1}\right)=k_{r}^{1}$. Then we have

$$
k_{r}^{1}=B\left(k_{r-1}^{1}\right)=B^{2}\left(k_{r-2}^{1}\right)=\ldots=B^{r}\left(k_{0}^{1}\right) .
$$

So each $k_{r}^{1}$ may be written in terms of the first such entry. Now we write Theorem 9 in the form

$$
S(n, r)=B^{r}\left(k_{0}^{1}\right) n+\frac{r}{2} B^{r-1}\left(k_{0}^{1}\right) n^{2}+\frac{r(r-1)}{3.2} B^{r-2}\left(k_{0}^{1}\right) n^{3}+\ldots+\frac{1}{r+1} B\left(k_{0}^{1}\right)
$$

and if we multiply through by $(r+1)$ this gives a result that resembles a binomial sum, which by addition and subtraction of a new term is easily summed:

$$
\begin{aligned}
& =(r+1) B^{r}\left(k_{0}^{1}\right) \cdot n+B^{r-1}\left(k_{0}^{1}\right) \cdot \frac{(r+1) \cdot r}{2} \cdot n^{2}+B^{r-2}\left(k_{0}^{1}\right) \cdot \frac{(r+1) \cdot r(r-1)}{3 \cdot 2} \cdot n^{3}+\ldots \\
& =B^{r+1}\left(k_{0}^{1}\right)+(r+1) B^{r}\left(k_{0}^{1}\right) \cdot n+\ldots B^{r-2}\left(k_{0}^{1}\right) \cdot \frac{(r+1) \cdot r(r-1)}{3 \cdot 2} \cdot n^{3}+\ldots-B^{r+1}\left(k_{0}^{1}\right) .
\end{aligned}
$$

So now we have

| Theorem 10 | $S(n, r)=\frac{\left((B+n)^{r+1}-B^{r+1}\right)\left(k_{0}^{1}\right)}{r+1}$. |
| :--- | :--- |

That automates the formula for $S(n, r)$. How do we find the entries in the first column? These are just the 'constants of integration'. We calculated them recursively by applying the fact that $S(1, r)=1^{r}$ and putting $n=1$ in the formula for $S(n, r)$ :

$$
k_{r}^{1} \cdot 1+k_{r}^{2} \cdot 1^{2}+k_{r}^{3} \cdot 1^{3}+\ldots+k_{r}^{r+1} \cdot 1^{r+1}=1^{r}=1
$$

knowing $k_{r}^{2}, k_{r}^{3}, k_{r}^{4}, \ldots$ meant that we could calculate the next entry $k_{r}^{1}$.
There's another result that automates the determination of these numbers as well. If we substitute $n=1$ into Theorem 10 we have $S(1, r)=1^{r}=1$ and so

## Theorem 11 <br> $$
r+1=\left((B+1)^{r+1}-B^{r+1}\right)\left(k_{0}^{1}\right) .
$$

This is a more convenient way of calculating successive $k_{r}^{1}$ 's. For example, when $r=1$

$$
\begin{array}{rlrl} 
& & 2 & =\left((B+1)^{2}-B^{2}\right)\left(k_{0}^{1}\right) \\
\Rightarrow & 2 & =B^{2}\left(k_{0}^{1}\right)+2 B\left(k_{0}^{1}\right)+k_{0}^{1}-B^{2}\left(k_{0}^{1}\right) \\
\Rightarrow & 2 & =k_{2}^{1}+2 k_{1}^{1}+1-k_{2}^{1} \\
\Rightarrow & k_{1}^{1} & =1 / 2
\end{array}=2 k_{1}^{1}+1
$$

$r=2$

$$
\begin{array}{rlrl} 
& & 3 & =\left((B+1)^{3}-B^{3}\right)\left(k_{0}^{1}\right) \\
\Rightarrow & 3 & =B^{3}\left(k_{0}^{1}\right)+3 B^{2}\left(k_{0}^{1}\right)+3 B\left(k_{0}^{1}\right)+k_{0}^{1}-B^{3}\left(k_{0}^{1}\right) \\
\Rightarrow & & 3 & =3 k_{2}^{1}+3 k_{1}^{1}+1 \\
\Rightarrow & k_{2}^{1} & =\frac{2-3 \cdot 1 / 2}{3}=1 / 6 .
\end{array}
$$

We already have the values for $r=3$ ( $k_{3}^{1}=0$ : as $r$ is odd, Theorem 6 says that $S(n, r)$ has zero coefficient of $n) ; r=4\left(k_{4}^{1}=-1 / 30\right)$ and $r=5\left(k_{5}^{1}=0\right.$, again $r$ is odd). We find that for $r=6,7,8,9$ and $10, k_{6}^{1}=1 / 42, k_{7}^{1}=0$ ( $r$ is odd ), $k_{8}^{1}=-1 / 30, k_{9}^{1}=0, k_{10}^{1}=5 / 66$. So we have :

| $S(n, r)$ | $n$ | $n^{2}$ | $n^{3}$ | $n^{4}$ | $n^{5}$ | $n^{6}$ | $n^{7}$ | $n^{8}$ | $n^{9}$ | $\ldots$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $r=0$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\ldots$ |
| $r=1$ | $1 / 2$ | $1 / 2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\ldots$ |
| $r=2$ | $1 / 6$ | $1 / 2$ | $1 / 3$ | 0 | 0 | 0 | 0 | 0 | 0 | $\ldots$ |
| $r=3$ | 0 | $1 / 4$ | $1 / 2$ | $1 / 4$ | 0 | 0 | 0 | 0 | 0 | $\ldots$ |
| $r=4$ | $-1 / 30$ | 0 | $1 / 3$ | $1 / 2$ | $1 / 5$ | 0 | 0 | 0 | 0 | $\ldots$ |
| $r=5$ | 0 | $-1 / 12$ | 0 | $5 / 12$ | $1 / 2$ | $1 / 6$ | 0 | 0 | 0 | $\ldots$ |
| $r=6$ | $1 / 42$ | 0 | $-1 / 6$ | 0 | $1 / 2$ | $1 / 2$ | $1 / 7$ | 0 | 0 | $\ldots$ |
| $r=7$ | 0 | $1 / 12$ | 0 | $-7 / 24$ | 0 | $1 / 12$ | $1 / 2$ | $1 / 8$ | 0 | $\ldots$ |
| $r=8$ | $-1 / 30$ | 0 | $2 / 9$ | 0 | $-7 / 15$ | 0 | $2 / 3$ | $1 / 2$ | $1 / 9$ | $\ldots$ |
| $r=9$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

We conclude with a generating function for the $k_{r}^{1}$ 's. They are generated by the relation

$$
r+1=\left((B+1)^{r+1}-B^{r+1}\right)\left(k_{0}^{1}\right)
$$

This suggests a search for a function that obeys a corresponding recursive relation. We have met functions that satisfy such relations already in the guise of the differential operator $D$ rather than $B$. Consider, then, the function

$$
f(x)=\frac{x}{1-e^{-x}} \Rightarrow e^{-x} f(x)=-x+f(x)
$$

Differentiating and rearranging as before, we find that

$$
e^{-x} D f(x)=-1-x+(D+1) f(x)
$$

By differentiating and generalizing this result when $x$ takes the zero value, in the usual way, we have

$$
\left.\left((D+1)^{r+1}-D^{r+1}\right) f(x)\right|_{x=0}=r+1
$$

which is exactly the recurrence relation satisfied by the $k_{r}^{1}$ 's. To conclude then, we have

Theorem 12

$$
k_{r}^{1}=\left.D^{r}\left(\frac{x}{1-e^{-x}}\right)\right|_{x=0}
$$

## Cube

## Patrick Lee

This delightful problem was set as one of the 'teacher pacifiers' during the 1998 season of the Royal Institution Bath and Bristol Mathematics Masterclasses. It is submitted with the permission of Dr Geoff Smith, the moderator of the RIBBMM who tells me that it was brought to his attention by Prof. Dave Johnson of the University of the West Indies.

Rotate a cube about one of its long diagonals. The shape you get looks like a munched apple with pointy ends. Calculate its volume and compare that with the volume of the ball which circumscribes the original cube. If you get the answer correct, a chill should run down your spine.

You can find this as well as the other problems that were set at http://www.bath.ac.uk/RIBBM/tough_probs.html.
My solution follows and I should be interested to know of any other (better?) ones.

This is trickier than the usual solid of revolution problem because the boundary that sweeps out the surface of the solid is not all in one plane.

The cube $O A B C D E F G$ (assumed to have side 1), is shown in Figure 1, with point $O$ at the origin and sides $O A$ and $O E$ aligned along the $x$-axis and $y$-axis respectively. $O C$ is the major diagonal about which the cube is to be rotated. For each point on the axis of rotation, we need to find the radius of the circle swept out by the point farthest from the axis on the surface of the cube intersected by a plane passing through that point and at right angles to the axis.

Let $T$ be a point on $O C$, characterized by a parameter $t$, such that the coordinates of $T$ are $(t, t, t)$, where $0<t<1$. Let $P$ be any other point, and consider the vectors $O T$ and $T P$. Now, since $O$ is the origin, the components of $O T$ are: $(t, t, t)$ and the components of $T P$ are $(x-t, y-t, z-t)$.

As the vectors are required to be orthogonal to each other, it is necessary that their dot product be equal to zero, i.e. $O T \cdot T P=t(x-t)+t(y-t)+$ $t(z-t)=0$, hence $t(x+y+z)-3 t^{2}=0$, i.e.

$$
\begin{equation*}
t=\frac{x+y+z}{3} \tag{1}
\end{equation*}
$$

Also the lengths of these vectors satisfy, respectively: $\|O T\|^{2}=3 t^{2}$ and

$$
\begin{equation*}
\|T P\|^{2}=(x-t)^{2}+(y-t)^{2}+(z-t)^{2} \tag{2}
\end{equation*}
$$

Write

$$
\begin{equation*}
l=\|O T\|=\sqrt{3} t \tag{3}
\end{equation*}
$$

and $r=\|T P\|$.
Now consider point $P$ to be located on the contour $O A D C$ so that $r$ is the radius of the solid of rotation at a distance $l$ from $O$ along its axis $O C$.

The value of $r$ as a function of $l$ (and hence the half-profile of the solid of revolution) was calculated by the above formulae, in a spreadsheet, and plotted in Figure 2.

Note that since the coordinates of points $A, E, G$ are $(1,0,0),(0,1,0)$ and $(0,0,1)$ respectively, by (1), they correspond to the same value of $t$ equal to $1 / 3$. Similarly points $B, D, F$ all correspond to a value of $t$ equal to $2 / 3$. Thus contours $O A D C, O A B C, O G F C, O G B C, O E F C$ and $O E D C$ are equivalent to each other. The same result can be arrived at by invoking symmetry.

Let point $P$ start at $O$ and move along contour $O A D C$. When it is at distance $x$ from $O$, we have:

$$
\begin{aligned}
& \text { By (1), } \quad t=\frac{x}{3} . \\
& \text { By (3), } \quad l=\sqrt{3} \frac{x}{3}=\frac{x}{\sqrt{3}} . \\
& \text { By (2), } \quad r^{2}=\left(x-\frac{x}{3}\right)^{2}+\left(\frac{-x}{3}\right)^{2}+\left(\frac{-x}{3}\right)^{2}=\frac{2}{3} x^{2} .
\end{aligned}
$$

The volume $\Delta V$ of the elementary disc with centre at $l$ and thickness $\Delta l$ generated by rotation of the cube is given by: $\Delta V=\pi r^{2} \Delta l$, where, from (3),

$$
\Delta l=\sqrt{3} \Delta t=\frac{\Delta x}{\sqrt{3}} .
$$

Hence

$$
\Delta V=\frac{2}{3} \pi x^{2} \frac{\Delta x}{\sqrt{3}}=\frac{2}{3 \sqrt{3}} \pi x^{2} \Delta x
$$

Thus the volume of the solid generated by the rotation of $O A$ is given by

$$
V_{O A}=\frac{2 \pi}{3 \sqrt{3}} \int_{0}^{1} x^{2} d x=\frac{2 \pi}{9 \sqrt{3}} .
$$

When $P$ is on $A D$, as shown in Figure 1, with coordinates $(1, y, 0)$, then, by
(1), the value of $t$ is $\frac{1+y}{3}$ and $l=\sqrt{3} \frac{1+y}{3}=\frac{1+y}{\sqrt{3}}$. By (2),

$$
\begin{aligned}
r^{2} & =\left(1-\frac{1+y}{3}\right)^{2}+\left(y-\frac{1+y}{3}\right)^{2}+\left(-\frac{1+y}{3}\right)^{2} \\
& =\left(\frac{2-y}{3}\right)^{2}+\left(\frac{2 y-1}{3}\right)^{2}+\left(\frac{1+y}{3}\right)^{2} \\
& =\frac{1}{9}\left(4-4 y+y^{2}+4 y^{2}-4 y+1+1+2 y+y^{2}\right) \\
& =\frac{1}{9}\left(6-6 y+6 y^{2}\right)=\frac{2}{3}\left(1-y+y^{2}\right) .
\end{aligned}
$$

By (1), $t=(1+y) / 3$, hence $\Delta t=\Delta y / 3$. The volume $\Delta V$ of the elementary disc with centre at $l$ and thickness $\Delta l$ generated by rotation of the cube is given by $\Delta V=\pi r^{2} \Delta l$ where, from (3), $\Delta l=\sqrt{3} \Delta t=\Delta y / \sqrt{3}$. Thus, volume $V_{A D}$ of solid generated by rotation of $A D$ is given by

$$
\begin{aligned}
V_{A D} & =\frac{2 \pi}{3} \int_{0}^{1}\left(1-y+y^{2}\right) \frac{d y}{\sqrt{3}} \\
& =\frac{2 \pi}{3 \sqrt{3}}\left[y-\frac{y^{2}}{2}+\frac{y^{3}}{3}\right]_{0}^{1} \\
& =\frac{2 \pi}{3 \sqrt{3}}\left(1-\frac{1}{2}+\frac{1}{3}\right)=\frac{5 \pi}{9 \sqrt{3}} .
\end{aligned}
$$

When $P$ is on $D C$ with coordinates $(1,1, z)$ then, by (1), the value of $t$ is $(2+z) / 3$, i.e.

$$
l=\sqrt{3} \frac{2+z}{3}=\frac{2+z}{\sqrt{3}} .
$$

By (2),

$$
\begin{aligned}
r^{2} & =\left(1-\frac{2+z}{3}\right)^{2}+\left(1-\frac{2+z}{3}\right)^{2}+\left(z-\frac{2+z}{3}\right)^{2} \\
& =2\left(\frac{1}{3}-\frac{z}{3}\right)^{2}+\left(\frac{2}{3} z-\frac{2}{3}\right)^{2} \\
& =\frac{2}{9}(1-z)^{2}+\frac{4}{9}(z-1)^{2}=\frac{2}{3}\left(z^{2}-2 z+1\right)
\end{aligned}
$$

The volume $\Delta V$ of the elementary disc with centre at 1 and thickness $\Delta l$ generated by rotation of the cube is given by $\Delta V=\pi r^{2} \Delta l$ where, from (3),
$\Delta l=\sqrt{3} \Delta t=\Delta z / \sqrt{3}$. Hence the volume $V_{D C}$ of the solid generated by rotation of $D C$ is given by

$$
\begin{aligned}
V_{D C} & =\frac{2 \pi}{3 \sqrt{3}} \int_{0}^{1}\left(z^{2}-2 z+1\right) \frac{d z}{\sqrt{3}} \\
& =\frac{2 \pi}{3 \sqrt{3}}\left[\frac{z^{3}}{3}-z^{2}+z\right]_{0}^{1}=\frac{2 \pi}{3 \sqrt{3}}\left(\frac{1}{3}-1+1\right)=\frac{2 \pi}{9 \sqrt{3}} .
\end{aligned}
$$

This is the same as $V_{O A}$, as would be expected by considerations of symmetry. Thus

$$
V_{\text {SOLID }}=2 \frac{2 \pi}{9 \sqrt{3}}+\frac{5 \pi}{9 \sqrt{3}}=\frac{\pi}{\sqrt{3}} .
$$

The radius of the circumscribing sphere is $\sqrt{3} / 2$; therefore its volume is $4 / 3 \pi(\sqrt{3} / 2)^{3}=\pi \sqrt{3} / 2$. Hence

$$
\frac{V_{\text {SOLID }}}{V_{\text {SPHERE }}}=\frac{\pi}{\sqrt{3}} \frac{2}{\pi \sqrt{3}}=\frac{2}{3} .
$$



Figure 1


Figure 2
Radius of the solid generated by rotating a cube of side 1 about a major diagonal

As a post-script to the cube problem, it is interesting to note that the ratio of the volume of a sphere to that of the smallest cylinder that can contain it is also $2 / 3$.

## Magic squares

## John Halsall

A recent contribution of Barbara Lee [M500 164 1] illustrated the means for constructing 'magic squares' of various sizes with various properties.

Whilst investigating the mysteries of matrices I began to speculate that perhaps magic squares could be multiplied to produce larger squares of equal magicity, and so it proved.

As indicated in the diagram, a $3 \times 3$ square can be combined with another $3 \times 3$ square to produce a $9 \times 9$ square. A $3 \times 3$ square can be combined with a $4 \times 4$ square to produce a $12 \times 12$ square. We might then combine a $12 \times 12$ square with a $9 \times 9$ square to produce a $108 \times 108$ square, but the mind has already begun to boggle.


When we have exhausted the possibilities of the magic squares we can amuse ourselves by looking at other figures.

Each line in this magic star totals 26 . But so also do the four numbers at the apices of the great rhombuses, and the sets of five numbers centred on the points of the star.


Rhombuses: $8+3+5+10,9+4+11+2$, etc.
Star vertices: $12_{+3+6}^{+4+1}, \mathrm{Y}_{+1}^{+1+4}+2$, etc.

JRH-I was trying to find my desk. One of the things I thought it was under was an insurance mailshot from Lloyds, which said:
'Fact-In 1997, one in every six people in the UK are likely to spend time in hospital ... (Source: Department of Health).'

Are likely? What do they mean?

## Grazing oxen

## Stephen Sparrow

It is not necessary to be a mathematician to spot the flaw in John Bull's model [M500 165 1], any agriculture or veterinary student should be able to do so.

The problem lies with the application of the limits. We can ignore the trivial objection that if one were to try to put an infinite number of cattle onto ten acres there wouldn't be enough room for them to get their necks down to graze, since this can be overcome by employing an infinitely efficient herdsman to cut and carry the grass!

Much more serious is the concept of infinite time. This merely represents a sustainable resource and this is where John's model introduces the error. The Newton model is not breaking down below 0.9 oxen per acre; it merely shows that this is the number required to keep the grass in a steady state. MST204 students will readily recognize the model as similar to that of sustainable fish stocks in the North Sea; merely replace fish by grass and fisherman by oxen.

John's doubts about the reality of the model are also a little wide of the mark. The model has been used for many years. When I was a veterinary student in the ' 60 s , a very sound rule of thumb, given to us by our animal husbandry lecturers, was a stocking density of 1 cow per acre for dairy farms. This is very close to Newton's model, which shows he knew a thing or two about livestock. Although modern fertilizers have improved the rate of grass production since that time, the amount of milk produced per cow has also dramatically increased, so that the per head demand for grass has also increased.

Clearly the model will only work on an annual basis as differential growth rates of grass throughout the year (a problem for farmers and gardeners alike) means that grass needs to be conserved during high grass production in May and June by making hay and silage to feed to the cows during low grass production in the winter months.

Having more than 0.9 oxen per acre would result in overgrazing and having fewer enables the farmer to sell his excess hay for silage production.

For those who would like to gain further understanding of the dairy farming model, I can recommend Radio 4 at 7p.m. each weekday!

On the inaugural flight to Morocco in 1919, they found out too late that the man who had agreed to fix up an airfield at Alicante had confused metres with square metres, producing a landing strip the size and shape of a tennis court.
[Daily Telegraph]

## Solution 164.2 - ABCD

Given that $a, b, c, d$ are all between 0 and 1 , prove that

$$
(1-a)(1-b)(1-c)(1-d)>1-a-b-c-d .
$$

## Simon Geard

First, here is my solution to the original problem.
Consider

$$
\begin{aligned}
(1-a)(1-b) & =(1-a-b)+a b \\
& >1-a-b
\end{aligned}
$$

since $a, b>0$. Therefore

$$
\begin{aligned}
(1-a)(1-b)(1-c) & >(1-a-b)(1-c) \\
& >(1-a-b-c)+(a+b) c \\
& >1-a-b-c
\end{aligned}
$$

since $a, b, c>0$, and

$$
\begin{aligned}
(1-a)(1-b)(1-c)(1-d) & >(1-a-b-c)(1-d) \quad(d<1) \\
& >(1-a-b-c-d)+(a+b+c) d ; \\
(1-a)(1-b)(1-c)(1-d) & >1-a-b-c-d,
\end{aligned}
$$

since $a, b, c, d>0$.
Note that the conditions $a, b<1$ have not been used.

Having only used $0<c$ and $d<1$, I started thinking about the general case and why $a$ and $b$ didn't need such stringent constraints.

I think the inequality given in the problem can be generalized to

$$
(1-a)(1-b) \prod_{i}\left(1-x_{i}\right)>1-a-b-\sum_{i} x_{i},
$$

where $0<x_{i}<1$ and $a b>0$ and $a, b \neq 1$.
The method I used in my previous solution has (I think) the germs of a proof by induction but I haven't tried it yet; an exercise for the reader?

Sorry about the disjointed nature of my replies to $\mathbf{1 6 4 . 2}$-my brain must have put on a few pounds over Christmas!

## Solution 162.1 - Nine nines

$$
\begin{aligned}
& x=12,345,678,987,654,321, \\
& y=1+2+3+4+5+6+7+8+9+8+7+6+5+4+3+2+1, \\
& x y=999,999,999^{2} . \text { Why? }
\end{aligned}
$$

## John Halsall

We have

$$
\begin{aligned}
x & =12,345,678,987,654,321=111,111,111^{2} \\
y & =1+2+3+4+5+6+7+8+9+8+7+6+5+4+3+2+1 \\
& =81=9^{2}
\end{aligned}
$$

Need I go on?

## Re. Problem 164.3-24 squares

## Chris Pile

Thank you for digging this one up again. I see that it was first printed in M500 37 and M500 38. (How long ago was that?)

The square packing problem was given wide publicity in Scientific American and the 'solution' given in M500 $\mathbf{3 7}$ was the best obtained. I have still not seen a better arrangement.

My extension to the problem was to consider packing the set of 24 equilateral triangles (of sides 1 unit to 24 units) into a triangle of side 70 units. I have never seen any other reference to this and again I can offer no improvement on the arrangement in M500 38. However, as this problem had not been widely publicized, and the area left uncovered is over three times that for the squares, I had expected that someone would improve the packing. As an open-ended problem I find that it makes an entertaining puzzle.

ADF - OK then; we formally reissue it as:

## Problem 167.1-24 triangles

## Chris Pile

What is the best packing of the 24 equilateral triangles of sides 1 to 24 on to a triangle of side 70 ?

## Problem 167.2 - Cows

## John Halsall

Fritz and Helmut sold a herd of cows and obtained for each cow as many pounds as there were cows in the herd. Weak in arithmetic, they decided that they would each take $£ 10$ in turn from the sum obtained. Fritz got the last $£ 10$ and there was less than $£ 10$ left for Helmut, so Fritz gave Helmut his pocket knife in compensation. Helmut knew what the knife was worth and was well satisfied.

How much was the knife worth?

## Letters to the Editors

## Sir James Lighthill

I recently attended a day on mathematics, classical music and speeches in honour of Sir James. I started the day in awe of the man, which increased as I realized that his work in two of the fields in which I am involved was merely a small part of his repertoire. All of the music played in the afternoon session had been performed by him in concerts at UCL. One of the speeches spoke of him learning foreign languages at incredible speed, and his knowledge of poetry and the fine arts was comprehensive. Perhaps one of the most significant expressions used was by David Crighton. Referring to Sir James' seminal work in 1955 on aero-acoustics he mentioned that it was an article which contained and needed no references. A great loss!

## Steve Otto

## Eggs

Dear Jeremy,
John Bull, writing on mathematical models for predicting the consumption of grazing oxen (M500 165), comments that both models use doubtful data and allow for fractional oxen.

When I was at school and aged about 12, we were presented with the data:

A hen and a half lays an egg and a half in a day and a half.
We were then invited to show what four hens would do in six days (or was it six hens in four days?). My father kept chickens and my mother made omelets, so that problem always worried me.

Colin Davies

## Countdown

Dear M500 Ed.,

Further to Ralph Hancock's thoughts on the Countdown numbers game [M500 165], I have two programs on my Acorn written by two different authors which solve the puzzle in two different ways. One goes through every possible combination of numbers and operators in one line, the other works more as a human brain might as Ralph Hancock suggests. Both programs are very clever and can find solutions to difficult puzzles; equally, both occasionally miss some fairly obvious solutions. The programs were given to me by a friend who I think found them on the internet, so I will be pleased to give further details if required.

## Gail Volans

## Lottery

## Eddie Kent

This is the beginning of a fax I received recently.
'LOTTERY SECRETS REVEALED. How to choose winning numbers. A service that will help you choose the best lottery numbers, explain the relevance of astrology and tell you your chances of winning. Don't believe it? For example the most common combination is three odd numbers (i.e. 7 9 11) and three even (i.e. 2, 4, 6). This means you increase your chances if you mix odd and even numbers. in fact the first 75 draws had NO instances of all odd or all even numbers ....'

It then goes on to list various telephone numbers you can ring to get information about different aspects of the lottery-A complete list of winning numbers; How often numbers based on birthdays win; The use of astrology; etc.; all at a pound a minute I suppose.

Surely if the numbers are picked at random you would expect three odd and three even, wouldn't you? And are there really people out there who believe that past numbers have any influence on the future? I did see a note in the paper some time ago about someone with nothing better to do who analysed an American version of the lottery, with 50 numbers instead of our 49 (why 49 anyway?). He showed that there was a $51: 49$ per cent advantage in choosing completely at random. This does not sound like a very significant result since he would have to ask people and anyone sensible would naturally lie about this.

I still feel that, apart possibly from senders of faxes like the above, people like me come out of this best. I put nothing on the lottery at all but still reap the benefit when surplus money is used to build a bar in my local park.

## M500 Mathematics Revision Week-end 1999 JRH

The 25th M500 Society Mathematics Revision Week-end will be held at ASTON UNIVERSITY, BIRMINGHAM over 17 - 19 SEPTEMBER 1999.

We plan to present most OU maths courses (MU120, MST121, M203, M206, M246, M261, M336, M337, M343, M346, M353, M355, M358, M372, M381, M433, MA290, MDST242, MS221, MS284, MST207, MST322, MT262, MT365), and also S271, S357, SM355 and T223. Tutorial sessions start at 19.30 on the Friday and finish at 17.00 on the Sunday. On the Saturday night there is a mathematical guest lecture, a disco, and folk singing. The Week-end is designed to help with revision and exam preparation, and is open to all OU students.

We offer the following tuition packages:

1. Full board ( 350 places): $£ 100$ members, $£ 110$ non-members.
2. Hot breakfast, packed lunch ( 100 places): £87 members, £97 nonmembers.
3. Continental breakfast, packed lunch (250 places): £87 members, £97 non-members.
4. Non-residential: £45 members, £23455 non-members.

The latest date we can accept bookings, even if places are still available, is 3 September 1999. For full details and an application form, send an SAE to Jeremy Humphries.

## Vic Parsons

We are sorry to learn that Vic Parsons, who used to tutor M245 and M246 at the M500 Revision Week-end, died suddenly on 2nd March.

Vic's association with the Open University was multi-faceted and to all aspects he brought enormous commitment, enthusiasm and enjoyment in helping students and colleagues. He very much personified the ideals of the OU community, always offering practical support for everyone.

He will be remembered as a colleague at M101 summer schools, working tirelessly on the students' behalf. He had a particular interest in helping those who found the going tough and successfully rescued many such.

He was highly committed to the OU all round and would do everything in his power to help. OU students were his highest priority and he was always quite happy to (and often did) cancel his personal obligations to stand in for a scheduled tutor at the last minute.

Our sympathy goes to his family, especially his widow Trish. He will be greatly missed.

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