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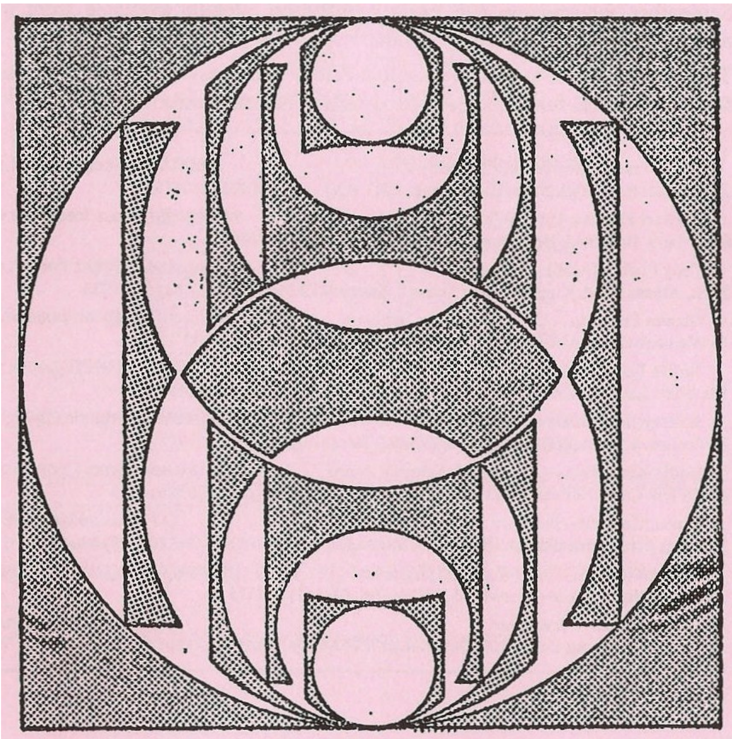

The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

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MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

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Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail. Camera-ready copy can be accepted if it follows the general format of the magazine.

A history of π

David Singmaster

This began as extracts from my various chronologies but has since been extended to cover the topic in greater depth.

Notes

π = 3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510 58209 74944 59230 78164 06286 20899 86280 34825 34211 70679 82148 08651 32823 06647 09384 46095 50582 23172 53594 08128 48111 74502 84102 70193 85211 05559 64462 29489 54930 38196

25/8	= 3.125
16/5	= 3.2
22/7	= 3.14285 71428 57143
223/71	= 3.14084 50704 22535
256/81	= 3.16049 38271 60494
355/113	= 3.14159 29203 53982
$\sqrt{10}$	= 3.16227 76601 68379

The continued fraction for π is: [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, 2, 6, ...]. The first 11 convergents are:

3/1	= 3.00000 00000 00000
22/7	= 3.14285 71428 57143
333/106	= 3.14150 94339 62264
355/113	= 3.14159 29203 53982
103993/33102	= 3.14159 26530 11903
104348/33215	= 3.14159 26539 21421
208341/66317	= 3.14159 26534 67437
312689/99532	= 3.14159 26536 18937
833719/265381	= 3.14159 26535 81078
1146408/364913	= 3.14159 26535 91404
4272943/1360120	= 3.14159 26535 89389

The value 3927/1250 = 3.1416 has continued fraction [3; 7, 16, 11], which has the notable convergents: 3; 22/7; 355/113; 3927/1250.

In sexagesimals, the value obtained by al-Kashi is: [3; 8, 29, 44, 0, 47, 25, 53, 7, 25]. The correct value is: [3; 8, 29, 44, 0, 47, 25, 53, 7, 24, 57, 36, 17, 43, 4, 29, 7, 10, 3, 41].

$e^{\pi\sqrt{163}}$ = 262 53741 26407 68743.99999 99999 992....

Tony Forbes gives 55 expressions which approximate π and gives the number of correct places.

I use the following symbols: C = circumference, A = area of a circle of radius r; V = volume of a sphere of radius r, $\phi = (1+\sqrt{5})/2$. Earlier authors did not always know that the constant values $C/2r$, A/r^2 and $3V/4r^3$ were the same, which sometimes confuses things.

The number of digits and the number of decimal places may differ by 1 since authors are not always clear whether they include the initial 3 or whether they consider rounding of final figures. Roger Webster has given me a sheet with details of computer calculations giving the number of digits computed and the number correct. These differ a bit from what I have recorded. I will give his values as: Webster: correct/computed if they differ from what I have, though I won't bother with the longer values where the number of correct places is essentially equal to the number of computed places except for a few places at the end.

A quadratrix is a curve which allows one to determine π. However, these cannot be constructed with ruler and compass.

Following Wrench, I label the following identities which are used by various calculators. Following Conway & Guy, I let $t_n = \tan^{-1}(1/n)$, so, e.g. $t_1 = \pi/4$. Conway and Guy call these *Gregory numbers*.

I. $\frac{\pi}{4} = 5t_7 + 2\tan^{-1}\frac{3}{79}$. Euler, 1755.

II. $\frac{\pi}{4} = 4t_5 - t_{70} + t_{99}$. Euler, 1764.

III. $\frac{\pi}{4} = t_2 + t_5 + t_8$. Von Strassnitzky, 1844.

IV. $\frac{\pi}{4} = t_2 + t_3$. Hutton, 1776 (another source says Euler knew this).

V. $\frac{\pi}{4} = 2t_3 + t_7$. Hutton, 1776 (another source says Euler knew this).

VI. $\frac{\pi}{4} = 3t_4 + t_{20} + t_{1985}$. West, 1810?; Loney, 1893.

VII. $\frac{\pi}{4} = 8t_{10} - t_{239} - 4t_{515}$. Klingstierna, 1730; West, 1810?

VIII. $\frac{\pi}{4} = 12t_{18} + 8t_{57} - 5t_{239}$. Størmer, 1896.

IX. $\frac{\pi}{4} = 4t_5 - t_{239}$. Machin, 1706.

X. $\frac{\pi}{4} = 5t_7 + 2t_{18} - 2t_{57}$. Euler, *c.* 1750.

Some References

Augustus De Morgan, *A Budget of Paradoxes* (1872); 2nd edition, edited by D. E. Smith, (1915), Books for Libraries Press, Freeport, NY, 1967.

J. W. Wrench Jr., 'The evolution of extended decimal approximations to π', *Mathematics Teacher* **53** (Dec 1960) 644–650. Good survey with 55 references, including original sources.

Petr Beckmann, *A History of* π. The Golem Press, Boulder, Colorado, (1970), 2nd ed., 1971.

Lam Lay–Yong & Ang Tian-Se, 'Circle measurements in ancient China', *Historia Mathematica* **13** (1986) 325–340. Good survey of the calculation of π in China.

Dario Castellanos, 'The ubiquitous π', *Mathematics Magazine* **61** (1988) 67–98 & 148–163. Good survey of methods of computing π.

Jonathan & Peter Borwein, 'Ramanujan, modular equations, and approximations to π, or how to compute one billion digits of π', *Amer. Math. Monthly* **96** (1989) 201–219.

Joel Chan, 'As easy as pi', *Math Horizons* **1** (Winter 1993) 18–19. Outlines some recent work on calculating π and gives several of the formulae used.

Tony Forbes, 'An assortment of approximations to π', M500 **145** (July 1995) 10–13.

John H. Conway & Richard K. Guy, *The Book of Numbers*, Copernicus (Springer-Verlag) 1996, pp. 241–248.

Eddie Kent, 'Table of computations of π from 2000 B.C. to now', M500 **159** (Dec 1997) 2. (This gives a number of early Chinese and Indian dates that I have not seen elsewhere and I will wait until I have more details before entering them.)

Alex D. D. Craik, 'Geometry, analysis, and the baptism of slaves: John West in Scotland and Jamaica', *Hist. Math.* **25:1** (1998) 29–74. West computed π, so Craik discusses this in general on pp. 63–64.

Boaz Tsaban & David Garber, 'On the Rabbinical approximation of π', *Hist. Math.* **25:1** (1998) 75–84.

The History

c. **–1720 /** ***c.*** **–1570.** Ahmes copies Rhind papyrus. Includes rule making $\pi = {}^{256}/_{81} = 3.16049$. This approximation is still used in cooking, where a 9″ diameter circular pan is considered equal to an 8″ side square pan. In fact the Egyptian picture may imply that the area of a circle of diameter 9 should be 63 which would give $\pi = 3^1/_9 = 3.11111$.

c. **–1700.** An Old Babylonian tablet gives $\pi = {}^{25}/_8 = 3.125$.

c. **–550.** The Old Testament (I Kings 7.23 and II Chronicles 4.2) indicates $\pi = 3$. Tsaban & Garber say this was written after -965 and 'not much later' than -561. However, the texts differ by one letter in the spelling of the word 'line-measure.' Applying Hebrew gematria to the letters, one spelling has number values 5, 6, 100 while the other has 6, 100. Taking the totals gives us 111 and 106 and ${}^{111}/_{106} \approx \pi/3$, indeed ${}^{333}/_{106} = 3.14150\ 94340$. (One source attributes this to Gaon of Vilna, but Tsaban & Garber say the earliest they can find it is in 1962!)

c. **–520.** Bryson, possibly a pupil of Pythagoras, estimates π by use of inscribed and circumscribed hexagons, obtaining $\pi = 3.031$. Apparently the first person to try this, he could not go to larger polygons because of the limited geometrical knowledge of his time.

c. **–500.** The Talmud gives $\pi = 3$.

c. **–430.** Hippocrates of Chios shows that areas of circles are proportional to the squares of their diameters, i.e. that $A = cr^2$ for some constant c, i.e. that π exists. He is also the first to find an area bounded by curves, leading to the hope that the circle could be squared.

–428. Death of Anaxagoras, first man accused of trying to square the circle.

c. **–420.** Trisectrix (and Quadratrix) of Hippias. See *c.* –350.

−414. Aristophanes' *The Birds* refers to circle squarers.

c. **–350.** The *Sulvasutras* include constructions which can be interpreted as implying 15 values of π, ranging from 2.56 to 3.31, including $\pi = 54 - 36\sqrt{2} = 3.08831$, ${}^{25}/_8$, ${}^{256}/_{81} = 3.16049$, ${}^{625}/_{196} = 3.18878$ and ${}^{16}/_5$. [R. C. Gupta, 'New Indian values of π from the Manava Sulba Sutra', *Centaurus* **31** (1988) 114–126.] Kaye ['The Trisatika of Sridharacarya', *Bibliotheca Math.* (**3**) **13** (1913) 203–217 & plate] says they had values of 3.0044 and 3.097.

c. **–350.** Dinostratus, brother of Menaechmus, shows that the Trisectrix of Hippias is also a Quadratrix.

c. **–335.** Eudemus: *History of Geometry.* (*c.* −350?) Though the original is lost, many excerpts have been preserved in other works. He says

Antiphon and Hippocrates (presumably of Chios) thought they had squared the circle. Actually Antiphon describes constructing 2^n-gons and carrying on indefinitely.

***c.* −300.** Euclid's *Elements* only gives $3 < \pi < 4$.

−250. Archimedes shows $A = rC/2$ and uses a 96-gon to show $^{223}/_{71} < 3^{1137}/_{8069} < \pi < 3^{1335}/_{9347} < {}^{22}/_{7}$, i.e. $3.140845 < 3.140910 < \pi < 3.142827 < 3.142857$. I also have that he got $3.141495 < \pi < 3.141697$ and $3.14103 < \pi < 3.14271$ by use of a 96-gon, but I don't know why there are different values. He may have found $^{333}/_{106} = 3.141509$.

−2C. Apollonius may have known $^{333}/_{106} = 3.141509$ and Ptolemy's $^{377}/_{120} = 3.141666\ldots$.

−1C. Jiu Zhang Suan Shu generally uses $\pi = 3$, but says the volume of a sphere is 9/16 the volume of the circumscribed cube, i.e. $V = {}^{9}/_{2}\, r^3$, a traditional ratio based on weighing, corresponding to $\pi = {}^{27}/_{8}$.

***c.* −25.** Vitruvius's *De architectura* is often said to have given $\pi = {}^{25}/_{8}$, but this is due to editorial tampering with the text which gives $\pi = 3$. [John Pottage, 'The Vitruvian value of π', *Isis* **59** (1968) 190–197.]

***c.* −20.** Liu Xin (= Liu Hsin) gives an improved value of π perhaps 3.1547.

***c.* 0.** The *Mahabharata* uses $\pi = 3$.

***c.* 120.** Zhang Heng (78–139) gives $\pi = {}^{365}/_{116} = 3.14655$ (or $^{92}/_{29} = 3.17241$) and $\pi = \sqrt{10} = 3.16228$. Another source says Chang Hing gives $\pi = {}^{142}/_{45} = 3.15556$, cf. *c.* 250.

***c.* 85 / *c.* 165.** Ptolemy: *Almagest*. He estimates $\pi = [3; 8, 30] = 3^{17}/_{120} = 3.1416666\ldots$.

2C? The *Mishnat Ha-Midot* gives $3^1/_7$.

3C. Chinese have 'Chih's value' of $3^1/_8$.

***c.* 250.** Wang Fan gives $\pi = {}^{142}/_{45} = 3.15556$. Another source attributes this value to Chang Hing in 120.

263. Liu Hui's 'method of circle division' estimates $3.141024 < \pi < 3.142704$ by use of a 192-gon, but he uses $^{157}/_{50} = 3.14$ for practical work. Using a 3072-gon, he may have estimated $\pi = 3.14160$, but some historians feel this was done by Zu (*c.* 480). Another source says Liu Hui misinterpreted the Jiu Zhang Suan Shu as giving $\pi = 3$ and then suggested $\sqrt{10}$.

4C. Al-Biruni says the *Pulisa Siddhanta* gives $\pi = 3$ and $3^{177}/_{1250} = 3.1416$.

***c.* 480.** Zu Chongzhi = Tsu Ch'ung-chih (430–501), also known as Wenyuan, estimates π by $^{22}/_{7}$ and by $^{355}/_{113}$ (= 3.14159292) and says

$3.1415926 < \pi < 3.1415927$ (see Liu, 263). He may have used a 24,576 sided polygon. He may have given $\pi = {}^{25}/_{8}$.

499. Aryabhata I gives $\pi = 3.1416$ (by examining a polygon of 384 sides) and $\pi = \sqrt{10}$. (Al-Biruni, quoting Brahmagupta, says the first is Ptolemy's value ${}^{3393}/_{1080} = 3{}^{17}/_{120} = 3.14166\ 66667$, but that another place has ${}^{3393}/_{1050}$, which al-Biruni assumes to be a copying error. Shukla says Aryabhata gives ${}^{62832}/_{20000}$ and this is the first time it occurs.) Says the volume of a sphere is $(\pi r^2)^{3/2}$, which is the volume of a circular cylinder of radius r and height $\sqrt{\pi} r$. This effectively makes $\pi = {}^{16}/_{9} = 1.555\ldots$!

628. Brahmagupta gives $\pi = 3{}^{1}/_{7}$, noting that this is nearly $\sqrt{10}$, but al-Biruni also says he gives ${}^{4800}/_{1581} = 3.03605\ 31309$.

629. Bhaskara I gives Aryabhata's formula for the volume of a sphere and also quotes another earlier formula: $V = {}^{9}/_{2}\, r^3$, corresponding to $\pi = {}^{27}/_{8} = 3.375$. He also gives the following estimate (converted to radian measure): $\sin x = 16x(\pi - x)/(5\pi^2 - 4x(\pi - x))$, which leads to $\pi = {}^{16}/_{5} = 3.2$. (Previously dated as 522.)

635. The Sui Shu of Tsu Ch'ung-chih gives $\pi = 3.1415927$. (?)

***c.* 780.** Al-Biruni gives $164909/52500 = 3.14112\ 38095$ as a value due to ibn Tariq.

***c.* 820.** Al-Khowarizmi mentions $\sqrt{10}$ as a value used by geometers.

850. Mahavira gives the volume of a sphere as ${}^{9}/_{2}\,{}^{9}/_{10}\, r^3$, corresponding to $\pi = {}^{243}/_{80} = 3.0375$.

***c.* 900.** Sridhara uses $\sqrt{10}$. He gives the volume of a sphere as $4{}^{19}/_{18}\, r^3$, corresponding to $\pi = {}^{19}/_{6} = 3.16666\ldots$, and this is repeated by Aryabhata II (*c.* 950) and Sripati (1039).

904. Vatesvara notes that Aryabhata's value of ${}^{3927}/_{1250} = 3.1416$ is better than $\sqrt{10}$.

973/1048. Al-Biruni determines π as 3.1417482.

***c.* 1000.** Madhavacandra estimates π as $\sqrt{10}$.

1030. Al-Biruni cites various Indian values for π, states it is irrational, says it is nearly ${}^{4800}/_{1527} = 3{}^{73}/_{509} = 3.14341\ 84676$ and also gives a value due to ibn Tariq, *c.* 780.

***c.* 1100.** Omar Khayyam (Abul-Fath Umar ibn Ibrahim al-Khayyami) asserts that values like $\sqrt{2}$ and π are actually numbers, rather than just ratios of magnitudes as in Greek thought.

1150. Bhaskara II gives ${}^{3927}/_{1250} = 3.1416$, ${}^{22}/_{7}$, $\sqrt{10}$ and ${}^{754}/_{240}$ ($= {}^{377}/_{120}$) $= 3.141666\ldots$. He correctly gives the volume of a sphere as (surface $\times D$)$/6$ and the correct formula when $\pi = {}^{22}/_{7}$.

***c.* 1190.** Maimonides (1135–1204) asserts that π cannot be known precisely (apparently meaning it is irrational) and gives $3^1/7$ as an approximate value.

1202. Fibonacci: *Liber Abaci.* He estimates $3.1410 < \pi < 3.1427$ or $\pi = {}^{864}/275 = 3.141818$ by use of a 96-gon.

1247. Qin Jiushao gives $\pi = \sqrt{10}$.

***c.* 1320.** Dante's *Paradiso*, XXXIII, 133–136, mentions circle squaring or measuring ('misurar lo cerchio').

14C. Zhao Youqin uses the ideas of Liu (263) and Zu (*c.* 480), reaching a 16384-gon, obtaining $\pi = 3.1415926$.

1424. Al-Kashi gives 2π as 6.28318 53071 79586 5, so $\pi = 3.14159\ 26535\ 89793\ 3$ by using a polygon of $3 \times 2^{28} = 805{,}306{,}368$ sides, making careful allowance for rounding errors. He actually does his work in sexagesimals, obtaining [6; 16, 59, 28, 34, 51, 46, 15, 50].

***c.* 1500.** Nilakantha gives the series usually called Leibniz's. In his commentary on Aryabhata I, he asserts that no rational value can be found for π.

1551. Rhaeticus (1514–1576) publishes tables of the six trigonometric functions for $10'$ intervals, to 7 figures. He then starts on tables with $10''$ intervals, completed and printed by his student Otho in 1596, *q.v.* The value of $\sin 10''$ implies $\pi = 3.14159\ 26523$.

1573. L. Valentin(e) Otho gives $\pi = {}^{355}/113$, as done by Zu Chongzhi (*c.* 480). Peurbach (1423–1469) also gives this value.

***c.* 1580.** Tycho Brahe estimates $\pi = 88/\sqrt{785} = 3.14085$.

1585. Adrien Anthoniszoon (= Metius?) estimates $\pi = {}^{355}/113$, as given by Zu Chongzhi (*c.* 480). He apparently added numerators and denominators of the approximants ${}^{333}/106$ and ${}^{377}/120$.

1593. François Viète (= Franciscus Vieta) (1540–1603). Uses a polygon of $6 \times 2^{16} = 393{,}216$ sides to determine π to 9 places—in his *Variorum de rebus mathematicis responsorum liber VIII.* (1597?) Finds his infinite product: $\dfrac{2}{\pi} = \sqrt{{}^1/2}\sqrt{{}^1/2 + {}^1/2\sqrt{{}^1/2}}\sqrt{{}^1/2 + {}^1/2\sqrt{{}^1/2 + {}^1/2\sqrt{{}^1/2}}}\ldots$. Indeed this is the first usage of an infinite product. (F. Rudio first proved that this converged in 1891.) He also gives $\pi = (12 + 6/\phi)/5 = 3.141640786$.

1593. Adriaen van Rooman (Adrianus Romanus) extends Vieta's work to 17 places: $\pi = 3.14159\ 26535\ 89793\ 23$. (Or 15 places, using a polygon of 2^{30} sides.)

1551/1596. The trigonometric tables of Rhaeticus and Otho have a value of $\sin 10''$ which implies $\pi = 3.14159\ 26523$.

1610. Ludolph van Ceulen computes π to 35 places using a polygon of 2^{62} sides. His tombstone says that $\pi > 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288$, but is less than the number with the last digit increased to 9. (Kent says he got 20 places in 1596.)

1621. Willebrord Snel(l) (van Roijen) improves the methods, so he can obtain van Ceulen's results (or gets 36 places) with a polygon of 2^{30} sides. Huygens proves Snell's method is correct in 1654.

1627. In Japan, Yoshida Koyu gives $\pi = 3.16$, probably derived from $\sqrt{10}$.

1630. Grienberger, using Snell's method, gets 39 places.

1632. In Japan, Imamura Chisho uses $\pi = 3.162$, probably derived from $\sqrt{10}$, but gives the volume of a sphere as $0.51D^3$, corresponding to $\pi = 3.06$.

1655. Wallis gives his product:

$$\frac{\pi}{2} = \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \frac{6}{7} \times \dots = \prod \frac{4m^2}{4m^2 - 1}.$$

***c.* 1658.** Brouncker gives a continued fraction for $4/\pi$ based on Wallis's product. $4/\pi = 1 + 1^2/(2 + 3^2/(2 + 5^2/(2 + 7^2/\dots)))$.

***c.* 1660?** Thomas Hobbes asserts $\pi = 16/5$.

1663. In Japan, Muramatsu Kudayu Mosei considers a polygon of $2^{15} = 32768$ sides inscribed in a circle of diameter 1 and finds the perimeter is $3.14159\ 26487\ 77698\ 86924\ 8$ and then cautiously says one should take $\pi = 3.14$. In discussing the volume of the sphere, he arbitrarily takes $\pi = 3.144$.

1667. James Gregory's *Vera Circuli et Hyperbolae Quadratura* gives first series expansions for trigonometric functions and proves π is irrational.

1670. In Japan, Sawaguchi Kazuyuki arbitrarily gives $\pi = 3.142$.

***c.* 1670?** Huygens gives $\pi = 3 + \sqrt{2}/10 = 3.1414213562$.

1671. James Gregory finds the series:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots,$$

though he didn't try $x = 1$ in this as done by Leibniz.

1672. $\pi = 355/113$ is first given in Japan by Ikeda Shoi.

1674. Leibniz finds $\pi/4 = 1/1 - 1/3 + 1/5 - 1/7 + \dots$, generally known as Leibniz's series, though it is a special case of Gregory's series, and is equivalent to Brouncker's expression—as shown by Euler.

1676. Newton finds the series

$$\sin^{-1} x \;=\; x + \frac{1}{2}\,\frac{x^3}{3} + \frac{1}{2}\,\frac{3}{4}\,\frac{x^5}{5} + \cdots.$$

He used $x = 1/2$ and obtained 14 places of π.

1699. Abraham Sharp computes 72 decimal places of π using the series of Gregory/Leibniz with $x = 1/\sqrt{3}$. Another source says he computed 75 places and checked it to 72 places by another method, but doesn't specify either method. (Craik dates this as 1741, but this may be a date of publication?)

1706. John Machin gives his identity IX for π and uses it with Gregory's series to compute π to 100 places.

1706. William Jones's *Synopsis palmariorum mathesos, or A New Introduction to the Mathematics* introduces symbol Π for what we now denote by π and first publishes Machin's 100 decimals; Π had previously been used for 'periphery' by Oughtred and David Gregory.

1712. In Japan, a book giving some of Seki Kowa's results gives a rather arbitrary interpolation to obtain $\pi = 3.14159\ 26535\ 9$.

1719. Fautet De Lagny extends Sharp's calculation to compute 127 decimals of π, with a unit error in the 113th place.

1722. In Japan, Takebe Kenko says he used Seki's ideas and a 1024-gon to obtain 42 places of π, though it seems it must have used a larger n-gon to do this. Takebe also describes using the square of the perimeter of the 512-gon to obtain 32 places of π^2 and gives continued fraction approximations and a series for the square of the arc length of an arc of height h in a circle of radius r. Taking $h = r$ or $2r$ gives values for π^2.

1730. S. Klingstierna discovers VII. See 1832, 1926, 1957.

***c.* 1730.** In Japan, Kurushima Yoshita gives $\pi^2 = 98548/9985$, correct to 9 places. Takuma Genzayemon obtains π correct to 25 places.

1733. Buffon devises his famous needle-dropping experiment to measure π, but he doesn't publish a full description until 1777.

1736. Euler finds $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.

1739. In Japan, Matsunaga Ryohitsu uses Newton's series to get 50 places of π, but this was not published until 1769, *q.v.*

1745. J. E. Montucla's *Histoire des recherches sur la quadrature du cercle* appears anonymously.

1748. Euler's *Introductio in Analysin Infinitorum* makes the Greek letter for π the standard notation. (He had used it in various ways from

1736.) He seems to be the first to observe that $t_1 = t_2 + t_3$ and he used this to find a series for π. He also extends his 1736 result to the general formula for $\sum 1/n^{2k}$. He gives π to 126 places.

***c.* 1750.** Euler gives

$$\frac{\pi}{2} = \frac{3}{2}\,\frac{5}{6}\,\frac{7}{6}\,\frac{11}{10}\,\frac{13}{14}\,\frac{17}{18}\,\frac{19}{18}\,\ldots,$$

where the numerators are the odd primes and the denominator corresponding to p is $p+1$ or $p-1$, whichever is not divisible by 4.

***c.* 1750.** Euler gives e to 15 places. Conway & Guy say he knew IV, V and X.

1755. Euler finds another series

$$\tan^{-1} x = \frac{x}{1+x^2}\left(1 + \frac{2}{3}\,\frac{x^2}{1+x^2} + \frac{2}{3}\,\frac{4}{5}\left(\frac{x^2}{1+x^2}\right)^2 + \ldots\right).$$

He gives I and uses it to compute 20 places of π in one hour. Conway and Guy give a different form, using $m = n^2 + 1$:

$$\tan^{-1}\frac{1}{n} = \frac{1}{\sqrt{m}}\left(1 + \frac{1}{2}\,\frac{1}{3m} + \frac{1}{2}\,\frac{3}{4}\,\frac{1}{5m^2} + \frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\,\frac{1}{7m^3} + \ldots\right).$$

Euler pointed out that since the values of m are 10 and 50 in using V and 50, 325, 3250 in using X, so that calculations are quite easy.

1755. French Academy of Sciences refuses to examine any more solutions of the quadrature problem. The Royal Society follows suit a few years later.

1761. (But not printed until 1768 and 1770.) Lambert shows π and e^m, for rational m, are irrational, though the proof requires some later amendments by Legendre. His method was to show that if x is rational and non-zero, then $\tan x$ is irrational. He also computes the continued fraction of π as: $[3; 7, 15, 1, 292, 1, 1, 1, 2, \ldots]$. and conjectures π is transcendental, as followed by Euler (1775) and Legendre (1794).

1764. Euler publishes II. Used in 1841.

1769. In Japan, Arima Raido publishes π to 50 places, as obtained but not published in 1739. He also gives the fraction 42822 45933 49304 / 13630 81215 70117 which gives π correct to 29 places. Another source says that the fraction 5419351/1725033, which is correct to 12 places, was known about this time.

1776. Hutton gives IV (used in 1853 & 1877) & V (used in 1789) and a series for $\pi/4$.

1777. Buffon fully describes his experiment.

1789. Georg von Vega uses Gregory's series and V to compute 143 decimals of π, 126 being correct. See 1794.

1794. Vega redoes his 1789 work, using I, to obtain 140 places, with 136 correct.

1794. Legendre's *Élements de Géométrie* amends Lambert's work and shows π^2 is irrational.

***c.* 1800.** Chu Hung uses Newton's series to get 40 places of π, but only 25 are correct.

1808. Specht finds $\sqrt{146} \times 13/50 = 3.141591953$.

1810? (Published 1838.) John West states that the easily proven result that $t_n = t_{n+c} + t_{n+d}$ if $cd = n^2 + 1$. Cf. 1880s, Carroll. He finds or states VI, VII, IX and $\pi/4 = 3t_7 + 2\tan^{-1}(2/11)$.

1818. Gauss develops his arithmetic-geometric mean for elliptic integrals— see 1976. At some time, he also studies the general form of arctangent relations and gives VIII, used in 1957–58.

1822. Thibaut finds 156 decimals of π.

1832. K. H. Schellbach rediscovers VII.

1841. William Rutherford computes 208 digits of π, using II, but the last 56 are wrong. (Kent dates this as 1824.)

1844. L. K. Schulz von Strassnitzky gives III. Johann Zacharias Dase uses this to compute π to 205 places, 200 being correct. This was done (largely or entirely?) in his head, taking two months, and was published in *Crelle's Journal* **27** (1844) 198.

1847. Thomas Clausen publishes 248 decimals of π, using V.

1853. W. Lehmann uses IX to compute 261 places of π and then independently checks this by use of V.

1853. Richter gives 330 places of π, going to 440 in 1854 and 500 in 1855, confirming Shanks's 1854 work (presumably corrected), but he doesn't say how he did it.

1854. William Rutherford continues computing π, using IX, getting 441 places. Shanks soon exceeds him and Shanks's 530 places are published in a paper of Rutherford.

1854. Rutherford's pupil William Shanks uses IX and computes π to 318, then 530, then 607 places, which he publishes later this year. However, it is wrong from the 528th place. This is not detected until 1945, *q.v.* Further, he made an error in correcting proofs which produced errors in places 460–462 and 513–515, and these were not corrected until his second paper of 1873.

1872. Edgar Frisby uses Euler's series and V to get 30 places of π.

1872. Augustus De Morgan notes a deficiency in the number of 7s in Shanks 607 places.

1873. Shanks continues his 1853 calculation of π to 707 places without noticing the error. He publishes this with the 1854 proof errors and then publishes a second paper with these corrected, but there is a new typographical error in the 326th place.

1877. Tseng Chi–hung uses IV to compute 100 places of π in a little over a month.

1880s? Conway and Guy say Lewis Carroll noted that $t_n = t_{n+c} + t_{n+d}$ if and only if $cd = n^2 + 1$. Cf. West, 1810?

1882. Lindemann shows π is transcendental by showing that if A_i and c_i are algebraic, then $\sum A_i \exp(c_i) \neq 0$, except in obvious cases. But $e^{i\pi} + e^0 = 0$. This appeared as: 'Über die Zahl π', *Math. Annalen* **20** (1882) 213–225.

1885. Weierstrass simplifies Lindemann's proof.

1890. Gerard Daniel [*Knowledge* **13** (1 Nov 1890) 257] suggests $C = 3D +$ (side of inscribed square)$/5 = 3.1414213562373\ldots$ and $3\sqrt{5}/5 + 2\sqrt{2}/3 + 6/7 = 3.14159\ 26852\ 2\ldots.$

1893. S. L. Loney publishes VI. Størmer also finds it in 1896. Used in 1944–1945.

1896. Størmer relates Gaussian integers to Gregory numbers $t_n = \tan^{-1}(1/n)$ and shows how to obtain a Gregory number as a sum of other Gregory numbers. From this it follows that the only two-term expressions for $\pi/4$ are IV, V, IX and $2t_2 - t_7$. This is described in Conway & Guy, but there is a misprint for VIII at the bottom of p. 246.

***c.* 1900.** The number e is known to 346 places.

1900. H. S. Uhler uses IX to get 282 places of π.

1901. Lazzerini uses Buffon's method with 3408 throws to get $\pi = 3.1415929$, a result which is several orders of magnitude more accurate than can reasonably be expected—see 1959.

1902. F. J. Duarte uses IX to compute 200 places of π. Published in 1908.

1909. 'If we take the world of geometrical relations, the thousandth decimal of pi sleeps there, though no one may ever try to compute it.' William James, *The Meaning of Truth*.

***c.* 1910.** Ramanujan discovers $355/113\,(1 - 0.0003/3533) = 3.14159$

26535 89794 3286, which is accurate to 15 places.

1914. Ramanujan discovers $(2143/22)^{1/4} = 3.14159\ 26525\ 82646\ 1252$.

1926. C. C. Camp uses VII to obtain $\pi/4$ to 56 places.

1929. Gelfond's work shows e^{π} is irrational.

1933. Uhler finds 333 places of π as a by-product of other work.

1938. D. H. Lehmer studies arc-cotangent relations and recommends use of VII and VIII and J. P. Ballantine substantiates this in 1939.

1944–45. D. F. Ferguson uses VI to compute π to 530 places by hand and discovers the error in Shanks's 1853 and 1873 results. (Craik says Ferguson records using a new series due to R. W. Morris in May 1944, which turns out to be VI, but another source says he used IX—perhaps he used both?)

1945–47. Ferguson uses a desk calculator to obtain 710 places in January 1947 and finds that Shanks apparently omitted a term from the 569th place onward. [*Math. Gaz.* **30** (1946) 89–90.] He extends to 808 places in September 1947.

1947. John W. Wrench Jr. and Levi B. Smith use IX to compute 818 places of π, agreeing with Ferguson's 710 places. (The same other source says they used VI (?).) In April 1947, 808 places were published. Ferguson continued his calculations and showed an error in Wrench's calculations from the 723rd place. These were corrected and a checked 808 place value was published in January 1948. Subsequently, Wrench and Smith obtained 1120 places by June 1949, but the following work superseded their work before they had checked it. They later extended to 1160 places in January 1956, 1157 agreeing with ENIAC.

1949. Labor Day weekend: von Neumann, George Reitwiesner, and N. C. Metropolis use IX and ENIAC at Aberdeen Proving Ground to compute 2037 digits (or places) of π in 70 hours (Webster: 2037/2038). They also compute 2016 digits of e in 36 hours and study the distribution of the digits in both π and e.

1954. November: S. C. Nicholson and J. Jeenel use IX and NORC to compute 3089 places of π in 13 minutes, as a test prior to delivery (Webster: 3092/3093; Kent: 3092).

1957. 31 March: G. E. Felton uses Pegasus at the Ferranti Computer centre, London, to compute 10,021 places of π in 33 hours, using VII. However, a check using VIII showed only 7480 places were correct, due to a machine error.

1958. 1 March: Felton corrects error and gets two independent values of π to 10,021 places, disagreeing only by 3 units in the last place. (Kent dates this as May.)

1958. January: François Genuys uses Gregory's series, IX and an IBM 704 at Paris Data Processing Center to compute 10K digits of π in 100 minutes.

1959. Norman T. Gridgeman analyses results reported by various needle-droppers and finds many of them too good to be true—especially Lazzerini's 1901 work.

1959. 20 July: Jean Guilloud uses Genuys's program on an IBM 704 at the Commissariat à l'Énergie Atomique in Paris to compute 16,167 places of π in 4.3 hours. Wrench found no significant deviation from randomness in the numbers of digits.

1961. Daniel Shanks (no relation to William Shanks) and John W. Wrench Jr. compute 100,265 places of π in 8.7 hours on an IBM 7090. The computation is done in two ways to provide a check. The formulae used were $\pi/4 = 6t_8 + 2t_{57} + t_{239}$ and VIII, both due to Størmer. Gardner dates this as 1957.

***c.* 1964.** Philip J. Davis asks a π calculator for an upper bound on the number of decimals to which π could ever be computed. The answer was one billion—but 1B was done in 1989.

1966. Jean Guilloud & J. Fillatoire use an IBM STRETCH at the Commissariat à l'Énergie Atomique in Paris to compute 250K places of π.

1967. Guilloud & Michele Dichampt find 500K places of π on a CDC 6600 in 44.75 hours. (28h. 10m. for first method; 16h. 35m. for verification.)

1973. Guilloud and Martine Bouyer compute 1M digits of π in less than a day on a CDC 7600. (Kent says it was 1,001,250 digits.) Gardner erroneously says it was an IBM 7600. The French Atomic Energy Commission published this in 1974 as a book of 400 pages! Chan says an extension of Machin's IX was used. In 1981–1982, Guilloud gets to 2,000,050 places.

1976. Eugene Salamin publishes first quadratically convergent algorithm for π, based on Gauss's 1818 arithmetic–geometric mean applied to elliptic integrals, which Salamin rediscovers.

1981. Kazunori Miyoshi & Kazuhika Nakayama compute 2,000,036 places of π on a FACOM M-200 in 137.3 hours.

1982/84. Yoshiaki Tamura (International Latitude Observatory) and Yasumasa Kanada (Univ. of Tokyo) use Salamin's method to compute π to 2^{21} (actually 2,097,144), 2^{22} (4,194,288) and 2^{23} (8,388,576) places on a HITAC M-28OH, the last case taking 6.8 hours. In 1984, they compute 2^{24} (16,777,206) places in under 30 hours. Multiplication is done via a Fast Fourier Transform. Tamura works alone on a MELCOM 900II for the first case, then is joined by Kanada and they use a HITAC M-280H for the later three cases.

1985/86. R. William Gosper, of Symbolics Inc., Palo Alto, uses an unverified formula of Ramanujan and a Symbolics 3670 to compute 17,526,200 digits of π. Its agreement with π completes the verification of Ramanujan's formula. Castellanos says Gosper found 17.5M terms of the continued fraction for π and then verified Kanada's results to $2^{24} = 16.8$M places.) The formula was

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\sqrt{8}(4n)!(1103 + 26390n)}{9801(n!)^4 396^{4n}}.$$

1986. January: David H. Bailey, at NASA Ames Research Center, applies Jonathan M. & Peter B. Borwein's extension of Salamin's 1976 method to compute 29,360,111 digits of π on a Cray-2 in 28 hours. The iteration used may have been the following, due to the Borweins.

$$\begin{aligned} y_0 &= \sqrt{2} - 1; \qquad x_0 = 6 - 4\sqrt{2}; \\ y_{n+1} &= \frac{1 - (1 - y_n^4)^{1/4}}{1 + (1 - y_n^4)^{1/4}}; \\ x_{n+1} &= (1 + y_{n+1})^4 x_n - 2^{2n+3} y_{n+1}(1 + y_{n+1} + y_{n+1}^2). \end{aligned}$$

Fifteen iterations will guarantee to give two billion digits of $\pi = \lim x_n^{-1}$.

1986/88. Kanada and Tamura uses HITAC S-810/20, HITAC S-810/20, NEC SX-2, then HITAC S-820/80 to extend Bailey's 1986 calculations to get 33M (33,554,414), then 67M (67,108,839), then 134M (134,217,700), then 201M (201,326,551) digits of π. The last takes about 6 hours.

1987. Hideaki Tomoyori memorizes 40,000 digits of π in 17 hours.

1987. The Borweins extend Ramanujan's formula (see 1985/86) to

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)! B(n)}{(n!)^3 (3n)! \left(5280(236674 + 30303\sqrt{61})\right)^{3n+3/2}},$$

where $B(n) = 212175710912\sqrt{61} + 1657145277365 + n(13773980892672\sqrt{61} + 107578229802750)$. This adds about 25 digits of accuracy per term.

1989. Gregory V. & David V. Chudnovsky at Columbia Univ. compute 480M (May, on a CRAY-2 and an IBM 3090/VF), then 525M (525,229,270) (June, on a IBM 3090), then 1.01B (1,011,196,691) (August, on an IBM 3090), then 1.13B digits of π. Unfortunately, they do not publish details of time used, nor do they run any verifications. (Webster and Kent don't mention the 1.13B case.)

In July, Kanada & Tamura find 536M (536,870,898) digits in 67h. 13m. on a HITAC S-820/80, then in November, 1.07B (1,073,741,799) digits in 74.5 hours on a HITAC S-820/80E.

1990? Afterward, the Chudnovskys assemble a supercomputer at home and compute 2.26B digits in late summer 1991. They use a Ramanujan-like series they developed in 1985:

$$\frac{1}{\pi} = \frac{12A}{B^{3/2}} \sum_{n=0}^{\infty} \left(\frac{13591409}{A} + n \right) \frac{(-1)^n (6n)!}{(3n)!(n!)^3 B^{3n}},$$

where $A = 545140134 = 2 \times 9 \times 7 \times 11 \times 19 \times 127 \times 163$ and $B = 640320 = 64 \times 3 \times 5 \times 23 \times 29$. This is supposed to be the fastest converging series with only integer terms.

***c.* 1990.** An English school teacher decides to experiment with computing π from Gregory's series: $\tan^{-1} x = x - x^3/3 + x^5/5 - x^7/7 + \ldots$, which clearly converges very slowly when $x = 1$. E.g., if we let S_n be the partial sum of $4 \tan^{-1} 1$ to n terms, then S_{100} is 3.131593 and has an error of one in the second place after the decimal point. However, he observed that the next four digits are correct! And S_{1000} has an error of one in the third place and the next seven places are correct. As n gets larger, one finds that the errors only occur in a few, widely spaced, places. Study has revealed that these are related to the Euler numbers which appear in the Maclaurin series for the secant. (Information from Roger Webster.)

1994. May: Chudnovskys obtain 4.044B digits.

1995. June: Takahashi & Kanada obtain 3.2B (3,221,225,466) digits.

1995. August: Takahashi & Kanada obtain 4.3B (4,294,967,286) digits.

1995. October: Takahashi & Kanada obtain 6.4B (6,442,450,938) digits.

1995. Hiroyuki Goto, a 24-year-old of Tokyo, memorizes 42,195 digits of π.

1996. Kanada finds 6.44B digits of π, using 5 days on a HITAC-S-3800/480. (But Kent dates this as October 1995, see above.) A TV programme in November 1996 mentions that the Chudnovskys have got to 8B digits, but this has not been announced yet and they have only done one calculation so far, whereas Kanada checked his calculation by a second method on a different machine.

1997. July: Takahashi & Kanada obtain 51.5B (51,539,600,000) digits.

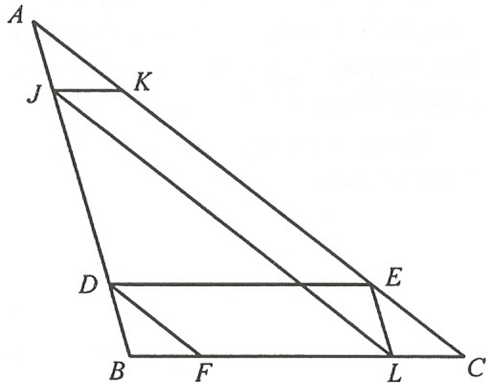
Solution 166.1 - A geometric theorem

> If, from any point on one side of a given triangle, a line be drawn parallel to a second side to meet the third side and then, from the same point on the first side, another line be drawn parallel to the third side to meet the second side, the parallelogram so formed is equal in area to the parallelogram whose adjacent sides are respectively equal to the remaining segments of the second and third sides of the given triangle.

Ralph Hancock

I stopped doing geometry at O-level and have to do it by the seat of my pants now.

The triangles were a bit confusingly drawn, so that it look as if *DKG* was the same as *ADE* and *BKH*. I have re-drawn the diagram, changing the proportions a bit.



Having drawn the first parallelogram *CDEF*, you then construct your second parallelogram *inside* the main triangle by marking a point J on AB so that $AJ = DB$, drawing a line parallel to BC to intersect AC at K, and drawing a line from J parallel to AC to intersect BC at L. You also draw a line EL parallel to AB; so your second parallelogram is *JKCL*. I don't think it needs proving here that EL and AB actually are parallel and that triangles AJK, *DBF*, and ELC are all identical.

Both parallelograms share ELC, so removing it from both of them reduces the area of the trapezia *JKEL* and *DBLE* by an equal amount. Add AJK to the first of these trapezia, *DBF* to the second. These two triangles are equal in area, so the area of the new parallelograms $AJEL$ and *DBEL* is increased by an equal amount.

Then turn the whole diagram anticlockwise till AB is horizontal. You now have two parallelograms on equal bases, AJ and *DB*, and of equal height because they share the same top line *EL*. So their area is the same, and therefore the area of the two parallelograms *CDEF* and *JKCL* is the same.

I bet there's a much nicer way of doing this.

David Brown

There are three cases to be considered, for which the proof is the same.

Given: Triangle ABC, with D any point on the side AB.

Construction: Draw DE parallel to BC to meet AC at E and DF parallel to AC to meet BC at F. Let $FC = DE = p$, $BF = q$. Let h be the height of ADE and l be the height of parallelogram $DECF$. On FD (or FD extended) cut off $FG = AE$. From B draw BH equal and parallel to FG. Join HG to cut AB at K. Then $BFGH$ is a parallelogram with adjacent sides respectively equal to the remaining segments of the sides AC and BC of ABC, not contained in the parallelogram $DHCF$.

Proof: HB is equal and parallel to AE; HK is equal and parallel to DE; therefore triangles HKB and AED are congruent; therefore the height of $\triangle HKB$ is equal to the corresponding height of $\triangle ADE$; therefore the height of parallelogram $HGFB$ (base BF) is equal to the height of $\triangle AED = h$.

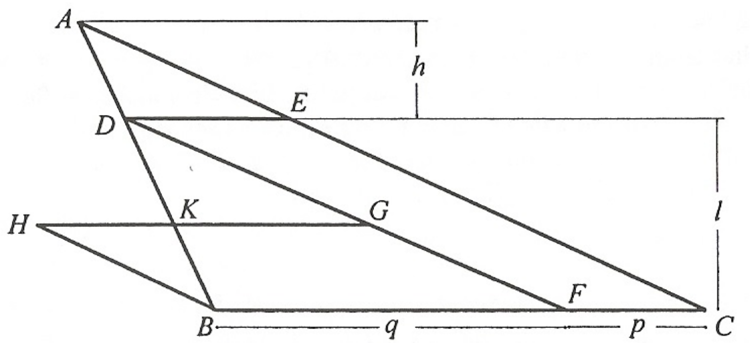
Considering areas we have $\triangle BDF + \triangle ADE +$ parallelogram $DECF = \triangle ABC$; therefore

$$\frac{ql}{2} + \frac{ph}{2} + pl \;=\; \frac{1}{2}(h+l)(q+p),$$

i.e.

$$ql + ph + 2pl \;=\; hq + lq + hp + lp.$$

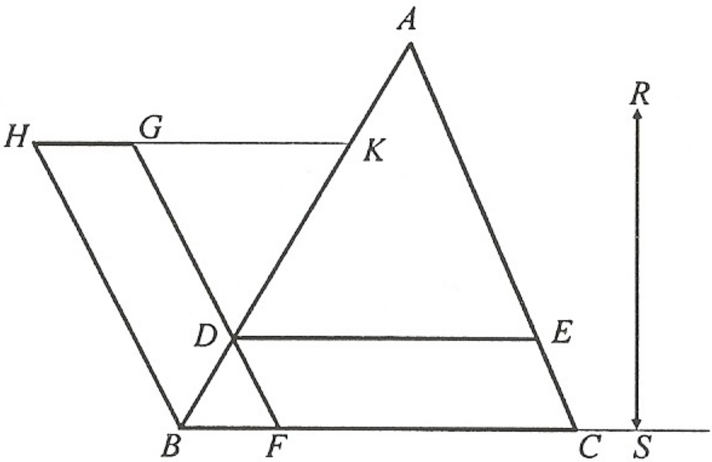
Hence $pl = qh$; i.e. parallelogram $DECF =$ parallelogram $HGFB$ in area. *QED*.



Case 1: $AD < DB$ is illustrated above.
Case 2: $AD = DB$.
Case 3: $AD > DB$; see previous page.

This theorem can be usefully applied in the geometric construction:

> *Given any parallelogram and a line of unspecified length, we can construct another parallelogram with a side equal to the given line such that it is equal in area to the first parallelogram, using straight edge and compasses only.*



For simplicity and convenience, I will use corresponding letters to name corresponding points in my example construction exercise.

Given a parallelogram $DECF$ and a line RS. With compass and straight edge only, we are required to construct a parallelogram equal in area to $DECF$, with a side equal to RS.

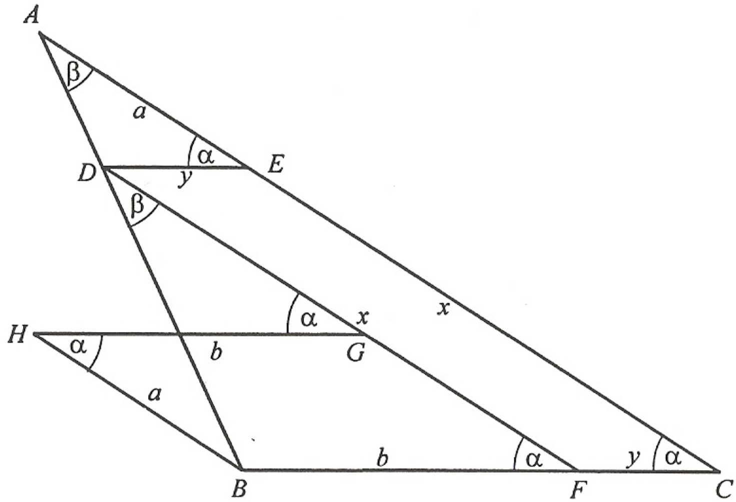
Construction: With centre E and radius equal to RS, find A on CE extended. Join AD and extend it to meet CF, extended, at B. With centre F and radius equal to RS, find G on FD (or FD extended). Join DG. With centre B and radius equal to FG, construct an arc so that it meets the arc of centre G and radius equal to FB at H.

Then $HGFB$ is the required parallelogram.

Proof: $GF = AE$ and $BF = BF$ (same line segment). But AE and BF are the remaining segments on the sides AC and BC of the triangle ABC, not contained in the parallelogram $DECF$ which has a vertex on the side AB of ABC. Hence, by my proposed theorem, the parm $HGFB$ is equal in area to parallelogram $DECF$, with side GF equal to the given line RS.

John Bull

To save lots of writing, label the angles and lines as shown. Angles shown as equal are all corresponding angles on a line crossing two parallel straight lines.



$\triangle AED$ is similar to $\triangle DFB$ by way of two equal angles.

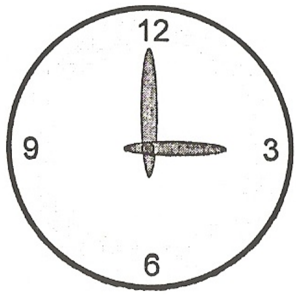
Therefore $\dfrac{b}{x} = \dfrac{y}{a}$, or $ab = xy$.

Therefore Area $DECF = xy \sin\alpha =$Area $BHGF = ab \sin\alpha$.

Dick Boardman, **Jonathan Griffiths** and **Ken Greatrix** sent similar solutions.

Problem 168.1 – Clock

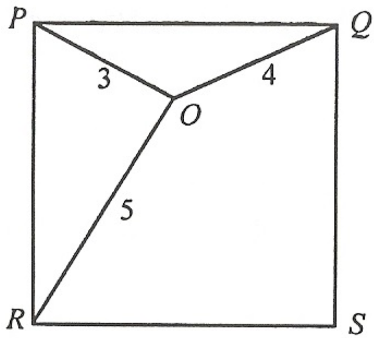
Grant Curry



The hour hand of a clock is three inches long. The minute hand is four inches. Determine the point at which the outermost points of the hands of the clock are travelling apart at the fastest speed.

Problem 168.2 – 345 square

Martyn Lawrence



$PQRS$ is a square.

$PO = 3$ units, $QO = 4$ units and $RO = 5$ units.

What is the length of side of the square?

Problem 168.3 – Fraction

John Halsall

We have

$$\frac{19}{95} = \frac{1}{5}.$$

Having dutifully reduced the answer to its lowest terms, I observed that instead of dividing numerator and denominator by 19 I might simply have struck out the two nines. My attention does tend to wander, but I'd found something more amusing than this damned maths course.

(a) $\dfrac{3544}{7531} = \dfrac{344}{731}.$ Just delete the fives.

(b) $\dfrac{2666}{6665} = \dfrac{266}{665} = \dfrac{26}{65} = \dfrac{2}{5}.$ Delete 6 thrice!

(c) $\dfrac{143185}{17018560} = \dfrac{1435}{170560}.$ Strike out 18.

Now here's one for you to do.

$$\frac{4251935345}{xxx1935xxxx} = \frac{425345}{xxxxxxx}. \qquad \text{Delete 1935.}$$

What is the denominator represented by $xxx1935xxxx$?

Have you noticed that when they give you an assignment it's always far more difficult than anything you did in the course?

A Cool Operator

Barry Lewis

Regular readers of M500 will know of my interest in the sums of the powers of the natural numbers

$$S(n,r) = 1^r + 2^r + 3^r + \cdots + n^r.$$

I will abbreviate this to S_r to indicate that we are interested in the sum of the r th powers, and that this will always be the sum of the first n such terms. What is immediately obvious from the early results is that

$$S_1 = \frac{n(n+1)}{2} \qquad \text{and} \qquad S_3 = \frac{n^2(n+1)^2}{4} \quad \text{so that } S_3 = S_1^2.$$

Recently Sebastian Hayes let me know of a similar result that Tony Forbes had given to him:

$$2S_3^2 = S_5 + S_7.$$

This article seeks to explore these relations and to develop a systematic means of generating them.

We begin with the definition of an operator - which I will call the Bernoulli Operator B - but note that it is *not* the same operator as I used in the evaluation of $S(n,r)$. We define B by the property that

$$B(S_r) = S_{r+1}.$$

So we have immediately

$$S_r = B(S_{r-1}) = B^2(S_{r-2}) = \cdots = B^r(S_0).$$

In particular we have

$$B(S_1) = S_2 \quad \text{so that } B\left(\frac{n(n+1)}{2}\right) = \frac{n(n+1)(2n+1)}{6}.$$

In fact B is a linear operator and we can manipulate it as if it were an operation such as differentiation ($\equiv D$) etc. So it obeys familiar algebraic rules. Now for an important result for our new object.

Lemma	If $f(x) = a_1x + a_2x^2 + \cdots + a_tx^t$ is any polynomial such that $f(0) = 0$ then $$f(B)(S_0) = \sum_{x=1}^{n} f(x).$$
Proof	If $f(x) = a_1x + a_2x^2 + \cdots + a_tx^t$ then $$\begin{aligned} f(B)(S_0) &= (a_1B + a_2B^2 + \cdots a_tB^t)(S_0) \\ &= a_1B(S_0) + a_2B^2(S_0) + \cdots + a_tB^t(S_0) \\ &= a_1S_1 + a_2S_2 + \cdots + a_tS_t \\ &= a_1\sum_{x=1}^{n} x + a_2\sum_{x=1}^{n} x^2 + \cdots + a_t\sum_{x=1}^{n} x^t \\ &= \sum_{x=1}^{n}(a_1x + a_2x^2 + \cdots + a_tx^t) \\ &= \sum_{x=1}^{n} f(x) \quad \text{as required.} \end{aligned}$$

Now for the main result.

Theorem	If $f(x) = a_1x + a_2x^2 + \cdots a_tx^t$ is any polynomial such that $f(0) = 0$ and n is a natural number then $$f(n) = \big(f(B) - f(B-1)\big)(S_0).$$
Proof	$$\begin{aligned} f(n) &= \sum_{x=1}^{n} f(x) - \sum_{x=1}^{n} f(x-1) \quad (\text{recall that } f(0) = 0) \\ &= \big(f(B)(S_0) - f(B-1)\big)(S_0) \\ &= \big(f(B) - f(B-1)\big)(S_0). \end{aligned}$$

This is a deceptively simple looking result - it is more powerful than it looks.

Now $S_1 = \dfrac{n(n+1)}{2}$ is a polynomial in n such that $S_1(0) = 0$ and so if we define

$$f(n) = S_1^m = \left(\frac{n(n+1)}{2}\right)^m = \frac{n^m(n+1)^m}{2^m}$$

then by our Theorem we have

$$\begin{aligned} S_1^m &= \left(\frac{B^m(B+1)^m}{2^m} - \frac{(B-1)^m B^m}{2^m}\right)(S_0) \\ &= \frac{B^m}{2^m}\big((B+1)^m - (B-1)^m\big)(S_0). \end{aligned}$$

- So when $m = 2$ we obtain

$$\begin{aligned} S_1^2 &= \frac{B^2}{2^2}\big((B+1)^2 - (B-1)^2\big)(S_0) \\ &= \frac{B^2}{2^2}\big(B^2 + 2B + 1 - B^2 + 2B - 1\big)(S_0) \\ &= \frac{B^2}{2^2}(4B)(S_0) = B^3(S_0) = S_3 \end{aligned}$$

i.e. $S_1^2 = S_3$ and that's where we started.

- When $m = 3$ we obtain

$$\begin{aligned} S_1^3 &= \frac{B^3}{2^3}\big((B+1)^3 - (B-1)^3\big)(S_0) \\ &= \frac{B^3}{2^3}\big(B^3 + 3B^2 + 3B + 1 - B^3 + 3B^2 - 3B + 1\big)(S_0) \\ &= \frac{B^3}{2^3}(6B^2 + 2)(S_0) = \frac{3}{4}B^5(S_0) + \frac{1}{4}B^3(S_0) \\ &= \frac{3}{4}S_5 + \frac{1}{4}S_3 \end{aligned}$$

i.e. $4S_1^3 = 3S_5 + S_3$ which is the other starting result.

- When $m = 4$ we obtain by identical methods

$$8S_1^4 = 4S_7 + 4S_5.$$

- Again, when $m = 5$ we get

$$16S_1^5 = 5S_9 + 10S_7 + S_5.$$

But we are not restricted to relations only for S_1. If we re-define the polynomial f in terms of S_2 we obtain the corresponding relations for it. So we set

$$f(n) = S_2^m = \left(\frac{n(n+1)(2n+1)}{6}\right)^m = \frac{n^m(n+1)^m(2n+1)^m}{6^m}.$$

The Theorem now gives

$$S_2^m = \left(\frac{B^m(B+1)^m(2B+1)^m}{6^m} - \frac{(B-1)^m B^m(2B-1)^m}{6^m}\right)(S_0)$$
$$= \frac{B^m}{6^m}\left((B+1)^m(2B+1)^m - (B-1)^m(2B-1)^m\right)(S_0).$$

- So when $m = 2$ we have

$$S_2^2 = \frac{B^2}{6^2}\left((B+1)^2(2B+1)^2 - (B-1)^2(2B-1)^2\right)(S_0)$$
$$= \frac{2}{3}B^5(S_0) + \frac{1}{3}B^3(S_0) = \frac{2}{3}S_5 + \frac{1}{3}S_3$$

ie $\quad 3S_2^2 = 2S_5 + S_3.$

- And when $m = 3$ we obtain

$$12S_2^3 = 4S_8 + 7S_6 + S_4$$

I've just started MST207, which comes with its *free* copy of MATHCAD Professional - it handles algebra like the above with ease and grace. Just to show it off a bit, I will list some more results.

- When $m = 4$ we obtain

$$54S_2^4 = 8S_{11} + 30S_9 + 15S_7 + S_5$$

- The corresponding results for S_3 are

$$4S_3^2 = 2S_7 + S_5$$
$$16S_3^3 = 3S_{11} + 10S_9 + 3S_7$$
$$64S_3^4 = 4S_{15} + 28S_{13} + 28S_{11} + 4S_9$$

....

while those for S_4 are

$$15S_4^2 = 6S_9 + 10S_7 - S_5$$
$$300S_4^3 = 36S_{14} + 195S_{12} + 88S_{10} - 20S_8 + S_6$$
$$6750S_4^4 = 216S_{19} + 2430S_{17} + 3942S_{15} + 415S_{13} - 282S_{11} + 30S_9 - S_7$$

Solution 166.3 – Boat

Dick Boardman

> A small boat which travels at constant speed of 2 metres per second through the water is racing up a channel (which runs north - south) against the tide. At a distance x metres from the western edge of the channel, the speed of the tide is $x/15$ metres per second due south. The boat starts on the western edge and must reach a buoy which is 500 metres north and 30 metres east. To reach the buoy, the boat must travel up the western edge until nearly at the buoy and then head out and past, eventually allowing the tide to carry it back to the buoy.
>
> *How far should it travel up the edge, and what path should it follow when heading out in order to minimize the time taken to reach the buoy?*

When I submitted this problem, I did not have a solution. However, seeing it in print prompted me to try to solve it once more. This time I had an inspiration.

Let the x axis run due east, let the y axis run due north; let the heading of the boat be θ (measured clockwise from north); let the velocity of the tide at point (x, y) be gx due south; let the velocity of the boat through the water be V.

Let the boat start at the origin and move towards the point (X, Y). Then

$$\frac{dy}{dt} = V\cos\theta - gx, \tag{1}$$

$$\frac{dx}{dt} = V\sin\theta. \tag{2}$$

Dividing these two equations gives

$$\frac{dy}{dx} = \cot\theta - \frac{gx}{V\sin\theta}.$$

Assuming that θ is a function of x we get that the change in y over the curved section of the boat's path is

$$y = \int \left(\cot\theta - \frac{gx}{V\sin\theta}\right) dx.$$

Furthermore, by (2),

$$t = \int \frac{dx}{V\sin\theta}.$$

Hence the total time is

$$\begin{aligned} t + \frac{B-y}{V} &= \int \frac{dx}{V\sin\theta} + \frac{1}{V}\left(B - \int \left(\cot\theta - \frac{gx}{V\sin\theta}\right) dx\right) \\ &= \frac{B}{V} + \frac{1}{V}\int \left(\frac{1+\dfrac{gx}{V}}{\sin\theta} - \cot\theta\right) dx. \end{aligned}$$

The integral may be minimized using the Euler–Lagrange formula from the calculus of variations. This specifies that we partially differentiate the contents of the integral with respect to θ and equate the result to zero. This gives

$$\left(1+\frac{gx}{V}\right)\operatorname{cosec}\theta\cot\theta - \operatorname{cosec}^2\theta = 0,$$

which simplifies to

$$\cos\theta = \frac{1}{1+\dfrac{gx}{V}}.$$

The integrations involved in the solution are very messy but can be carried out using the symbolic mathematics package MATHEMATICA.

Putting $V = 2$ and $g = 1/15$, so that $\cos\theta = 1/(1+x/30)$, and using the well-known formula $\sin\theta = \sqrt{1-(\cos\theta)^2}$, we obtain from (2)

$$t = \int_0^X \frac{dx}{V\sin\theta} = \frac{1}{2}\sqrt{X}\sqrt{60+X}. \tag{3}$$

Now divide (1) by (2), substitute for $\sin\theta$ and $\cos\theta$ and integrate:

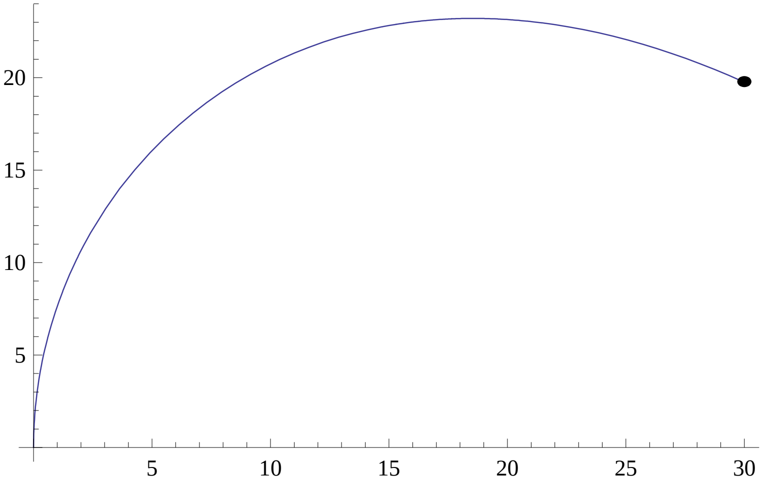
$$Y = \int_0^X \frac{V\cos\theta - gx}{V\sin\theta} dx = \frac{(30-X)\sqrt{X}\sqrt{X+60}}{60} + 30\operatorname{cosech}^{-1}\frac{2\sqrt{15}}{\sqrt{X}}.$$

With $B = 500$ the distance to the turning point is $500 - Y$ evaluated at $X = 30$, which works out at 480.246. Hence the total time is

$$\frac{500-Y}{V} + t = \frac{480.246}{2} + \frac{1}{2}\sqrt{30}\sqrt{90} = 266.104$$

on substituting $X = 30$ in (3).

The values for minimum total time and for the turning point match closely the values obtained by my 'hill climbing' program.



The path of the boat from the point at which it leaves the western edge of the channel

Book review

Barbara Lee

e: *The Story of a Number* by Eli Maor (Princeton University Press, ISBN 0-691-05854, paperback 1998, £12)

Being less than 500 years old, e is one of our youngest numbers. This book describes the mathematical developments that lead to the establishment of e, and to functions and expressions involving e. It covers a period of time from the Pythagoreans and their discovery that $\sqrt{2}$ is irrational to Hilbert and transcendental numbers.

Among the fifteen chapters we find Napier and his development of logarithms, Newton and Leibniz, the Bernoulli family, Euler, complex numbers and complex analysis, hyperbolic functions, and infinite series; in fact, almost everything related to e is included. Twelve short items on various topics are interspersed between chapters. They include logarithmic spirals, equations of motion, and Euler's formula. These short items, together with the eight appendixes, contain most of the mathematics in the book, all of which is clearly explained.

The book is well written, and holds your interest even if you are already familiar with the mathematical concepts.

Euler relation

Colin Davies

It occurred to me long ago that $e^{2\pi ni} = 1$, for all $n \in \mathbb{Z}^+$. The interesting question is what happens for the more general $n \in \mathbb{R}$. For values of n between integers, $e^{2\pi ni}$ presumably takes on a range of values like solutions to a polynomial, but drops back to the single value 1 when n is a positive integer. It is therefore rather like quadratics with the appropriate constant values that ensure that $b^2 = 4ac$, which have only one solution; for example, $ax^2 + 2x + 1 = 0$.

Now $1^k = 1$ for all $k \in \mathbb{Z}^+$. If $k = 3/2$, 1^k has values of 1 or -1. If $k = 5/4$, 1^k has values 1, -1, i, $-i$.

What happens if k is some other rational number that has a denominator that creates a root when used as an exponent? This is presumably explained in the previous articles on the 'Roots of Unity', but I failed to follow those articles after the first one or two.

But suppose k is real but not rational, and therefore does not contain a denominator and presumably does not form a root when used as an exponent. What values do $1^{\sqrt{2}}$ or 1^{π} have?

Correction. In David Brown's article, 'Euler relation' [M500 **166** 24], the equation near the bottom of the page should read

$$\frac{1}{2i}\log i \;=\; \frac{\pi}{4} \;\to\; \log i \;=\; \frac{\pi i}{2}.$$

M500 Mathematics Revision Week-end 1999

JRH

The 25th M500 Society Mathematics Revision Week-end will be held at ASTON UNIVERSITY, BIRMINGHAM over 17 – 19 SEPTEMBER 1999.

We plan to present most OU maths courses. Tutorial sessions start at 19.30 on the Friday and finish at 17.00 on the Sunday. On the Saturday night there is a mathematical guest lecture, a disco, and folk singing. The Week-end is designed to help with revision and exam preparation, and is open to all OU students.

The latest date we can accept bookings, even if places are still available, is 3 September 1999. For full details and an application form, send an SAE to **Jeremy Humphries**.

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