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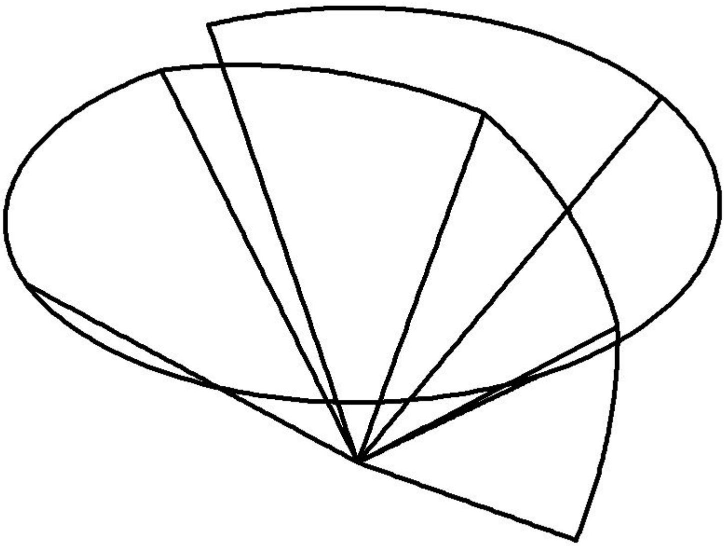
# M500 170



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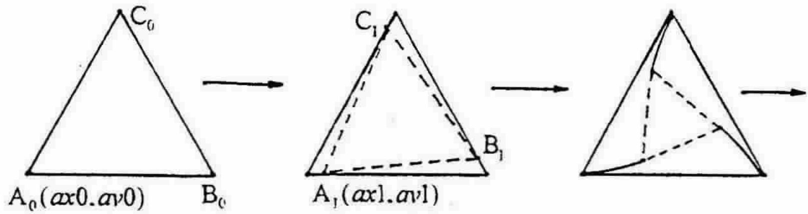
# A Mathematical Pursuit

**Barry Lewis**

Three dogs chase each other - or to bring it right up to date, three aircraft; an American aircraft chasing an Iraqi aircraft, chasing a British aircraft, chasing the American aircraft - what are the paths they follow ? Let us assume that they run with the same speed and that they start off from the vertices of an equilateral triangle. We can develop an algorithm to compute their paths by displacing each of the dogs by a small amount; and by a small amount again; ... .




Already their path is beginning to emerge. At each stage, each pursuing dog is moved towards the pursued dog by a fraction *f* of their separation. So we have, for the coordinates of the vertex, $A_1$

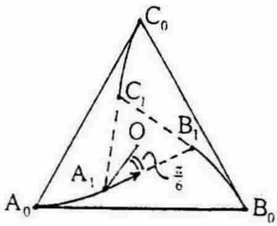
$$ax1 = \frac{f.ax0 + bx0}{(f+1)} \quad \text{and} \quad ay1 = \frac{f.ay0 + by0}{(f+1)}$$

and similarly for the vertices $B_1$ and $C_1$. This leads to the following QBASIC program. Change the value of `f` to change the speed of the program to suit your PC.

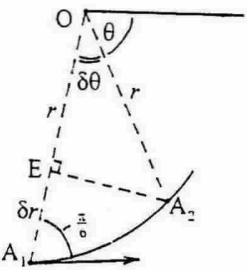
```
10 SCREEN 12
15 f=1000
20 WINDOW (0,0)-(2,2)
30 ax0=0:ay0=0
31 bx0=2:by0=0
32 cx0=1:cy0=SQR(3)
40 LINE (ax0,ay0)-(bx0,by0)
41 LINE (bx0,by0)-(cx0,cy0)
42 LINE (cx0,cy0)-(axo,ay0)
50 ax1=(f*ax0+bx0)/(f+1):ay1=(f*ay0+by0)/(f+1)
51 PSET (ax1,ay1),3
```

```
60 bx1=(f*bx0+cx0)/(f+1):by1=(f*by0+cy0)/(f+1)
61 PSET (bx1,by1),6
70 cx1=(f*cx0+ax0)/(f+1):cy1=(f*cy0+ay0)/(f+1)
71 PSET (cx1,cy1),9
80 ax0=ax1:ay0=ay1
81 bx0=bx1:by0=by1
82 cx0=cx1:cy0=cy1
90 GOTO 50
```

**1. Equiangular Spirals.** What are the equations of these curves ?




At any instant, the direction in which each dog runs makes a constant angle of $\frac{\pi}{6}$ with the line joining the dog to O, the centroid of the triangle, and a convenient origin for polar coordinates. O is also the point on which the dogs converge.




So if $A_1$ and $A_2$ are two neighbouring points on the curve through A, and E is the foot of the perpendicular from $A_2$ onto the line $OA_1$, and the (differential) lengths and angles are as shown, then

$$EA_2 = r\delta\theta$$
$$EA_1 = \delta r$$

so that

$$\tan\frac{\pi}{6} = \frac{EA_2}{EA_1} \Rightarrow \frac{1}{\sqrt{3}} = \frac{r\delta\theta}{\delta r}.$$

In the limit as $A_2 \to A_1$ we have

$$\frac{1}{\sqrt{3}} = -r\frac{d\theta}{dr}$$

(the minus sign because $r$ increases as $\theta$ decreases ) and integration gives

$$\int \frac{dr}{r} = -\sqrt{3}\int d\theta \Rightarrow \log_e r = -\sqrt{3}\theta + \text{constant}$$

ie $\quad r = k_a e^{-\sqrt{3}\theta}$

where $k_a$ is a constant for A. Clearly the curve through B is of the same form, with $\theta$ replaced by $\theta - \frac{2\pi}{3}$, giving

$$r = k_a e^{-\sqrt{3}(\theta - \frac{2\pi}{3})} = k_a e^{\frac{2\sqrt{3}\pi}{3}} e^{-\sqrt{3}\theta} = k_b e^{-\sqrt{3}\theta}$$

and $r = k_c e^{-\sqrt{3}\theta}$ as the curve through C.

Curves such as these are called *Equiangular Spirals* - equiangular because the tangent makes a constant angle with the radius vector, and spiral because, well, because of the way it spirals. But the curves that emerge from the program *don't* seem to spiral, they look as if they converge directly on O. Let's use the equation and an enlarged viewing `WINDOW` to see if this impression is confirmed. Recall that to change from polar coordinates $(r,\theta)$ to cartesian coordinates $(x,y)$ we use the transformation $x = r\cos\theta$ and $y = r\sin\theta$.

```
10 SCREEN 12
20 WINDOW (-5,-5)-(5,5)
30 t0=-3
40 LINE (-5,0)-(5,0)
41 LINE (0,-5)-(0,5)
50 t1=t0+.001
51 x = EXP(-t*SQR(3))*COS(t)
52 y = EXP(-t*SQR(3))*SIN(t)
53 PSET (x,y),3
80 t0=t1
90 GOTO 50
```

But the image we obtain is highly misleading - whatever viewing `WINDOW` is used. To understand the problem, consider two points P and Q on this curve separated by a polar angle of $2\pi$. Then we have

$$r_P = e^{-\sqrt{3}\theta} \text{ and } r_Q = e^{-\sqrt{3}(\theta-2\pi)}$$

$$\Rightarrow \frac{r_Q}{r_P} = \frac{e^{-\sqrt{3}(\theta-2\pi)}}{e^{-\sqrt{3}\theta}} = \frac{e^{-\sqrt{3}\theta}\, e^{2\sqrt{3}\pi}}{e^{-\sqrt{3}\theta}} = e^{2\sqrt{3}\pi} > 53252$$

Such a scaling is impossible to view on a screen. To see the true nature of the curve, we will use a similar curve, scaled so that its corresponding ratio is 2. Change these lines in the program.

```
51 x = EXP(-t*LOG(2)/(2*31415))*COS(t)
52 y = EXP(-t*LOG(2)/(2*31415))*SIN(t)
```

Now we see what really happens near O - the dogs pursue a path whose curvature becomes tighter and tighter as they get nearer and nearer to each other and O. Ultimately they all disappear up their own singularity.

Another property of the equiangular spiral is its self similarity. Suppose we apply the transformation $(x,y) \mapsto (mx,my)$, that is, an enlargement by a factor of $m$, then the image of

$$r = ke^{-\sqrt{3}\theta} \quad \text{is} \quad r = mke^{-\sqrt{3}\theta} = ke^{-\sqrt{3}\theta+\log m}$$

which is the same equiangular spiral advanced by the 'angle' $\log_e m$. Here, $m > 0$, otherwise the spiral would be reversed in direction.

There is another property in store. Intuitatively, we imagine that such a curve - that winds round the origin an infinite number of times - must have an infinite length. Not so. The differential length of a curve is given by

$$ds^2 = dx^2 + dy^2 \quad \text{so that} \quad s = \int_{x_1}^{x_2} \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} dx$$

But $\quad y = r\sin\theta = ke^{-\sqrt{3}\theta}\sin\theta$

$$\Rightarrow \quad \frac{dy}{d\theta} = -\sqrt{3}ke^{-\sqrt{3}\theta}\sin\theta + ke^{-\sqrt{3}\theta}\cos\theta = ke^{-\sqrt{3}\theta}(-\sqrt{3}\sin\theta + \cos\theta).$$

Similarly

$$\frac{dx}{d\theta} = ke^{-\sqrt{3}\theta}(-\sqrt{3}\cos\theta - \sin\theta)$$

$$\Rightarrow \quad dx = ke^{-\sqrt{3}\theta}(-\sqrt{3}\cos\theta - \sin\theta)\,d\theta.$$

So now we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{(-\sqrt{3}\sin\theta + \cos\theta)}{(-\sqrt{3}\cos\theta - \sin\theta)}$$ and after some simplification

$$\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} = \frac{2}{(-\sqrt{3}\cos\theta - \sin\theta)}.$$

So finally, we have

$$s = \int_{\theta_1}^{\theta_2} \frac{2}{(-\sqrt{3}\cos\theta - \sin\theta)} ke^{-\sqrt{3}\theta}(-\sqrt{3}\cos\theta - \sin\theta)\,d\theta$$

$$= \int_{\theta_1}^{\theta_2} ke^{-\sqrt{3}\theta}\,d\theta = \left[\frac{-2ke^{-\sqrt{3}\theta}}{\sqrt{3}}\right]_{\theta_2}^{\theta_1}$$

We can evaluate this for one of the paths - say that for the dog A. But we need to find the constant $k_a$. The curve $r = k_a e^{-\sqrt{3}\theta}$ passes through the point $A_0(a; -\frac{5\pi}{6})$ where $a$ is the distance between $A_0$ and O. So

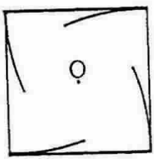
$$a = k_a e^{-\sqrt{3}(-\frac{5\pi}{6})} \Rightarrow k_a = ae^{-\frac{5\sqrt{3}\pi}{6}} \text{ and then}$$
$$r = ae^{-\frac{5\sqrt{3}\pi}{6}} e^{-\sqrt{3}\theta}.$$

The distance along the curve of pursuit between $A_0$ and O is

$$s = \left[ -\frac{2ae^{-\frac{5\sqrt{3}\pi}{6}}}{\sqrt{3}} e^{-\sqrt{3}\theta} \right]_{-\frac{5\pi}{6}}^{\infty} = 0 + \frac{2ae^{-\frac{5\sqrt{3}\pi}{6}}}{\sqrt{3}} e^{\frac{5\sqrt{3}\pi}{6}}$$
$$\Rightarrow \quad s = \frac{2a}{\sqrt{3}}.$$

Notice that the side of the equilateral triangle is $\sqrt{3}a$ so we have

$$\frac{\text{curve length}}{\text{side of triangle}} = \frac{\frac{2a}{\sqrt{3}}}{\sqrt{3}a} = \frac{2}{3} \qquad \dots \quad \textbf{(1)}$$




It is easy to see, and prove, that the equation of the curves of pursuit for a square are of the form

$$r = k_a e^{-\theta}$$

and in general for an $n$ sided polygon

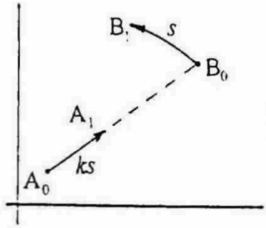
$$r = k_a e^{-\cot\left(\frac{n-2}{2n}\right)\theta}.$$

Equation **(1)** for such an $n$ sided ploygon becomes

$$\frac{\text{curve length}}{\text{side of polygon}} = \frac{1}{1-\cos\left(\frac{2\pi}{n}\right)}$$

so that the square has the pursuit curves whose length is equal to its sides. It is worthwhile adding lines to the program to deal with more values of $n$. What happens as $n \to \infty$?

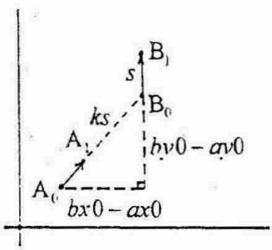
**2. Other Curves of Pursuit.** We have not considered two dogs chasing each other because without further restrictions, the problem is trivial. So consider the case where one of the dogs is constrained to follow a particular path. We investigate what happens for some particular paths and as before, we'll examine the problem on computer first before attempting analytic solutions.




Suppose that dog A chases dog B, and that dog B is constrained to follow a particular path - not specified at the moment.

Suppose also that dog A runs at $k$ times the speed of dog B. If $A_0(ax0, ay0)$, $A_1(ax1, ay1)$ and $B_0(bx0, by0)$, $B_1(bx1, by1)$ are two successive positions of the two dogs, then if dog B runs a distance $s$ along its specified path, then dog A runs $ks$ along the line joining $A_0$ to $B_0$. Now we consider some specific paths for dog B.

**The Specified Path is the Vertical Line $x = 1$**




From the geometry we have

$$ax1 = ax0 + \frac{(bx0 - ax0)ks}{\left((by0 - ay0)^2 + (bx0 - ax0)^2\right)^{\frac{1}{2}}}$$

$$ay1 = ay0 + \frac{(by0 - ay0)ks}{\left((by0 - ay0)^2 + (bx0 - ax0)^2\right)^{\frac{1}{2}}}$$

If we displace dog B by $\frac{1}{f}$ ( ie $s = \frac{1}{f}$) at each iteration, then we have the following program - change the value of f to suit your PC.

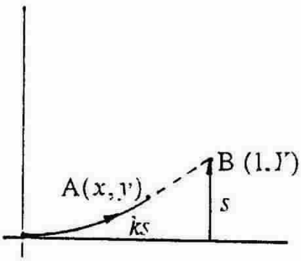
```
10 SCREEN 12
15 f=1000
20 WINDOW (-1,-1)-(2,2)
25 k = 1
30 ax0=0:ay0=0
31 bx0=1:by0=0
40 LINE (-1,0)-(2,0)
41 LINE(0,-1)-(0,2)
```

```
50 ax1=ax0+(bx0-ax0)*k/f*SQR((by0-ay0)^2+(bx0-
axo)^2)
51 ay1=ay0+(by0-ay0)*k/f*SQR((by0-ay0)^2+(bx0-
ax0)^2)
52 PSET (ax1,ay1),3
60 bx1=bx0:by1=(1/f)+by0
61 PSET (bx1,by1),6
80 ax1=ax0:ay0=ay1
81 bx1=bx0:by0=by1
90 GOTO 50
```

If you want to see what happens as $k$ varies, use this.

```
25 INPUT "What speed ratio do you want ",k
```

What is the threshold speed ratio at which dog A *does/does not* catch dog B ? Remember to alter the viewing `WINDOW` to follow the action to extremes. Can we find the equations of these paths ? We can, but not without some differential geometry.




Suppose at some instant that the dogs are as shown. Then we have the following equations

$$\frac{dy}{dx} = \frac{Y-y}{1-x} \quad \text{(Dog A runs } \textit{towards} \text{ dog B.)}$$

$$ds = k\,dY \qquad \text{(Dog A runs } k \text{ times as } \textit{fast} \text{ as dog B.)}$$

So we have

$$Y = y + (1-x)\frac{dy}{dx} \quad \Rightarrow \quad dY = dy - dx.\frac{dy}{dx} + (1-x)d\left(\frac{dy}{dx}\right)$$
$$\Rightarrow \quad dY = (1-x)d\left(\frac{dy}{dx}\right)$$

and

$$ds^2 = k^2dY^2 \quad \Rightarrow \quad dx^2 + dy^2 = k^2dY^2$$

Putting these together, we have

$$dx^2 + dy^2 = k^2(1-x)^2\left(d\left(\frac{dy}{dx}\right)\right)^2$$

$$\Rightarrow \qquad 1+\left(\frac{dy}{dx}\right)^2 = k^2(1-x)^2\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)^2$$

$$\text{ie} \qquad 1+\left(\frac{dy}{dx}\right)^2 = k^2(1-x)^2\left(\frac{d^2y}{dx^2}\right)^2 .$$

This is the differential equation that this curve of pursuit must satisfy. Moreover, the equation may be integrated once by substituting $p = \dfrac{dy}{dx}$ and then separating variables. Another integration then gives the following results;

$$k \neq 1: \qquad 2y = \frac{k(1-x)^{\frac{(k-1)}{k}}}{1-k} + \frac{k(1-x)^{\frac{(k+1)}{k}}}{1+k} - \frac{2k}{1-k^2}$$

$$k = 1: \qquad \pm 4y = (1-x)^2 - 2\log(1-x) - 1.$$

**The Specified Path is the Unit Circle, centre (2,0)**

It is easy to adapt the program to incorporate this new requirement. All we need to do is to change the path of B; the other changes are to scale the screen so that the circle looks 'circular'.

```
20 WINDOW (-1,-1)-(3,1.9)
40 LINE (-1,0)-(3,0)
41 LINE (0,-1)-(0,1)
60 bx1=bx0+by0/f*SQR(by0^2+(2-bx0)^2)
61 by1=by0+(2-bx0)/f*SQR(by0^2+(2-bx0)^2)
62 PSET (bx1,by1),6
```

The analytic solution of this problem was not obtained until 1926. Nowhere near solution is the pursuit curves when B follows other paths - other conic sections for example. However, such paths may be investigated by incorporating into the program an appropriately changed algorithm for calculating the coordinates of B.

# How stupid can you get?

## David Leng

First of all, for the benefit of readers who are just starting their foundation course, let us go over the meaning of the term *binary operation*. Incidentally, the *bin* in binary rhymes with *vine* not *tin*. Apart from the fact that binary means two, this has nothing to do with the system of counting using only 0s and 1s which is also called binary. What happens is that we take *two* thingies and do something with them to produce a third thingy. Actually *thingy* is not an approved mathematical term; *mathematical entity* or *element* would be more respectable way of putting it. In most cases, and certainly for the purposes of this article the element is a *number* and this makes it easier to understand. So for instance, addition is a binary operation since we can take two numbers, say two and three, do something with them, add them in this case, and produce a third number, five. Since $6 \div 3 = 2$ division is another binary operation. Quite often we want to talk about a binary operation in the abstract (it's a disconcerting habit mathematicians have) so we invent a symbol such as $*$ to represent a general binary operation and come up with expressions such as $a * b = c$.

Again for the benefit of the freshers, we had better go over the meaning of the term *associative*. A binary operation $*$ is said to be *associative* if for all elements $a$, $b$, and $c$, $a * (b * c) = (a * b) * c$. So, for instance, addition is associative, an example being $(2 + 3) + 4 = 2 + (3 + 4)$. Division is an example of a binary operation which is non-associative. This is proved by the example $(24 \div 6) \div 2 = 2$ whereas $24 \div (6 \div 2) = 8$.

At this stage the old lags can wake up and everyone can please answer the following question, preferably honestly. Without thinking about it, and without first doing any examples, is the binary operation *raising to the power of* associative? It must have been about 48 years ago when I was first introduced to this binary operation, although we were not told that it was a binary operation. Since that time I have used maths continuously, taught maths, taught about binary operations and associativity, and I am now well into an OU degree, all the courses of which have had an M in their code. During all this time no one told me the answer to the above question and although I could easily have supplied the answer myself after a few moments thought, I had never considered it, so got it wrong. The operation is not, of course, associative as consideration of the following example shows

$$(4^3)^2 = 4096 \quad \text{but} \quad 4^{(3^2)} = 262144.$$

Using ^ as one of the commonly used symbols for raising to the power of, the above shows us that

$$(4 \,\hat{}\, 3) \,\hat{}\, 2 \neq 4 \,\hat{}\, (3 \,\hat{}\, 2)$$

and in general $(a \mathbin{\hat{}} b) \mathbin{\hat{}} c \neq a \mathbin{\hat{}} (b \mathbin{\hat{}} c)$.

This means, of course, that an expression like $a^{b^c}$ is meaningless unless some convention is understood. I have never been taught about such a convention and have never seen one mentioned in any of the numerous maths books I have looked at. When it comes to concrete examples I think that I must have instinctively known the convention, namely that $a^{b^c}$ means $a^{(b^c)}$ and not $(a^b)^c$. For one thing, I would never dream of keying $e^{x^2}$ into my calculator (if anybody is interested, a Texas TI-81 using an equation operating system) as $(e^x)^2$. For another, the $(a^b)^c$ interpretation would be redundant since it could more easily be written as $a^{(bc)}$ or just $a^{bc}$. No, it was the more abstract question of the interpretation of $a \mathbin{\hat{}} b \mathbin{\hat{}} c$ which caused my momentary stupidity.

On my, and presumably anybody else's, calculator, keying in $e^{x^2}$ just as it stands, using the squaring key and not the power key gives the correct answer. Using the power key does not, and keying in, for instance, $4 \mathbin{\hat{}} 3 \mathbin{\hat{}} 2$ gives the wrong answer of 4096. This is because calculators, and compilers as well for that matter, use the convention that binary operators of equal precedent are applied in the order in which they are keyed in. In other words $a * b * c$ is always interpreted as $(a * b) * c$ and $a * b \bullet c$ as $(a * b) \bullet c$, where $*$ and $\bullet$ have equal priority. This presumably means that the calculator manufacturers assume that their customers are thoroughly conversant with all the conventions. However, even they sometimes stoop to debatable devices. My calculator has a unary minus sign as well as a binary minus sign. However, it gives a higher precedence to raising to a power than to unary minus. I do not agree with this and when a Texas rep. pointed it out to me he explained that when the calculator was being designed, Texas took a straw poll of students on a, presumably American, campus and followed the majority verdict! The QBASIC compiler agrees with Texas but then it does not distinguish between unary and binary minus so the author of the compiler is forgiven. Incidentally, whilst on the subject of operator precedence, my calculator gives priority to implied multiplication over multiplication using the multiplication sign, which is fine if you know about it but is a source of wrong answers if you do not. The same convention is used by Sharps and I suspect others. You had better check what yours does, hadn't you?

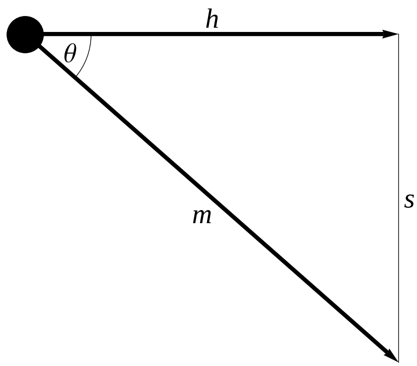
I am sorry if this article has bored readers who have never had the slightest trouble with its subject matter but hope they will think it worthwhile if it has helped someone. I am surely not the only maths student who lapses into—shall I be kind and say occasional—stupidities. Perhaps some of you might like to submit articles with the same title. Self knowledge suggests that I for one shall be at least tempted to do so.

# Solution 168.1 – Clock

> The hour hand of a clock is three inches long. The minute hand is four inches. Determine when the outermost points of the hands of the clock are travelling apart at the fastest speed.

## Ken Greatrix

My first solution to this problem is by computer. Translating the ends of the hands into Cartesian coordinates and calculating the distance between them, for an incremental angle. Then by plotting a graph of the increase in distance I obtain an angle of 41.41°, which gives a time of 3:24.




I then checked this answer by analytical means. It costs little to generalize, so let $h$ and $m$ be the lengths of the hands, $h < m$, and let $\theta$ be the angle between them. By the cosine rule, the distance between the ends of the hands is

$$s = \sqrt{h^2 + m^2 - 2hm\cos\theta}$$

and the speed of separation is

$$\frac{ds}{dt} = \frac{ds}{d\theta}\frac{d\theta}{dt} = \frac{hm\sin\theta}{s}\frac{d\theta}{dt}.$$

To find the maximum I differentiate and set equal to zero. If the clock is running correctly, $d\theta/dt$ is constant. Hence

$$\begin{aligned}
\frac{d^2s}{dt^2} &= \left(-\frac{1}{s^2}\frac{h^2m^2\sin^2\theta}{s} + \frac{hm\cos\theta}{s}\right)\frac{d\theta}{dt} \\
&= -\frac{hm}{s^3}\left(hm(1-\cos^2\theta) - s^2\cos\theta\right)\frac{d\theta}{dt} \\
&= -\frac{hm}{s^3}\left(hm\cos^2\theta - (h^2+m^2)\cos\theta + hm\right)\frac{d\theta}{dt}.
\end{aligned}$$

This is zero when

$$hm\cos^2\theta - (h^2+m^2)\cos\theta + hm = 0;$$

i.e. when

$$\left(\cos\theta - \frac{h}{m}\right)\left(\cos\theta - \frac{m}{h}\right) = 0.$$

So

$$\cos\theta = \frac{h}{m} \quad \text{or} \quad \frac{m}{h}$$

and I choose $h/m$ because it is less than one.

With $h = 3$ and $m = 4$ this gives $\cos\theta = 0.75$ and $\theta = 41.4096°$. The minute hand moves at 360° per hour and the hour hand at 30°, so their difference is 330° per hour. Starting from 3:00 (as in the original diagram), the minute hand moves $90° + 41.4096°$ in 28.8927 minutes and is then moving away (point-wise) from the hour hand at the fastest speed.

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## ADF

This is very interesting. The maximum occurs when the hands make a right-angled triangle, with the (longer) minute hand as hypotenuse. Furthermore, it is possible that there may be an alternative to the differentiate- and-equate-to-zero method of solution. In my experience it is unusual for a quadratic equation to factorize. The simple answer suggests that there ought to be a 'linear' way of looking at the problem. Any ideas?

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## John Bull

### Solution A

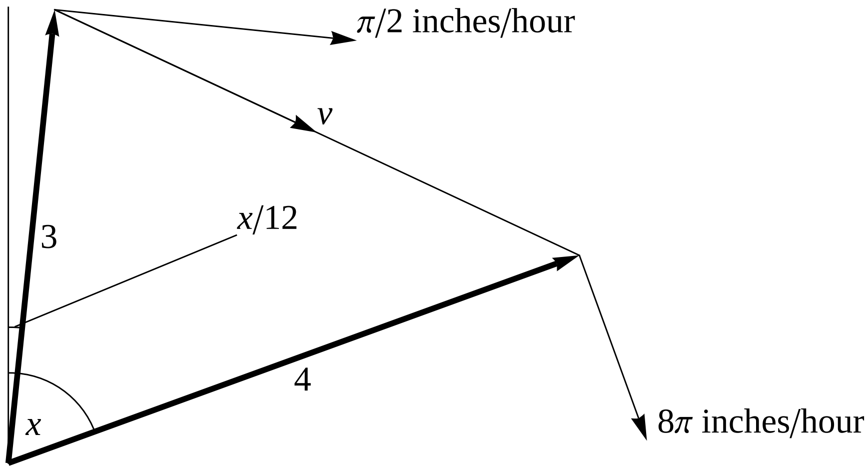
This solution interprets 'travelling apart at the fastest speed' as 'the greatest magnitude of the relative velocity of the tips of the hands'.

Measure the angles of the hands clockwise from 12 o'clock. When the minute hand turns through $x$ radians, the hour hand turns through $x/12$ radians. At any instant, the speed of the tip of the minute hand is $4 \times 2\pi$ inches per hour. The speed of the tip of the hour hand is $3 \times 2\pi/12$ inches per hour (in our quaint imperial units!).

If $v$ is the relative velocity of the tips of the hands, we have

$$\begin{aligned} v^2 &= \left(\frac{\pi}{2}\right)^2 + (8\pi)^2 - 2\frac{\pi}{2}8\pi\cos\left(x - \frac{x}{12}\right) \\ &= \frac{\pi^2}{4} + 64\pi^2 - 8\pi^2\cos\left(\frac{11}{12}x\right) \end{aligned}$$

and $v$ is a maximum when $\cos(11x/12) = -1$; that is when $11x/12 = \pi + 2\pi n$, or when $x = 12/11(2n+1)\pi$, where $n = 0, 1, 2, \ldots$.




Hence, the tips of the hands travel apart fastest (at 26.7 inches per hour) when the hands are directly opposite (which we might have guessed). It occurs when the hour hand is at an angle $\dfrac{(2n+1)\pi}{11}$, with the minute hand at $\dfrac{12}{11}(2n+1)\pi$, or to express this as times, at:

| | | | |
|---|---|---|---|
| 1. | 12 hr, 32 mins, 44 secs | 7. | 7 hr, 5 mins, 27 secs |
| 2. | 1 hr, 38 mins, 11 secs | 8. | 8 hr, 10 mins, 55 secs |
| 3. | 2 hr, 43 mins, 38 secs | 9. | 9 hr, 16 mins, 22 secs |
| 4. | 3 hr, 49 mins, 5 secs | 10. | 10 hr, 21 mins, 49 secs |
| 5. | 4 hr, 54 mins, 33 secs | 11. | 11 hr, 27 mins, 16 secs |
| 6. | 6 o'clock precisely | 12. | back to 1. again. |

**Solution B**

This solution interprets 'travelling apart at the fastest speed' as 'the greatest rate of increase of the distance between the tips of the hands'.

Measure the angles of the hands clockwise from 12 o'clock. When the minute hand turns through $x$ radians, the hour hand turns through $x/12$ radians. The angle between the hands is $x - x/12 = 11x/12$ radians.

The maximum occurs when the cosine of the angle between the hands is 3/4. [See Ken Greatrix's solution on page 11 for the details.] Hence the greatest or least speed of separation is when

$$x \;=\; \frac{12}{11}\left(\cos^{-1}\left(\frac{3}{4}\right) + 2\pi n\right).$$

This is when the minute hand is at $12/11(\pm 0.7227 + 2\pi n)$ radians and the hour hand is at $1/11(\pm 0.7227 + 2\pi n)$, where $n = 0, 1, 2, \ldots$.

It can be seen by inspection that the plus signs give times when the hands are separating, and the negative signs give times when the hands are approaching. Hence the times when the when the tips of the hands are moving apart at the fastest speed are:

| | | | |
|---|---|---|---|
| 1. | 12 hr, 07 mins, 32 secs | 7. | 6 hr, 40 mins, 15 secs |
| 2. | 1 hr, 12 mins, 59 secs | 8. | 7 hr, 45 mins, 43 secs |
| 3. | 2 hr, 18 mins, 26 secs | 9. | 8 hr, 51 mins, 10 secs |
| 4. | 3 hr, 23 mins, 54 secs | 10. | 9 hr, 56 mins, 37 secs |
| 5. | 4 hr, 29 mins, 21 secs | 11. | 11 hr, 02 mins, 04 secs |
| 6. | 5 hr, 34 mins, 48 secs | 12. | back to 1. again. |

# Problem 170.1 – Interesting integral

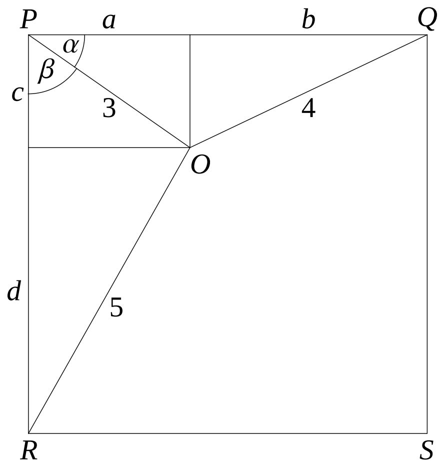
## ADF

Show that

$$\int_0^1 \frac{dx}{x^x} \;=\; \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

# Solution 168.2 – 345 square

$PQRS$ is a square. $PO = 3$ units, $QO = 4$ units and $RO = 5$ units. What is the length of side of the square?




## Martyn Lawrence

The following is one method of solving the problem.

Draw a line from point $O$, perpendicular to line $PQ$, to divide $PQ$ into $a$ and $b$.

Draw a line from point $O$, perpendicular to line $PR$, to divide $PR$ into $c$ and $d$.

From Pythagoras's Theorem:

$$\begin{aligned} a^2 + c^2 &= 3^2 = 9, && (1)\\ a^2 + d^2 &= 5^2 = 25, && (2)\\ b^2 + c^2 &= 4^2 = 16. && (3) \end{aligned}$$

Subtract (1) from (3):

$$b^2 - a^2 = 7. \qquad (4)$$

From (1): $c = \sqrt{9 - a^2}$. From (2): $d = \sqrt{25 - a^2}$. From (4): $b = \sqrt{7 + a^2}$. As the sides of a square are equal, $a + b = c + d$, or $(a + b) - (c + d) = 0$. Therefore, substituting in the above,

$$a + \sqrt{7 + a^2} - \left(\sqrt{9 - a^2} + \sqrt{25 - a^2}\right) = 0.$$

Solving the above yields that $a = 2.459284$. Therefore $b = 3.612212$, $c = 1.718116$, $d = 4.353381$ and

$$\begin{aligned} a + b &= 2.45928 + 3.61221 = 6.07149,\\ c + d &= 1.71811 + 4.35338 = 6.07149. \end{aligned}$$

Therefore the length of side of the square is 6.07149.

## Christine White

Call the sides of the square $t$ and let $\angle OPQ = \alpha$, $\angle OPR = \beta$. Using the cosine formula,

$$\cos\alpha = \frac{3^2 + t^2 - 4^2}{2 \times 3 \times t} \quad \text{and} \quad \cos\beta = \frac{3^2 + t^2 - 5^2}{2 \times 3 \times t}.$$

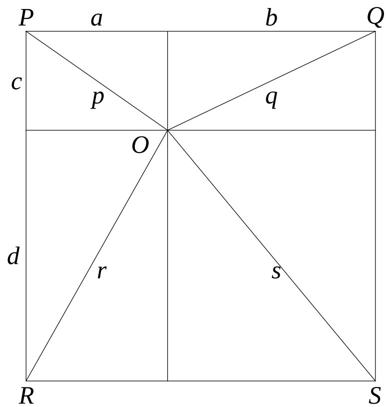
But since $\beta = 90^\circ - \alpha$, $\cos\beta = \sin\alpha$ and since $\cos^2\alpha + \sin^2\alpha = 1$, the two expressions combine to give

$$\begin{aligned}
\frac{(9 + t^2 - 16)^2}{36t^2} + \frac{(9 + t^2 - 25)^2}{36t^2} &= 1,\\
(t^2 - 7)^2 + (t^2 - 16)^2 &= 36t^2,\\
t^4 - 14t^2 + 49 + t^4 - 32t^2 + 256 &= 36t^2,\\
2t^4 - 82t^2 + 305 &= 0,\\
t^2 = \frac{82 \pm \sqrt{82^2 - 4 \times 2 \times 305}}{4} &= \frac{82 \pm \sqrt{4284}}{4},
\end{aligned}$$

leading to $t^2 = 36.863$ or $t^2 = 4.137$ (to 3 decimal places). Clearly, $t$ must be greater than 3 and less than 7. So $t = \sqrt{36.863} = 6.071$.

This led me to wonder how far $O$ is from the fourth corner, and then to a more general problem.

> Given any square and any point $O$ inside it, a distance $p$, $q$, $r$ and $s$ from the four corners, what is the connection between $p$, $q$, $r$ and $s$?




Drawing lines through $O$ parallel to the sides produces right-angled triangles with sides combinations of $a$, $b$, $c$ and $d$. So

$$\begin{aligned}
p^2 + s^2 &= a^2 + c^2 + b^2 + d^2,\\
q^2 + r^2 &= b^2 + c^2 + a^2 + d^2.
\end{aligned}$$

Hence we have $p^2 + s^2 = q^2 + r^2$, immediately, and this holds for a rectangle too.

## Paul Terry

Let the length of the side of the square be $t$ units. Join the diagonal from $R$ to $Q$. This has length $\sqrt{2}t$. Then $A$ = area of triangle $PQR = t^2/2$ = area of three triangles $POQ$, $ROQ$ and $POR$. Using Heron's Formula

$$\begin{aligned} B &= \text{area of } POQ = \left(-\frac{t^4}{16} + \frac{50t^2}{16} - \frac{49}{16}\right)^{1/2}, \\ C &= \text{area of } ROQ = \left(-\frac{4t^4}{16} + \frac{164t^2}{16} - \frac{81}{16}\right)^{1/2}, \\ D &= \text{area of } POR = \left(-\frac{t^4}{16} + \frac{68t^2}{16} - \frac{256}{16}\right)^{1/2}. \end{aligned}$$

Putting $A = (B + C + D)$, and iterating, yields

$$t = 6.07149624\ldots.$$

As a check, call the angles $OPQ$ and $OPR$ $\alpha$ and $\beta$. Since $PQRS$ is a square, $\alpha + \beta$ is a right angle. Then using the cosine formula

$$\begin{aligned} \cos\alpha &= \frac{(6.07149624)^2 + 9 - 16}{6 \times 6.07149624} = 0.819761318 \to \alpha = 34.93909^\circ, \\ \cos\beta &= \frac{(6.07149624)^2 + 9 - 25}{6 \times 6.07149624} = 0.572705248 \to \beta = 55.06091^\circ. \end{aligned}$$

So $\alpha + \beta = 90^\circ$ as required.

---

## John Bull

Maybe there is a wizzo trick way to solve this, but the obvious direct way is quite short.

Let the lengths be as shown in the diagram on page 15, where $t$ is a side. From each of the rectangles, using Pythagoras,

$$c^2 + a^2 = 3^2 = 9, \tag{1}$$

$$(t-c)^2 + a^2 = 5^2 = 25, \quad t^2 - 2ct + c^2 + a^2 = 25, \tag{2}$$

$$(t-a)^2 + c^2 = 4^2 = 16, \quad t^2 - 2at + a^2 + c^2 = 16. \tag{3}$$

From (2) and (1), $t^2 - 2ct = 16$, $2ct = t^2 - 16$. From (3) and (1), $t^2 - 2at = 7$, $2at = t^2 - 7$.

Substitute $c$ and $a$ back into (1). Then

$$(t^2-16)^2+(t^2-7)^2 \;=\; 36t^2, \qquad 2t^4-82t^2+305 \;=\; 0.$$

This gives

$$t^2 \;=\; 36.863068 \;\text{ or }\; 4.136932, \qquad t \;=\; 6.071496 \;\text{ or }\; 2.033945.$$

Hence the solution with the diagram given is 6.071496, but the other solution is valid if the point $O$ is permitted to be outside the square.

---

## John Reade

If $O$ is zero in the complex plane and $OP$ is the real axis then we can take $P=3$, $Q=4e^{-i\theta}$, $R=5e^{i\phi}$, where $\theta=\angle POQ$, $\phi=\angle POR$.

For $P$, $Q$, $R$ to be vertices of a square of a square, we require

$$Q-P \;=\; i(R-P),$$

which gives

$$4e^{-i\theta}-3 \;=\; i(5e^{i\phi}-3),$$

which on equating real and imaginary parts gives

$$4\cos\theta+5\sin\phi \;=\; 3, \qquad 4\sin\theta+5\cos\phi \;=\; 3.$$

Squaring and adding we obtain

$$\sin(\theta+\phi) \;=\; -\frac{23}{40}.$$

We also have

$$\begin{aligned}
|Q-R|^2 &= |4e^{-i\theta}-5e^{i\phi}|^2\\
&= (4\cos\theta-5\cos\phi)^2+4\sin\theta+5\sin\phi)^2\\
&= 41-40\cos(\theta+\phi)\\
&= 41\pm 40\sqrt{1-(23/40)^2}\\
&= 41\pm\sqrt{1071}.
\end{aligned}$$

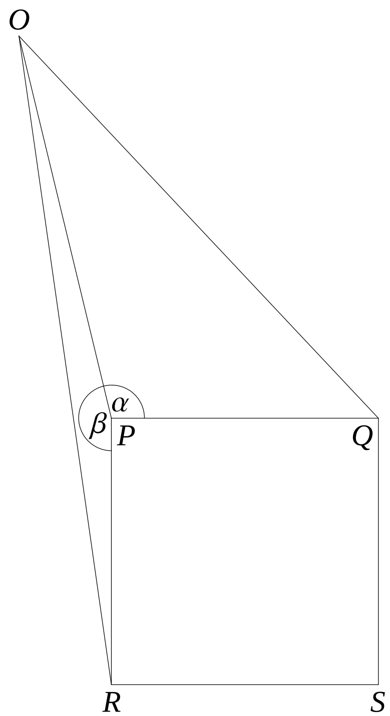
Therefore

$$t^2 \;=\; \frac{41\pm\sqrt{1071}}{2},$$

etc.

---

## Patrick Lee




Let the side of the square be $t$ and let angles $OPQ$ and $OPR$ be $\alpha$ and $\beta$, respectively. By the cosine formula applied to triangle $OPQ$,

$$\cos\alpha = \frac{4^2 - 3^2 - t^2}{6t}.$$

Similarly in triangle $OPR$,

$$\cos\beta = \frac{5^2 - 3^2 - t^2}{6t}.$$

But we also have $\cos\beta = \pm\sin\alpha$ and $\cos^2\alpha + \sin^2\alpha = 1$. Hence

$$\left(\frac{7 - t^2}{6t}\right)^2 + \left(\frac{16 - t^2}{6t}\right)^2 = 1.$$

This quadratic in $t^2$ can be solved to give $t = 6.0715$ or $t = 2.0339$. The smaller value corresponds to the arrangement with $O$ outside the square, in which case $\cos\beta = -\sin\alpha$.

---

## Simon Geard

One method of solving this problem is to use coordinate geometry.

To do this first note that the point $O$ is located at the intersection of three circles centred on $R$, $P$ and $Q$. If we fix $R$ at $(0, 0)$ and use $t$ for the length $PQ$ (and $PR$) then we get the following:

$$\begin{aligned} x^2 + y^2 &= 25, \\ x^2 + (y - t)^2 &= 9, \\ (x - t)^2 + (y - t)^2 &= 16. \end{aligned}$$

These can be combined to give $x$ and $y$ in terms of $t$:

$$x = \frac{t^2 - 7}{2t} \quad \text{and} \quad y = \frac{t^2 + 16}{2t}.$$

Combining these with the first equation gives a quadratic in $t^2$:

$$2t^2 - 82t + 305 \;=\; 0,$$

which has solutions $t = \left(\dfrac{41 \pm \sqrt{1071}}{2}\right)^{1/2}$.

Also solved by **R. M. Boardman** and **Peter Fletcher**.

# Problem 170.2 – Rational square

## R. M. Boardman

Using the diagram for problem 168.2 (page 15), find values for $OP$, $OQ$, $OQ$ and the side of the square that are all integers. This is equivalent to finding a point which is a rational distance from three of the four corners of a unit square.

A well-known unsolved problem is to find a point which is a rational distance from from all four points of a unit square.

# Problem 170.3 – Reciprocals

## ADF

When is it true that

$$\frac{1}{a+b} \;=\; \frac{1}{a} + \frac{1}{b}?$$

What about

$$\frac{1}{a+b+c} \;=\; \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

$$\frac{1}{a+b+c+d} \;=\; \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}, \quad \ldots?$$

# Xmas quiz

## JRH

1. On which day did Good King Wences first look out?
2. At which time of the day did tame shepherds watch their flocks?
3. What happened away in a subsidiary ger?

Any more?

# Solution 168.3 – Fraction

Find decimal digits $a$, $b$, ..., $g$ to make this equality work:

$$\frac{4251935345}{abc1935defg} = \frac{425345}{abcdefg}.$$

## Peter Fletcher

The fraction can be written as

$$\frac{4251935345}{100000000x + 19350000 + y} = \frac{425345}{10000x + y},$$

where $100 \le x \le 999$ and $0 \le y \le 9999$, $x$, $y$ integers. The inequalities restrict $x$ to three digits and $y$ to four. Also $y$ can have leading zeros but $x$ cannot. Both sides are divisible by $5 \times 97 = 485$; hence

$$\begin{aligned} 8766877(10000x + y) &= 877(100000000x + 19350000 + y), \\ 8766000y &= 31230000x + 16969950000, \\ 487y &= 1735x + 942775. \end{aligned}$$

The right-hand side is divisible by 5, which means that $y$ must be divisible by 5. Write $v = y/5$. Then $v$ is an integer and $487v = 347x + 188555$. Solving for $v$ and $x$ in turn gives:

$$v = \frac{347x + 188555}{487}, \qquad \text{(i)} \qquad\qquad x = \frac{487v - 188555}{347}. \qquad \text{(ii)}$$

Starting with $x = 999$, $v$ can be found from (i) and it is not an integer. The integer part of this $v$ can be substituted into (ii) to give a new $x$, which is not an integer. This process is repeated until $v$ and $x$ are both integers. In this way, with the help of a spreadsheet, it was found that $x = 912$, $v = 1037$ and $y = 5185$. The required denominator is therefore 91219355185.

---

## John Bull

Express the question as

$$\frac{4251935345}{100000000x + 19350000 + y} = \frac{425345}{10000x + y},$$

where $y$ has three digits and $x$ has four digits. This simplifies to

$$425151000y = 15146550000x + 8230425750000.$$

Obviously we can divide by 10000 throughout, but we just happen to notice that it also has common factors of 9 and 97, giving $487y = 1735x + 942775$. This Diophantine equation can be solved by one of the methods in M381, but it can be seen by inspection (and some cheating) that there are two solutions: $x = 425$, $y = 3450$ and $x = 912$, $y = 5185$.

Hence we have

$$\frac{4251935345}{42519353450} = \frac{425345}{4253450} \text{ and } \frac{4251935345}{91219355185} = \frac{425345}{9125185},$$

and to help check the second solution, notice the common factor of 877:

$$\frac{425345}{9125185} = \frac{97 \times 877 \times 5}{2081 \times 877 \times 5}.$$

So the answer to the question is 42519353450 or 91219355185.

---

Also solved by **Paul Terry** and **R. M. Boardman**.

---

# Euler relation

## ADF

Colin Davies [M500 **168** 29] asks how to interpret expressions like $1^\pi$ and $1^{\sqrt{2}}$; i.e. 1 raised to the power of an irrational number. I think the only sensible approach involves the definition of the complex exponential function:

$$\exp(i\theta) = e^{i\theta} = \cos\theta + i\sin\theta. \tag{1}$$

It follows immediately from this definition that whenever $n$ is any integer, $1 = \cos 2\pi n + i \sin 2\pi n = e^{2\pi i n}$. Hence if $\theta$ is any number,

$$1^\theta = e^{2\pi i n\theta} = \cos 2\pi n\theta + i \sin 2\pi n\theta,$$

$n = 0, \pm1, \pm2, \ldots$. Thus $1^\theta$ takes every possible value of $\cos 2\pi n\theta + i\sin 2\pi n\theta$ as $n$ runs through the integers. If $\theta$ is an integer, $\cos 2\pi n\theta + i \sin 2\pi n\theta$ is always 1. If $\theta$ is rational, $\theta = a/b$ with $a, b$ integers, $\gcd(a,b) = 1$, then $\cos 2\pi n\theta + i\sin 2\pi n\theta$ has precisely $b$ different values because the cosine and sine functions repeat at intervals of $2\pi$. However, if $\theta$ is irrational, the $\cos 2\pi n\theta + i \sin 2\pi n\theta$ are all different and $1^\theta$ is infinitely-valued. The answer to one of Colin's questions is therefore

$$1^\pi = \{1, \cos 2\pi^2 \pm i\sin 2\pi^2,\ \cos 4\pi^2 \pm i \sin 4\pi^2,\ \cos 6\pi^2 \pm i\sin 6\pi^2,\ \ldots\}.$$

By the way, I was first told of the amazing formula $e^{\pi i} = -1$ when I was at school, about a hundred years ago. I honestly thought that there was something deep and mysterious going on here. Imagine my disappointment to learn, some years later, that it is in fact a trivial result: Put $\theta = \pi$ in (1).

---

# *One Million Places of* $\pi$

## Ken Greatrix

In the mid-70s the then *Guinness Book of Records* described *One Million Places of* $\pi$ [*Un Million de Décimales de* $\pi$ by Jean Guilloud & Martine Bouyer, Commissariat à l'Énergie Atomique, 1973] as the world's most boring book. This jumped on to my list of must-haves (that's my anorak/street-cred ratio gone sky-high!). Needless to say I acquired the book (it was sent free of charge by the authors). Apart from 400 pages of the million places, there are also about a dozen pages of information, some of which I now give below.

I refer to David Singmaster's 1973 entry in 'A history of $\pi$', M500 **168**. The computer used was a CDC 7600 (using the COMPASS language) but strangely the book arrived in an IBM-liveried box, in which I have always kept it. Perhaps this is the source of the IBM 7600 misconception. There was also a translation of some of the extra pages, which explain the formulae used and the statistical tests done to validate the accuracy of the calculations.

According to Guilloud and Bouyer, the formula used (David's VIII) was

$$\frac{\pi}{64} \;=\; \frac{3}{4}\arctan\frac{1}{18} + \frac{1}{2}\arctan\frac{1}{57} - \frac{5}{16}\arctan\frac{1}{239},$$

where each term is calculated from

$$\arctan\frac{1}{K} \;=\; \sum_{p=0}^{\infty}\frac{(-1)^p}{(2p+1)K^{2p+1}}.$$

The general term of this series is found by the recurrence relation.

$$V_{n+1} \;=\; -\left(\frac{2n+1}{2n+3}\right)\frac{1}{K^2}V_n.$$

There is no mention of the extra 1250 digits (Kent) but each term would need to be evaluated to sufficient accuracy, so extra digits may have been calculated. The above calculation was checked by means of the Størmer formula:

$$\frac{\pi}{64} \;=\; \frac{3}{8}\arctan\frac{1}{8} + \frac{1}{8}\arctan\frac{1}{57} + \frac{1}{16}\arctan\frac{1}{239},$$

which David quotes in his 1961 entry for Daniel Shanks.

The total time taken to calculate each term and convert the values to decimal was 23 hours 18 minutes. The term $\arctan 1/8$ required an extra 13 hours 40 minutes.

The last page gives a brief history of the value of $\pi$.

| | |
|---|---|
| The Bible (Book of Kings) | 3 |
| Ptolemy (150 BC) | 3.1416 |
| Tsu Chhung-Chih (*sic*) (same era) | 3.1415927 |
| Al Kashi (1414) | 16 places |
| Ludolph van Ceulen (1596) | 35 |
| Abraham Sharp (1699) | 72 |
| Fautet de Lagny (1719) | 127 |
| Baron Georg von Vega (1794) | 136 |
| Zacharias Dahse (1844) | 200 |
| Thomas Clausen (1847) | 248 |
| W. Lehmann (1853) | 261 |
| William Shanks (1873)* | 707 |
| D. F. Ferguson & J. W. Wrench jnr (1947) | 808 (desk calculator) |
| Smith & J. W. Wrench jnr. (1949) | 1120 (desk calculator) |
| Georges W. Reitwiesner (1949) | 2037 (ENIAC) |
| S. C. Nicholson & J. Jeenel (1954) | 3089 |
| (NORC) François Genuys (January 1958) | 10K (IBM 704, Paris) |
| G. E. Felton (March 1958) | 10021 (PEGASUS, London) |
| D. Shanks & J. W. Wrench jnr (August 1961) | 100265 (IBM 7090, USA) |
| J. Guilloud & J. Filliatre (February 1966) | 250K (IBM 7030, Paris) |
| J. Guilloud & M. Dichampt (February 1967) | 500K (CDC 6600, Paris) |

* Le calcul de 1947 a montré que les décimales de Shanks ètaient fausses à partir de la 528ème. (The 1947 calculation showed that Shanks's value was in error at the 528th decimal.)

There are also references to three publications within the text:

'Calculation of $\pi$ to 100K decimals' by Daniel Shanks and John W. Wrench jnr, *Mathematics of Computation*, vol. **16** no. 2 (January 1962).

'10K decimals of $\pi$' by François Denuys (*sic*), *Chiffres*, Vol. **1** (1966).

*The Art of Computer Programming* by D. E. Knuth.

---

The famous number $\pi$, the nonrepeating, nonterminating decimal that begins 3.146..., sprang up in the study of circles.—*The Man Who Loved Only Numbers*, Hoffman, page 209. Spotted by everybody!

# Fermat's Last Theorem

## Bob Margolis

Peter Griffith's contribution ['Fermat's Last Theorem: A simple proof based on irrational numbers', M500 **169**] worried me a bit (partly because I wasn't sure that I followed the logic) so I sat down to think carefully about his approach. What follows is intended to be *constructive* criticism, and is certainly not directed personally at Peter.

Whilst a 'pure' proof, without recourse to complex numbers, might be more satisfying, it is no more nor less valid than one which does use complex numbers. However, I applaud the idea of searching for a 'pure' proof. A similar debate has occurred at various times with respect to proofs in group theory; those which use representation theory have been regarded as less 'desirable' than pure group-theoretic proofs.

I found the structure of the article rather hard to follow, which probably speaks volumes about my ageing neurons! I think that the plan is as follows. Assume that there exist integers $a$, $b$, $c$ and $n > 2$ such that $a^n + b^n = c^n$, then this inevitably leads to a contradiction.

I found the early mention of rationals a bit of a distraction, since if rationals exist satisfying the above, multiplying through by the $n$th power square of their least common denominator promptly produces integer solutions.

A number of the 'simple to prove' results need stating rather more carefully. For example,

$$t - 1 \ < \ (t^n - 1)^{1/n} \ < \ t$$

is not true for all integers $t$ and $n$. Indeed, for $n = -2$ and $t = 2$, the expression $(t^n - 1)^{1/n}$ is undefined. I think that it is true for $n > 0$ and $t > 2$. (For $n = 1$ and $t = 2$, the right-hand inequality becomes equality.)

Similarly, the second result is: if $n > 0$ and $t > 1$ are integers, then

$$t \ < \ (t^n + 1)^{1/n} \ < \ t + 1.$$

A counterexample to the original statement is provided by $n = -2$ and $t = 2$ because

$$(t^n + 1)^{1/n} = \left(\frac{1}{4} + 1\right)^{-1/2} = \left(\frac{5}{4}\right)^{-1/2} = \left(\frac{4}{5}\right)^{1/2} = \frac{2}{\sqrt{5}} < 2 = t.$$

All this is intended to indicate is that one must be very careful when making assertions, else some pedant (me) will appear brandishing pathological counterexamples.

The irrationality assertions I am happy with, although whether the proofs are 'simple' depends a little on one's mathematical experience. However, the logic where these are put to use is inverted. The sequence which I assembled looked a bit like this.

Suppose there exists a set of (positive) integers breaking Fermat's Last Theorem. That is,

$$c^n \;=\; a^n + b^n, \qquad c > a > b > 1, \qquad n > 2.$$

Then

$$c \;=\; (a^n + b^n)^{1/n} \;<\; (a^n + a^n)^{1/n} \;=\; 2^{1/n}a \;<\; e^{1/n}a.$$

So far, so good. I'm afraid that what Peter asserts next simply doesn't follow. He deduces from $c < e^{1/n}a$, and the fact that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e,$$

that $c < (1 + 1/n)a$.

Unfortunately, the inequalities are against us. It is fairly clear (from the series for $e$) that

$$1 \;<\; \left(1+\frac{1}{n}\right)^n \;<\; e.$$

It follows that we have two inequalities:

$$1+\frac{1}{n} \;<\; e^{1/n}, \quad \text{and} \quad c \;<\; e^{1/n}a.$$

We *cannot* deduce the asserted inequality from this. All we can do (and this requires some analysis) is say the assertion will be true *for sufficiently large $n$*. At this point, I think that the rest of the argument has collapsed.

I'd like to make a further point. The tail end of Peter's argument is probabilistic. This also doesn't hold water. Just because there is a very small probability of something occurring, it doesn't mean that it won't. (See all queues for lottery tickets.) His 'and so on ... ' conceals the adding up of an infinite number of infinitesimal probabilities and, as Newton and others found, such sums can be non-zero. More positively, can anyone fix the holes that I've detected?

I found the discussion of the $n = 2$ case very interesting. For the sake of accuracy, there are some points to make.

- The special case relevant is the identity $(r^2-q^2)^2+(2rq)^2 = (r^2+q^2)^2$.
- For $r$ and $q$ integers, the results will be consistently *integers*, not just rationals.
- There are corresponding identities for higher powers, but they have no connection with Fermat's Last Theorem as there are too many terms.

I regret to say it, but there appear to be some problems which have failed to yield to attempts to find 'elementary' solutions. An impressive phalanx of professional and amateur mathematicians have tackled Fermat (and Goldbach's Conjecture). One has a proof (but not an elementary one), the other does not (yet). Many years after the Odd Order Theorem (all finite groups of odd order are soluble) was first proved, the proofs are still far from elementary. The proof of the Four-colour Theorem has been simplified somewhat, but not hugely.

But: please keep chipping away.

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# Letters to the Editors

## Feedback

Dear Tony Forbes,

I have recently joined the M500 Society and magazine **169** arrived yesterday, the previous edition arrived a few weeks ago. Most of the contents is way above my head but I was quite tempted to ask for an Italian paperback edition of *Fermat's Last Theorem* (struggling with the language and, I suspect, mathematical content would have been good for the soul). But the contact address was an e-mail so I dropped what was no more than a passing thought.

Ledger White writes about *Countdown* in **169** and gives an e-mail contact so, should I have wished, pursuing this would have been difficult.

If the MOUTHS 4/99 list is representative of the membership then 13/20 (65%) can use e-mail and 7/20 (35%) can use alternatives like snail mail. It is not apparent to me how non-e-mail users can efficiently pursue topics of interest. In the next edition could there be something in the 'small print' to clarify, ...

Yours sincerely,

**Josephine Stubbs** MDST242 student

PS. ... or—as some of us are getting on a bit—large print might be kinder.

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[**ADF**—I see the problem. In the past, everybody was on the MOUTHS list, so communication was always easy. Not so nowadays. Authors who are not on MOUTHS: If you wish to correspond with readers, I need permission (explicitly and clearly) to publish your postal address with your article. Or, better still, *please get yourself on the list.*]

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## Lottery

Dear Jeremy,

I take the same view as Eddie Kent [M500 **167**] that it is better to benefit of the public largesse from the lottery than to contribute to it. It seems to me that just one of the many ways of dying unexpectedly is in a traffic accident; about ten people do it that way every day. However as less than that number tend to have a big lottery win in a week: I am unlikely to survive long enough to collect a win.

*Problem:* What proportion of people who die in traffic accidents miss collecting a big lottery win by that method?

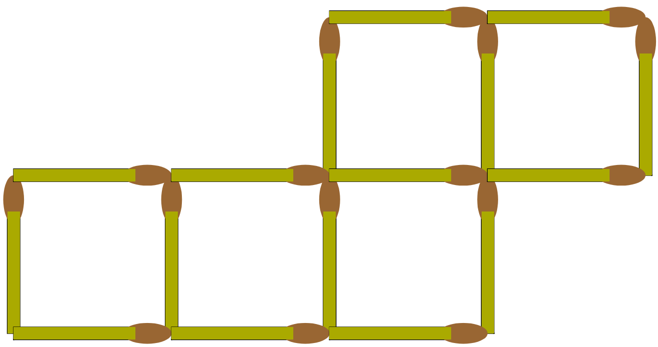
I was surprised that the first 75 draws had no instances of all even or all odd numbers. There are evidently 24 even numbers and 25 odd, but the chance of getting an even or odd run of 6 balls is near enough $1/2^5 = 1/32$. So the probability of not having such a run in 75 trials is $(31/32)^{75}$ = about 1/11.

**Colin Davies**

# Sixteen matches

## EK

I think this is clear. It is 16 matches arranged to make 5 squares. You have to move two matches and end with four squares.



'Only two things are infinite, the universe and human stupidity, and I'm not sure about the former.'— Albert Einstein.

# Twenty-five years ago

## Extracts from M500 16

**Eddie Kent**—As an attempt to enumerate the physical characteristics of a mathematician I offer the following: they are hairy (That is, they generally tend to have some hair somewhere on their body). Notice the converse is not necessarily true. I don't think any more can be said with certainty. This is probably enough, however, since it excludes all but a finite number of computers.

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**Sinbad**—M500 was never exclusively intended to be a showcase for high fliers, but rather an exercise in communication and mutual assistance between ordinary guys, like you and me and our (allegedly) chubby ed.

In my non-introspective scale of values it is far more important that an Arts student should actually want to read a mathematics news sheet, than that some far-off professor in a distant ivory tower might chunter disapprovingly into his beard.

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**Ed**—34-28-40.

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**A Glassblower for $5^1/_2$ Years**—'A Computer programmer stands in the same relation to mathematics as a glassblower does to physics', said an unknown professor of mathematics.

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# Winter Week-end

## Norma Rosier

This is an annual residential Weekend to dispel the withdrawal symptoms due to courses finishing in October and not starting again until February. It is an opportunity to get together with friends, old and new, and do some interesting mathematics.

The **nineteenth M500 Society WINTER WEEK-END** will be held at **Nottingham University** from **Friday 7th to Sunday 9th January, 2000**. Ian Harrison is running it and the theme will be announced later. It promises to be as much fun as ever!

Cost (not yet fixed): approximately £120 for M500 members, £125 for non- members. This includes accommodation and all meals from dinner on Friday to lunch on Sunday. Please send a stamped, addressed envelope for booking form to **Norma Rosier** after September 12th, when all details should be known.