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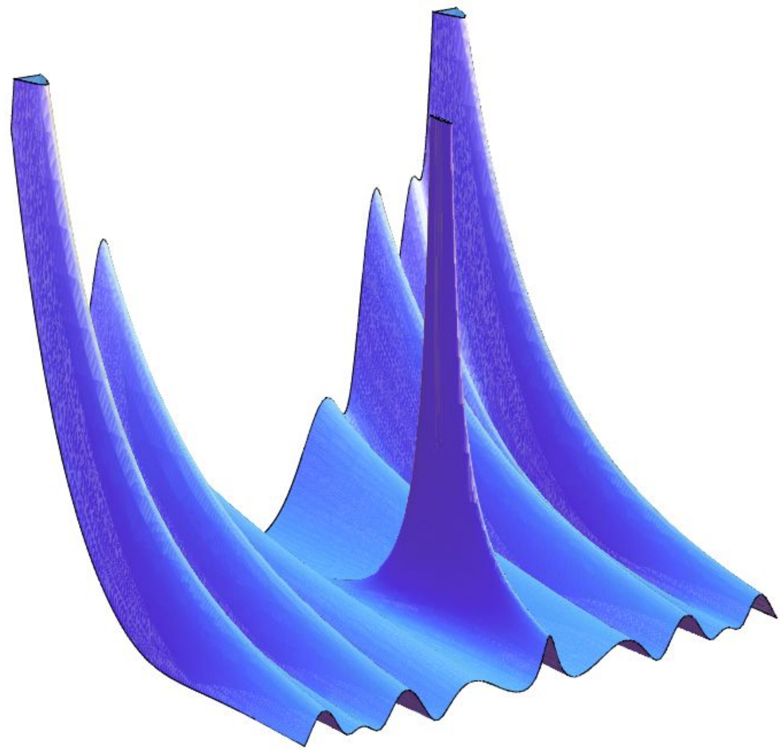
M500 175



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The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

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The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

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Complex geometry

Barry Lewis

Sebastian Hayes's original problem [M500 **169** 2] was new to me, but it inspired some very interesting results and generalisations.

We will need the following basic results about complex numbers in our investigation.

Results

- Sum of conjugates $\overline{(z+w)} = \bar{z} + \bar{w}$.
- Powers of conjugates $\overline{(z^n)} = (\bar{z})^n$.
- The length of a complex number z is denoted by $|z|$ and we have
$$|z|^2 = z.\bar{z}.$$
- If the complex number z lies on the unit circle then $|z|^2 = 1^2 = 1$ so that $z.\bar{z} = 1$.
- If ω is an nth root of 1, then it is a root of the equation $z^n - 1 = 0$. If $z \neq 1$ then dividing by $z - 1$ gives
$$1 + z + z^2 + \cdots + z^{n-1} = 0$$
and substituting ω for z gives
$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$$
i.e. $$\sum_{r=0}^{n-1} \omega^r = 0$$
and $$\sum_{r=0}^{n-1} \omega^{n-r} = 0$$
since this is the same sum in reverse order.

But let me briefly recap the result that was a generalization of Sebastian's starting point. Consider a regular n-sided polygon inscribed in a unit circle. Assume that its vertices are the points $A_0, A_1, \cdots, A_{n-1}$. The roots of the equation

$$z^{n-1} + z^{n-2} + \cdots + z + 1 = 0$$

are just $\omega, \omega^2, \cdots, \omega^{n-1}$ where each term is a complex n th root of unity. These, together with 1 (which is itself one of the points A – and for convenience, I denote it by A_0) are simply the vertices of the regular polygon in question.

So the roots of the equation

$$(1-w)^{n-1} + (1-w)^{n-2} + \cdots + (1-w) + 1 = 0$$

are

$$1-\omega, 1-\omega^2, \cdots, 1-\omega^{n-1}$$

and the product of these is just ($\pm$) the constant term in the equation, so that

$$(1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1}) = (-1)^{n-1} n$$

and taking absolute values gives the result:

$$|1-\omega|.|1-\omega^2|.\cdots.|1-\omega^{n-1}| = n.$$

But these terms are simply the chord lengths:

$$|1-\omega| = (A_0 A), \cdots, |1-\omega^{n-1}| = (A_0 A_{n-1}).$$

So this gives us the following result.

Theorem 1

If the vertices of a regular n-sided polygon inscribed in a unit circle are $A_0, A_1, A_2, \cdots, A_{n-1}$ then

$$(A_0 A_1).(A_0 A_2).\cdots.(A_0 A_{n-1}) = n..$$

But there are other results to be found by similar 'complex' methods. Suppose that P is any other point on the unit circle, again with a regular n-sided polygon inscribed within. Let P be represented by the complex number z.

Then we have:

$$\begin{aligned}\sum_{r=0}^{n-1}(PA_r)^2 &= \sum_{r=0}^{n-1}|z-\omega^r|^2 \\ &= \sum_{r=0}^{n-1}(z-\omega^r).\overline{(z-\omega^r)} \\ &= \sum_{r=0}^{n-1}(z-\omega^r).(\bar{z}-\overline{\omega^r}).\end{aligned}$$

But by the geometry of the polygon, we have

$$\overline{\omega^r} = \omega^{n-r}$$

and

$$\begin{aligned}\sum_{r=0}^{n-1}(PA_r)^2 &= \sum_{r=0}^{n-1}(z-\omega^r).(\bar{z}-\omega^{n-r}) \\ &= \sum_{r=0}^{n-1}(z\bar{z}-z\omega^{n-r}-\bar{z}\omega^r+\omega^r\omega^{n-r}) \\ &= \sum_{r=0}^{n-1}(2)-z\sum_{r=0}^{n-1}\omega^{n-r}-\bar{z}\sum_{r=0}^{n-1}\omega^r.\end{aligned}$$

Each of the last two sums is zero, so we have

$$\sum_{r=0}^{n-1}(PA_r)^2 = \sum_{r=0}^{n-1}2 = 2n.$$

We can state this as another result.

Theorem 2

If the vertices of a regular n-sided polygon inscribed in a unit circle are $A_0, A_1, A_2, \cdots, A_{n-1}$ and P is any other point on the circle, then

$$(PA_0)^2+(PA_1)^2+\cdots+(PA_{n-1})^2 = 2n.$$

There is a corresponding result for $(PA_0)^4 + (PA_1)^4 + \cdots + (PA_{n-1})^4$:

$$\sum_{r=0}^{n-1}(PA_r)^4 = \sum_{r=0}^{n-1}(z-\omega^r)^2.(\bar{z}-\overline{\omega^r})^2$$

$$= \sum_{r=0}^{n-1}(z\bar{z} - z\omega^{n-r} - \bar{z}\omega^r + \omega^r\omega^{n-r})^2$$

$$= \sum_{r=0}^{n-1}(2 - z\omega^{n-r} - \bar{z}\omega^r)^2$$

$$= \sum_{r=0}^{n-1}(4 + z^2\omega^{2(n-r)} + \bar{z}^2\omega^{2r} - 4z\omega^{n-r} - 4\bar{z}\omega^r + 2z\bar{z}\omega^{n-r}\omega^r).$$

This is simply

$$= \sum_{r=0}^{n-1}(6 + z^2(\omega^2)^{n-r} + \bar{z}^2(\omega^2)^r - 4z\omega^{n-r} - 4\bar{z}\omega^r)$$

$$= 6n + z^2\sum_{r=0}^{n-1}(\omega^2)^{n-r} + \bar{z}^2\sum_{r=0}^{n-1}(\omega^2)^r - 4z\sum_{r=0}^{n-1}\omega^{n-r} - 4\bar{z}\sum_{r=0}^{n-1}\omega^r.$$

The latter two sums we know already are zero; what about the other two ? We have

$$\sum_{r=0}^{n-1}\omega^r = 0$$

and squaring this we get

$$\left(\sum_{r=0}^{n-1}\omega^r\right)^2 = 0$$

$$\Rightarrow \quad \sum_{r=0}^{n-1}(\omega^r)^2 + 2\left(\sum_{i,j=0}^{n-1}\omega^i\omega^j\right) = 0$$

$$\Rightarrow \quad \sum_{r=0}^{n-1}(\omega^2)^r + 2\left(\sum_{i,j=0}^{n-1}\omega^i\omega^j\right) = 0.$$

Now ω^i, ω^j are the roots of the equation $z^n - 1 = 0$ and so the sum of the product of them, taken two at a time is $(\pm)$ the coefficient of z^{n-2}.

But this coefficient is just zero and so

$$\sum_{r=0}^{n-1} (\omega^2)^r = 0.$$

This also means that $\sum_{r=0}^{n-1} (\omega^2)^{n-r} = 0$. So now we have the following result.

Theorem 3

If the vertices of a regular n-sided polygon inscribed in a unit circle are $A_0, A_1, A_2, \cdots, A_{n-1}$ and P is any other point on the circle, then

$$(PA_0)^4 + (PA_1)^4 + \cdots (PA_{n-1})^4 = 6n.$$

Can we generalize this to non-regular polygons ? Consider any triangle with vertices, A_0, A_1, and A_2, which are represented in the complex plane by the complex numbers a_0, a_1, and a_2. Let G be the point represented by by,

$$\tfrac{1}{3}(a_0 + a_1 + a_2).$$

We refer to G as the *centroid* of the triangle. Then we have

$$\begin{aligned}(GA_0)^2 &= \left(\tfrac{1}{3}(a_0 + a_1 + a_2) - a_0\right)\overline{\left(\tfrac{1}{3}(a_0 + a_1 + a_2) - a_0\right)} \\ &= \tfrac{1}{9}\left(a_1 + a_2 - 2a_0\right)\left(\overline{a_1} + \overline{a_2} - 2\overline{a_0}\right) \\ \Rightarrow \quad 9(GA_0)^2 &= ((a_1 - a_0) + (a_2 - a_0))\left((\overline{a_1} - \overline{a_0}) + (\overline{a_2} - \overline{a_0})\right).\end{aligned}$$

In a similar way we have:

$$\begin{aligned}9(GA_1)^2 &= ((a_0 - a_1) + (a_2 - a_1))\left((\overline{a_0} - \overline{a_1}) + (\overline{a_2} - \overline{a_1})\right), \\ 9(GA_2)^2 &= ((a_0 - a_2) + (a_1 - a_2))\left((\overline{a_0} - \overline{a_2}) + (\overline{a_1} - \overline{a_2})\right).\end{aligned}$$

If we multiply out the first of these, we get:

$$\begin{aligned}9(GA_0)^2 &= (a_1-a_0).(\overline{a_1}-\overline{a_0})+(a_2-a_0).(\overline{a_2}-\overline{a_0})\\ &\quad + (a_1-a_0).(\overline{a_2}-\overline{a_0})+(a_2-a_0).(\overline{a_1}-\overline{a_0})\\ &= (A_0A_1)^2+(A_0A_2)^2+(a_1-a_0).(\overline{a_2}-\overline{a_0})+(a_2-a_0).(\overline{a_1}-\overline{a_0})\end{aligned}$$

so that each expression has two squares and two 'cross terms'. In full, we have:

$$9(GA_0)^2 = (A_0A_1)^2+(A_0A_2)^2+(a_2-a_0).(\overline{a_1}-\overline{a_0})+(a_1-a_0).(\overline{a_2}-\overline{a_0}),$$

$$9(GA_1)^2 = (A_0A_1)^2+(A_1A_2)^2+(a_0-a_1).(\overline{a_2}-\overline{a_1})+(a_2-a_1).(\overline{a_0}-\overline{a_1}),$$

$$9(GA_2)^2 = (A_0A_2)^2+(A_1A_2)^2+(a_0-a_2).(\overline{a_1}-\overline{a_2})+(a_1-a_2).(\overline{a_0}-\overline{a_2}).$$

Adding these gives

$$9\sum_{r=0}^{2}(GA_r)^2 = 2(A_0A_1)^2+2(A_0A_2)^2+2(A_1A_2)^2+(\text{sum of 'cross terms'}).$$

Consider $(a_1-a_0)(\overline{a_2}-\overline{a_0})$ in the first line. There is a term like it in the second line $(a_0-a_1)(\overline{a_2}-\overline{a_1})$. If we add these two terms we get

$$(a_1-a_0)(\overline{a_2}-\overline{a_0}-\overline{a_2}+\overline{a_1}) = (a_1-a_0)(\overline{a_1}-\overline{a_0}) = (A_0A_1)^2.$$

This is no coincidence. From the first such term in the first line and the first such term in the third line we get

$$(a_2-a_0)(\overline{a_1}-\overline{a_0}-\overline{a_1}+\overline{a_2}) = (a_2-a_0)(\overline{a_2}-\overline{a_0}) = (A_0A_2)^2$$

and from the last such term in the second and third lines we get,

$$(a_2-a_1)(\overline{a_0}-\overline{a_1}-\overline{a_0}+\overline{a_2}) = (a_2-a_1)(\overline{a_2}-\overline{a_1}) = (A_1A_2)^2.$$

So now we have

$$9\sum_{r=0}^{2}(GA_r)^2 = 3(A_0A_1)^2 + 3(A_0A_2)^2 + 3(A_1A_2)^2$$

$$\Rightarrow \quad 3\sum_{r=0}^{2}(GA_r)^2 = (A_0A_1)^2 + (A_0A_2)^2 + (A_1A_2)^2$$

which we set out as another result.

Theorem 4

If the vertices of any triangle are A_0, A_1, A_2 and G is its centroid then

$$3\left((GA_0)^2 + (GA_1)^2 + (GA_2)^2\right) = (A_0A_1)^2 + (A_0A_2)^2 + (A_1A_2)^2.$$

What about a quadrilateral? Let the vertices of the quadrilateral be A_0, A_1, A_2 and A_3 represented by the complex numbers a_0, a_1, a_2 and a_3. Also let,

$$G = \tfrac{1}{4}\left(a_0 + a_1 + a_2 + a_3\right).$$

Working in the same way as before, we have

$$(GA_0)^2 = \left(\tfrac{1}{4}(a_0 + a_1 + a_2 + a_3) - a_0\right)\overline{\left(\tfrac{1}{4}(a_0 + a_1 + a_2 + a_3) - a_0\right)}$$

and so

$$16(GA_0)^2 = \left((a_1 - a_0) + (a_2 - a_0) + (a_3 - a_0)\right)\overline{\left((a_1 - a_0) + (a_2 - a_0) + (a_3 - a_0)\right)},$$

$$= \left((a_1 - a_0) + (a_2 - a_0) + (a_3 - a_0)\right)\left((\overline{a_1} - \overline{a_0}) + (\overline{a_2} - \overline{a_0}) + (\overline{a_3} - \overline{a_0})\right),$$

$$16(GA_1)^2 = \left((a_0 - a_1) + (a_2 - a_1) + (a_3 - a_1)\right)\left((\overline{a_0} - \overline{a_1}) + (\overline{a_2} - \overline{a_1}) + (\overline{a_3} - \overline{a_1})\right),$$

$$16(GA_2)^2 = \left((a_0 - a_2) + (a_1 - a_2) + (a_3 - a_2)\right)\left((\overline{a_0} - \overline{a_2}) + (\overline{a_1} - \overline{a_2}) + (\overline{a_3} - \overline{a_2})\right),$$

$$16(GA_3)^2 = \left((a_0 - a_3) + (a_1 - a_3) + (a_2 - a_3)\right)\left((\overline{a_0} - \overline{a_3}) + (\overline{a_1} - \overline{a_3}) + (\overline{a_2} - \overline{a_3})\right).$$

When we multiply out each line, we get

3 squares and $3.2 = 6$ 'cross terms'.

Consider the cross term $(a_2 - a_0)(\overline{a_3} - \overline{a_0})$ in the first line. The corresponding terms like it will be of the form,

$$(a_0 - a_2)(.......)$$

which occurs in the third line – that for $(GA_2)^2$. There are two possibilities:

$$(a_0 - a_2)(\overline{a_1} - \overline{a_2}) \text{ and } (a_0 - a_2)(\overline{a_3} - \overline{a_2}).$$

The latter is the one we want since,

$$\begin{aligned}
&(a_2 - a_0)(\overline{a_3} - \overline{a_0}) \;+\; (a_0 - a_2)(\overline{a_3} - \overline{a_2}) \\
&\qquad = (a_0 - a_2)(\overline{a_3} - \overline{a_0} - \overline{a_3} + \overline{a_2}) = (a_0 - a_2)(\overline{a_2} - \overline{a_0}) \\
&\qquad = (A_0A_2)^2.
\end{aligned}$$

Every cross term is linked to exactly one other in this way. So now we know that the whole array, when we add all the lines, consists of,

$$\begin{aligned}
4(3 \text{ squares} + 6 \text{ cross terms}) &= 12 \text{ squares} + \tfrac{24}{2} \text{ squares} \\
&= 24 \text{ squares.}
\end{aligned}$$

But there are just 6 such squares:

$$(A_0A_1)^2, (A_0A_2)^2, (A_0A_3)^2, (A_1A_2)^2, (A_1A_3)^2 \text{ and } (A_2A_3)^2$$

and so we have

$$16\sum_{r=0}^{3}(GA_r)^2 = 4\begin{pmatrix} (A_0A_1)^2 + (A_0A_2)^2 + (A_0A_3)^2 \\ +(A_1A_2)^2 + (A_1A_3)^2 + (A_2A_3)^2 \end{pmatrix}.$$

This is our next result.

Theorem 5

If the vertices of any quadrilateral are A_0, A_1, A_2, A_3 represented by the complex numbers a_0, a_1, a_2, a_3 and G is the point,

$$G = \tfrac{1}{4}(a_0 + a_1 + a_2 + a_3)$$

then

$$4\sum_{r=0}^{3}(GA_r)^2 = (A_0A_1)^2 + (A_0A_2)^2 + (A_0A_3)^2$$
$$+ (A_1A_2)^2 + (A_1A_3)^2 + (A_2A_3)^2.$$

The method generalizes and we have the following.

Theorem 6

If the vertices of any n sided polygon are $A_0, A_1, \cdots, A_{n-1}$ represented by the complex numbers $a_0, a_1, \cdots, a_{n-1}$ and G is the point,

$$G = \frac{1}{n}(a_0 + a_1 + \cdots + a_{n-1})$$

then

$$n\sum_{r=0}^{n-1}(GA_r)^2 = \sum_{\substack{i,j=0 \\ i<j}}^{n-1}(A_iA_j)^2 \ .$$

Now suppose that P is any other point, represented by z, and that the origin is O. So we have

$$(PA_0)^2 = (z - a_0)(\overline{z} - \overline{a_0}) = z\overline{z} - z\overline{a_0} - \overline{z}a_0 + a_0\overline{a_0}$$
$$= (OP)^2 - z\overline{a_0} - \overline{z}a_0 + (OA)^2.$$

Summing over all the vertices of our n-sided polygon, we get

$$\sum_{r=0}^{n-1}(PA_r)^2 = n(OP)^2 - z\sum_{r=0}^{n-1}\overline{a_r} - \bar{z}\sum_{r=0}^{n-1}a_r + \sum_{r=0}^{n-1}(OA_r)^2$$

$$= n(OP)^2 - z\overline{\left(\sum_{r=0}^{n-1}a_r\right)} - \bar{z}\sum_{r=0}^{n-1}a_r + \sum_{r=0}^{n-1}(OA_r)^2$$

But we have some flexibility over our choice of origin O. Suppose that we choose it to be at the point G, so that $G \equiv O$. Then

$$\frac{1}{n}\sum_{r=0}^{n-1}a_r = 0 \quad \Rightarrow \quad \sum_{r=0}^{n-1}a_r = 0$$

and we have:

$$\sum_{r=0}^{n-1}(PA_r)^2 = n(GP)^2 + \sum_{r=0}^{n-1}(GA_r)^2$$

and using **Theorem 6** we have our final result.

Theorem 7

If the vertices of any n sided polygon are $A_0, A_1, \cdots, A_{n-1}$ represented by the complex numbers $a_0, a_1, \cdots, a_{n-1}$ and G is the point defined by

$$G = \tfrac{1}{n}(a_0 + a_1 + \cdots + a_{n-1})$$

and P is any other point then

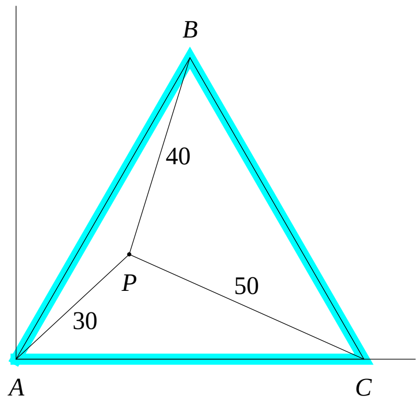
$$\sum_{r=0}^{n-1}(PA_r)^2 = n(GP)^2 + \frac{1}{n}\sum_{\substack{i,j=0\\ i<j}}^{n-1}(A_iA_j)^2 .$$

And what is the significance of this result ? One way of expressing it is in the following form. Given any set of points in the plane whose centroid is G and any other point P.

Then if P describes a circle whose centre is G, the sum of the squares of the distances of P from each of the points is constant.

Solution 172.1 – 345 triangle

An equilateral triangle encloses a point. The point is 30 metres from one corner, 40 metres from another corner, and 50 metres from the remaining corner. What is the length of the triangle's side?



The problem is popular and interesting. We have already published three (possibly four) different approaches [M500 **174**, 17–21]. We make no apology for keeping it open.

Elsie Page

My solution is obtained algebraically using coordinate geometry and the intersection of circles.

Triangle ABC is equilateral and P is the point. Let C have coordinates $(2s, 0)$. Then B is the point $(s, \sqrt{3}s)$. Let P be (x, y). Then

$$x^2 + y^2 = 900, \tag{1}$$
$$(x - 2s)^2 + y^2 = 2500, \tag{2}$$
$$(x - s)^2 + (y - \sqrt{3}s)^2 = 1600. \tag{3}$$

From (1) and (2) we obtain

$$x = \frac{s^2 - 400}{s}; \tag{4}$$

from (2), (3) and (4),

$$y = \frac{s^2 + 50}{\sqrt{3}s}. \tag{5}$$

From (1), (4) and (5) we obtain $s^4 - 1250s^2 + 120625 = 0$. Whence $2s = 67.66\,\mathrm{m}$.

John Bull

I published this problem in 1995 [M500 **142**, page 13]. As I pointed out then, a geometrical solution can be found in *Mathematical Quickies* by Charles W. Trigg, Dover, ISBN 0-486-24949-2, Problem 201. The solution in the book works out rather neatly because of the particular choice of numbers $3 : 4 : 5$.

In the previous article I reduced the problem to a single equation for numerical solution, which of itself presented an interesting problem. This time I give a boringly straightforward analytical solution requiring no more than a bit of tedious algebra. [ADF—The details are omitted. John's solution is similar to Elsie's on page 11 except that John first scales the problem by a factor of 10 and obtains a simpler quartic equation in s: $16s^4 - 200s^2 + 193 = 0$.]

I also came across the problem in another book, *Which Way did the Bicycle Go? ... and Other Interesting Mathematical Mysteries* by Joseph Konhauser, Dan Velleman and Stan Wagon (Mathematical Association of America, Dolciani Mathematical Expositions – 18, 1996). The solution provided by L. -S. Hahn (for distances 3, 4, 5) is $\sqrt{25 + 12\sqrt{3}}$. Hahn then goes on to point out an interesting duality with a slightly different problem: As before, triangle ABC is equilateral, $AP = 3$, $BP = 4$. But now the side of the triangle is 5. What is CP? The surprising result is $\sqrt{25 + 12\sqrt{3}}$, the same answer as for the original problem.

Bryan Orman

No doubt others have 'solved' this problem. I'm surprised that mathematicians resort to the computer to tackle basic problems that can be solved by elementary trigonometry, but I'm from the old school!

An equilateral triangle has a point with Pythagorean-triple distances from its vertices.

Triple $\{a, b, c\}$, $c^2 = a^2 + b^2$, $AP = a$, $BP = b$, $CP = c$. Triangle side t.

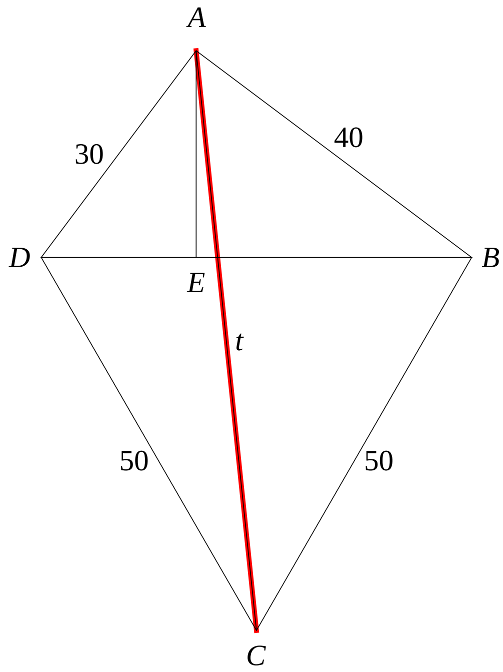
$$t^2 = c^2 - 2ab\cos APB.$$

But angle APB is always 150 degrees; so $t^2 = c^2 + ab\sqrt{3}$.

This last formula indicates that there is a higher structure to the result. Others might know, I don't!

C.C.T.V. Have you remembered to pay for your fuel. YOUR ON CAMERA —Notice at a petrol station. [Spotted by JRH.]

Andrew Baker



Find the length of the triangle side, t, say.

Take the triangles in the diagram on page 11 that are bounded by sides $(40, 50, t)$ and $(30, 50, t)$. Rearrange them as shown here on the right. I have flipped them over, though this is not necessary, relabelled the vertices, and added two extra lines. The original triangles are ABC and ADC; AC has length t.

Angle ACB and angle ACD correspond to the two angles in the bottom right hand corner of the original diagram, so $\angle BCD = \pi/3$.

Since $BC = CD$, triangle BCD is equilateral, and $BD = 50$.

Since triangle ABD has sides 30, 40, 50, it is Pythagorean, and $\angle BAD = \pi/2$. Line AE is the perpendicular dropped from A to BD. Then triangles ABD, ABE and ADE are all similar. Hence $DE = 18$ and $AE = 24$.

I found it most useful to consider this diagram as being on the (x, y)-plane, with D being at $(0, 0)$ and BD on the x-axis. Then $A = (18, 24)$ and $C = (25, -25\sqrt{3})$. Thus

$$\begin{aligned} t^2 &= (25-18)^2 + (-25\sqrt{3} - 24)^2 \\ &= 49 + 1875 + 1200\sqrt{3} + 576 \\ &= 1200\sqrt{3} + 2500. \end{aligned}$$

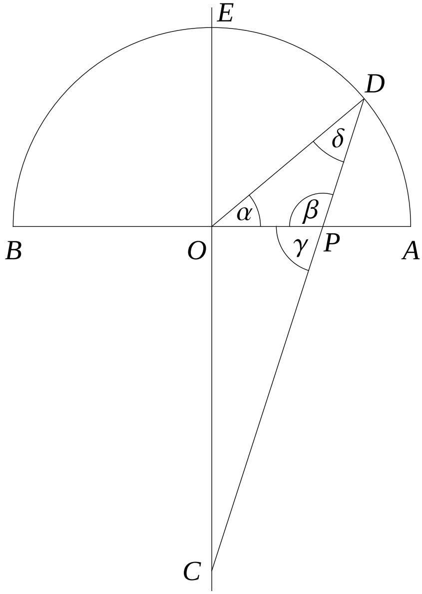
Hence $t = 10\sqrt{12\sqrt{3} + 25} \approx 67.66$.

Using this same argument, we can generalize this to any case where the lengths of the 'spokes' are in a Pythagorean relationship (but not otherwise). That is, if the lengths are a, b, c, where $a^2 + b^2 = c^2$, then $t^2 = ab\sqrt{3} + c^2$.

A better protractor

Ken Greatrix

Recall Dirk Bouwens's construction [M500 **171** 18], right, where we assume that $OA = 1$. If $OC = \sqrt{3}$, the 'conjecture' is that $\alpha = AP \times 90^\circ$, at least approximately.



At first I thought this might be an ancient Greek idea but I couldn't find a valid construction. So I decided on a more modern approach.

Let $OC = c$ and $AP = \lambda$. As in Alan Slomson's article [M500 **173** 16], we have

$$\begin{aligned}\sin\gamma &= \frac{c}{\sqrt{(1-\lambda)^2+c^2}},\\ \sin\beta &= \sin\gamma,\\ \cos\beta &= -\cos\gamma\\ &= \frac{\lambda-1}{\sqrt{(1-\lambda)^2+c^2}},\end{aligned}$$

$$\sin\delta = (1-\lambda)\sin\beta, \qquad \cos\delta = \frac{\sqrt{(1-\lambda)^2+c^2-(1-\lambda)^2c^2}}{\sqrt{(1-\lambda)^2+c^2}},$$

$$\alpha = \pi-\beta-\delta$$

and

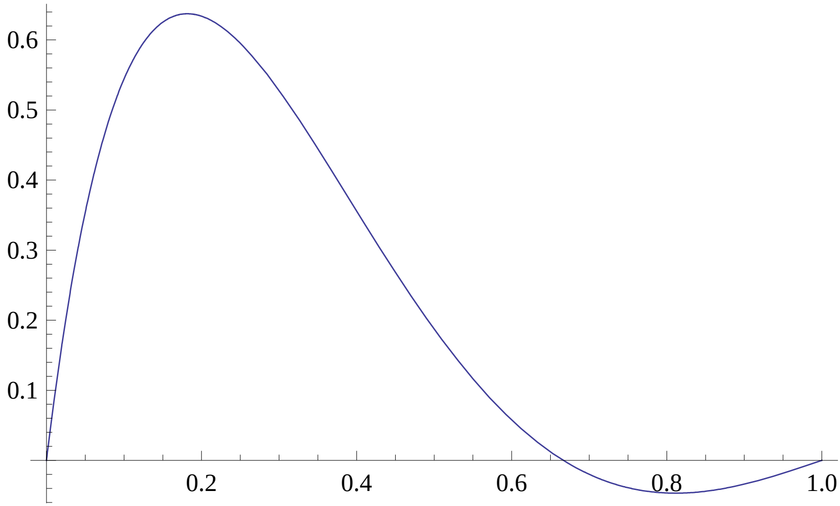
$$\sin\alpha = \sin\beta\cos\delta+\cos\beta\sin\delta.$$

Hence

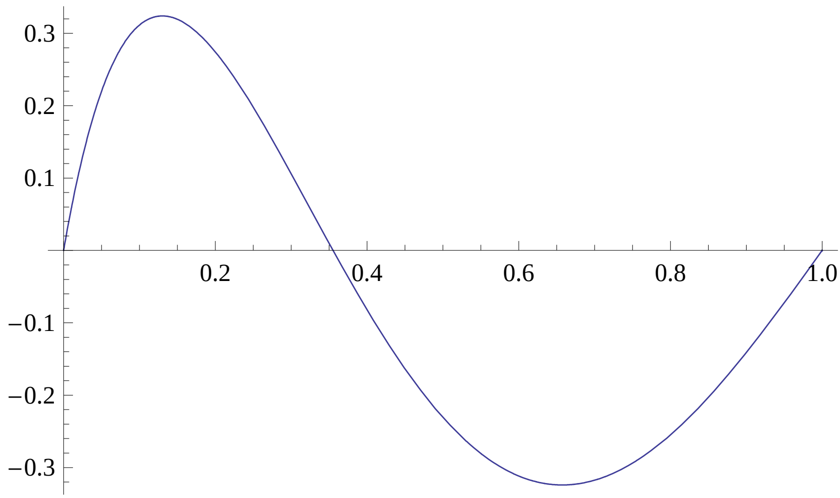
$$\begin{aligned}\sin\alpha &= \frac{c\sqrt{(1-\lambda)^2+c^2-(1-\lambda)^2c^2}}{(1-\lambda)^2+c^2}-\frac{c(1-\lambda)^2}{(1-\lambda)^2+c^2},\\ &= \sin^{-1}\left(c\,\frac{\sqrt{(c^2-1)(2-\lambda)\lambda+1}-(1-\lambda)^2}{(1-\lambda)^2+c^2}\right).\end{aligned}$$

Now let $c = \sqrt{3}$ and plot $\alpha - 90\lambda$ to obtain the first graph on the next page.

However, if $c = 1.67721$, it turns out that an overall more accurate protractor results.



$\alpha - 90\lambda, \quad c = \sqrt{3}$



$\alpha - 90\lambda, \quad c = 1.67721$

Pascal's pyramid

Robin Marks

This paper presents a 3-dimensional version of Pascal's triangle, which I have named 'Pascal's pyramid'. We will see that the numbers in the Pascal's pyramid are linked to the numbers in Pascal's triangle. First we shall use 3-dimensional position vectors to determine the positions for displaying the numbers that we will obtain.

Let $\{\boldsymbol{e}, \boldsymbol{f}, \boldsymbol{g}\}$ be a set of 3-dimensional position vectors, all different. Starting at the origin (0, 0, 0) we can add one vector to go one step away from the origin to any one of the three positions $\boldsymbol{e}$, $\boldsymbol{f}$ or $\boldsymbol{g}$. We can then add a second vector to end up at one of the six positions $\boldsymbol{e} + \boldsymbol{e}$, $\boldsymbol{f} + \boldsymbol{f}$, $\boldsymbol{g} + \boldsymbol{g}$, $\boldsymbol{e} + \boldsymbol{f}$, $\boldsymbol{e} + \boldsymbol{g}$, or $\boldsymbol{f} + \boldsymbol{g}$. Note that although we can get to the positions $\boldsymbol{e} + \boldsymbol{e}$, $\boldsymbol{f} + \boldsymbol{f}$, $\boldsymbol{g} + \boldsymbol{g}$ by only one route, we can get to the other positions, for example $\boldsymbol{e} + \boldsymbol{f}$, by two routes since vector addition is commutative; that is, $\boldsymbol{e} + \boldsymbol{f} = \boldsymbol{f} + \boldsymbol{e}$.

We can then take a third step to end up at one of the ten positions $\boldsymbol{e} + \boldsymbol{e} + \boldsymbol{e}$, $\boldsymbol{f} + \boldsymbol{f} + \boldsymbol{f}$, $\boldsymbol{g} + \boldsymbol{g} + \boldsymbol{g}$, $\boldsymbol{e} + \boldsymbol{e} + \boldsymbol{f}$, ... The number of ways of choosing a route with p steps in direction $\boldsymbol{e}$, q steps in direction $\boldsymbol{f}$, and r steps in direction $\boldsymbol{g}$, is given by

$$\frac{(p+q+r)!}{p!\,q!\,r!}.$$

For example, the following table shows one route of six steps from the origin to the position $\boldsymbol{e} + \boldsymbol{e} + \boldsymbol{e} + \boldsymbol{f} + \boldsymbol{f} + \boldsymbol{g}$.

	vector		
step	$\boldsymbol{e}$	$\boldsymbol{f}$	$\boldsymbol{g}$
1	#		
2			#
3	#		
4	#		
5		#	
6		#	

The symbol # marks the vector added at each step. Here $p = 3$, $q = 2$, and $r = 1$. There are $(3+2+1)!/(3!\,2!\,1!) = 60$ different routes from the origin leading to this position.

We can now construct Pascal's pyramid. For each position vector sum we first plot the position in space. At that position we write the total number of different routes from the origin to the position. The following tables give the vectors involved and the number of routes for up to three steps.

Steps	Position vector sum	Number of routes from the origin
0	$\mathbf{0}$	$\frac{0!}{0!\,0!\,0!}$
1	$\boldsymbol{e}$	$\frac{1!}{1!\,0!\,0!}$
	$\boldsymbol{f}$ $\quad$ $\boldsymbol{g}$	$\frac{1!}{0!\,1!\,0!}$ $\quad$ $\frac{1!}{0!\,0!\,1!}$
2	$2\boldsymbol{e}$	$\frac{2!}{2!\,0!\,0!}$
	$\boldsymbol{e}+\boldsymbol{f}$ $\quad$ $\boldsymbol{e}+\boldsymbol{g}$	$\frac{2!}{1!\,1!\,0!}$ $\quad$ $\frac{2!}{1!\,0!\,1!}$
	$2\boldsymbol{f}$ $\quad$ $\boldsymbol{f}+\boldsymbol{g}$ $\quad$ $2\boldsymbol{g}$	$\frac{2!}{0!\,2!\,0!}$ $\quad$ $\frac{2!}{0!\,1!\,1!}$ $\quad$ $\frac{2!}{0!\,0!\,2!}$
3	$3\boldsymbol{e}$	$\frac{3!}{3!\,0!\,0!}$
	$2\boldsymbol{e}+\boldsymbol{f}$ $\quad$ $2\boldsymbol{e}+\boldsymbol{g}$	$\frac{3!}{2!\,1!\,0!}$ $\quad$ $\frac{3!}{2!\,0!\,1!}$
	$\boldsymbol{e}+2\boldsymbol{f}$ $\quad$ $\boldsymbol{e}+\boldsymbol{f}+\boldsymbol{g}$ $\quad$ $\boldsymbol{e}+2\boldsymbol{g}$	$\frac{3!}{1!\,2!\,0!}$ $\quad$ $\frac{3!}{1!\,1!\,1!}$ $\quad$ $\frac{3!}{1!\,0!\,2!}$
	$3\boldsymbol{f}$ $\quad$ $2\boldsymbol{f}+\boldsymbol{g}$ $\quad$ $\boldsymbol{f}+2\boldsymbol{g}$ $\quad$ $3\boldsymbol{g}$	$\frac{3!}{0!\,3!\,0!}$ $\quad$ $\frac{3!}{0!\,2!\,1!}$ $\quad$ $\frac{3!}{0!\,1!\,2!}$ $\quad$ $\frac{3!}{0!\,0!\,3!}$

On the next page we present Pascal's pyramid set out with vectors $\mathbf{e} = (0, 5, 0)$, $\mathbf{f} = (-1, 6, 0)$, $\mathbf{g} = (1, 6, 0)$. (We make it 2-dimensional so that it fits on the 2-dimensional page!)

Some properties of Pascal's pyramid. The numbers at step n add up to 3^n. If we picture the entire Pascal's pyramid as a solid, each of the surfaces around the apex consists of a Pascal's triangle. The numbers in Pascal's pyramid can also be obtained from the coefficients of the terms of the trinomial expansion of $(a + b + c)^n$, $n = 0, 1, 2, \ldots$; for example,

$$(a + b + c)^2 \;=\; a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

Steps	
1	1
2	1
	2 2
	1 2 1
3	1
	3 3
	3 6 3
	1 3 3 1
4	1
	4 4
	6 12 6
	4 12 12 4
	1 4 6 4 1
5	1
	5 5
	10 20 10
	10 30 30 10
	5 20 30 20 5
	1 5 10 10 5 1
6	1
	6 6
	15 30 15
	20 60 60 20
	15 60 90 60 15
	6 30 60 60 30 6
	1 6 15 20 15 6 1

Solution 172.2 – Chords again

A regular polygon of n sides is inscribed in a unit circle. See page 21. Suppose that A_1, A_2, …, A_n are the vertices of this polygon. Prove that if P is any other point on the unit circle then

$$(PA_1)^2 + (PA_2)^2 + \cdots + (PA_n)^2 \;=\; 2n.$$

(There was another part to the problem but it asked you to prove something that is actually false. Thanks to Sebastian for spotting the error.)

Sebastian Hayes

Theorem *The sum of the squares of the distances from any point on the circumference of a unit circle to the vertices of a regular n-gon is 2n.*

Lemma 1 (a) *If a set of points is symmetric about the y-axis, the sum of the cosines of these points is zero.*

(b) *If a set of points is symmetric about the x-axis, the sum of the sines of these points is zero.*

(c) *If a set of points is antisymmetric about the origin, i.e. points lie two by two on diagonals through the origin, and at an equal distance from the origin, the sum of the sines and the cosines of these points is zero.*

Proof. Examine the graphs of the sine and cosine functions.

Alternatively: In polar coordinates, the position of a point is given by $(R_1 \cos A, R_1 \sin A)$. If a pair of points is symmetrical about the y-axis, we must have

$$R_1 \cos A + R_2 \cos B \;=\; 0,$$

with $R_1 = R_2$. Thus, unless $R_1 = 0$, $\cos A + \cos B = 0$. Similarly for points equidistant from the x-axis and on either side of it. Points on diagonals and equidistant from the origin (and on either side of it) satisfy both conditions.

Lemma 2 *The sum of the cosines of a set of points is constant about any axis passing through the same point taken as origin.*

Proof. By Pythagoras, the square of the distance from any two points is in polar coordinates

$$(R_1 \cos A - R_2 \cos B)^2 + (R_1 \sin A - R_2 \sin B)^2.$$

If the points lie on a unit circle, $R_1 = R_2 = 1$, and if the first point is $(1, 0)$, angle $A = 0$, so the distance squared reduces to $2(1 - \cos B)$. The sum from

$(1, 0)$ to n points (including itself) is

$$2n - 2(\cos A + \cos B + \dots) \tag{1}$$

and, if the point itself is excluded, is

$$2(n-1) - 2(\cos B + \cos C + \dots). \tag{2}$$

Now the distance from one point to another does not change if we rotate the system, and so the bracketed sum of cosines must remain the same—though in general the individual values will change.

In mechanics, Lemma 2 appears as the definition of equilibrium: *If a body, or system of bodies, is in equilibrium, the algebraic sum of the moments taken about any axis passing through the centre of mass is zero.*

We now apply the case to a regular n-gon inscribed in a unit circle.

If n is even, we arrange matters so that one vertex lies at $(1, 0)$. This will not affect the sum of the cosines (Lemma 2). The polygon so placed is symmetrical about the y-axis and so the sum of the cosines is zero (Lemma 1 (a)).

If n is a multiple of four, two vertices will lie on the y-axis (not about it) but this does not affect the sum since the cosines here are zero. (The moment of a force applied through the centre of mass of a body is zero.)

Thus, by (1) the sum of the distances squared from one vertex to all vertices (including itself) is $2n$.

If n is odd, the set of points is no longer symmetrical about the y-axis if we place a vertex at $(1, 0)$. However, if we rotate the polygon until a vertex lies at $(0, -1)$ the set, excluding this point, is symmetrical about the y-axis. Also, the cosine of the $(0, -1)$ vertex is zero, and so the sum of the cosines is zero (Lemma 1 (a)). Thus, by (1), the sum of the distances squared is $2n$ in this position, and so the sum is zero in any position with the same point as origin (Lemma 2).

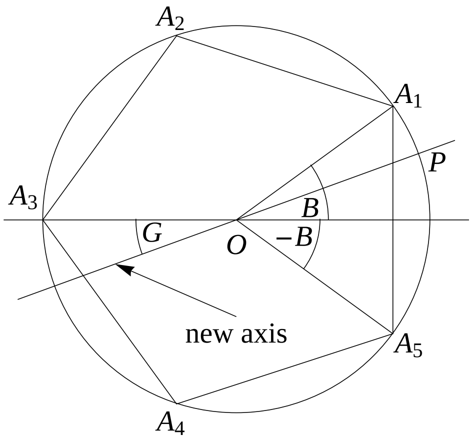
This covers distances squared from one vertex to the others.

If we consider the distances squared from a point on the circumference which lies midway between two vertices the same arguments apply. For an n-gon with n even, we situate the point at $(1, 0)$ and take the distances to the n vertices of the polygon—this time we do not take into account the point itself and its cosine. Since the set of vertices is symmetrical about the y-axis the distances squared sum to $2n$. For n odd, we situate the mid-point at $(0, -1)$.

Note that the arithmetic mean of the squares of the distances from a vertex (or midpoint) to all the vertices is constant for all n—it is just $2n/n = 2$, which is the diameter of the circle. This strikes me as remarkable and one ought to be able to give a strictly geometric proof.

There remains the case of distances squared from an arbitrary point situated between a vertex and a mid-point, say at angle G with respect to the x-axis when $(1, 0)$ is a mid-point. Then the relevant angle to the first vertex proceeding anti-clockwise will be $B - G$ and the corresponding angle clockwise will be

$-B + (-G) = -(B + G)$.



In the case of n even, the vertices of a polygon form an antisymmetric set and they remain antisymmetric under rotation, so the sum of the cosines and the sines will still be zero (Lemma 1 (c) and Lemma 2).

For n odd, the set of points is no longer antisymmetric and we have to evaluate the bracketed sum in

$$2n - 2(\cos(G - B) + \cos(G - C) + \dots) \tag{3}$$

which, using the double angle formula is

$$\begin{aligned}&\cos G \cos B + \sin G \sin B + \cos G \cos C + \sin G \sin C + \dots \\ &\quad = \cos G(\cos B + \cos C + \dots) + \sin G(\sin B + \sin C + \dots).\end{aligned}$$

But, since the sum of the cosines of the vertices of any polygon is zero, we need only bother about $\sin B + \sin C + \dots$. There will be an odd number of such sines—since the case of n even has already been dealt with. This set is symmetric about the x-axis if we place a vertex at $(-1, 0)$ and exclude this vertex. Thus the sum of the sines of these vertices is zero (Lemma 1 (b)) and the sine of the excluded vertex at $(-1, 0)$ is zero. Also, by an extension of Lemma 2 to sines, this sum will remain zero in any position. Thus the sum (3) above is $2n - 2(0 + 0) = 2n$.

The above introduces the agreeable series

$$\cos\frac{360}{n} + \cos\frac{2\cdot 360}{n} + \cos\frac{3\cdot 360}{n} + \dots + \cos\frac{n\cdot 360}{n} \;=\; 0,$$

$$\begin{aligned}
\cos\frac{180}{n} + \cos\frac{3\cdot 180}{n} + \cos\frac{5\cdot 180}{n} + \cdots + \cos\frac{(2n-1)\cdot 180}{n} &= 0,\\
\sin\frac{360}{n} + \sin\frac{2\cdot 360}{n} + \sin\frac{3\cdot 360}{n} + \cdots + \sin\frac{n\cdot 360}{n} &= 0,\\
\sin\frac{180}{n} + \sin\frac{3\cdot 180}{n} + \sin\frac{5\cdot 180}{n} + \cdots + \sin\frac{(2n-1)\cdot 180}{n} &= 0,
\end{aligned}$$

$n = 2, 3, \ldots,$ from which one can derive without a calculator many important sines and cosines bearing in mind that $\cos 0 = 1$ and $\cos 180 = -1$. Thus $\cos 60$ must be $1/2$ since $2\cos 60$ must balance $\cos 180 = -1$. Using the first and third series we have, in terms of complex numbers,

$$1 + c + c^2 + \cdots + c^{n-1} = 0,$$

where c is any nth root of unity other than unity itself.

Sum of the squares of the chords of a regular n-gon is $2n$

The theorem about polygons is a special case of a more general theorem concerning the squares of the distances from a set of points $A_1, A_2, \ldots, A_n$ to a fixed point P. Writing the distance OP as p, the angle it makes with the x-axis as P, the distance OA_r as a_r and the angle OA makes with the x-axis as A_r and applying the Cosine Rule, we have

$$(A_rP)^2 = p^2 + a_r^2 - 2a_rp\cos(A_r - P).$$

The formula works whether angle A_r is less or greater than angle P, since

$$\cos(A_r - P) = \cos(P - A_r)$$

and for a point on the x-axis itself we have $\cos(0 - P) = \cos P$. Therefore

$$\sum_{r=1}^{n}(A_rP)^2 = np^2 + \sum_{r=1}^{n} a_r^2 - 2p\sum_{r=1}^{n} a_r^2\cos(A_r - P).$$

If all the points $A_1, A_2, \ldots, A_n$ lie on the circumference of a circle, then $a_1 = a_2 = \cdots = a_n = R$ and if they represent the vertices of a regular polygon with a vertex set at $(1, 0)$ the angles are respectively $360/n$, $2\cdot 360/n$, $\ldots$, $n\cdot 360/n$ so that the sum of the cosines of $A_r - P$ is zero and we obtain

$$\sum_{r=1}^{n}(A_rP)^2 = n(p^2 + R^2).$$

If P is actually on the circumference of the circle and $R = 1$, we obtain $2n$ as in the theorem.

If we consider the sum of the distances from one vertex to all the others (not the sum of distances squared) we find we have to evaluate

$$\left(2\left(1-\cos\frac{360}{n}\right)\right)^{1/2} + \left(2\left(1-\cos\frac{2\cdot 360}{n}\right)\right)^{1/2} + \cdots + \left(2\left(1-\cos\frac{n\cdot 360}{n}\right)\right)^{1/2}$$

which, using the addition formula $1 - \cos 2\theta = \sin^2\theta$, reduces to

$$2\left(\sin\frac{180}{n} + \sin\frac{2\cdot 180}{n} + \cdots + \sin\frac{(n-1)\cdot 180}{n}\right).$$

All these sines are positive (angle < 180) and symmetrical about the y-axis, but this gives no ready method of evaluating them in terms of n.

If we are dealing with the product of the chords from one vertex to the others (the original problem), we have to evaluate

$$\left(2^{n-1}\left(2\sin^2\frac{180}{n}\right)\left(2\sin^2\frac{2\cdot 180}{n}\right)\ldots\left(2\sin^2\frac{(n-1)\cdot 180}{n}\right)\right)^{1/2}$$
$$= 2^{n-1}\prod_{r=1}^{n-1}\sin\frac{r\cdot 180}{n}.$$

We can conjecture that the above product of sines is $n/2^{n-1}$ and this can be verified for a few values of n. Thus the grand product is n, and the product of the squares is n^2 in line with the result obtained by Barry Lewis and others using complex numbers. In theory, then, this would be an 'elementary' method of proving the original theorem. But I see no easy proof that the product in question is $n/2^{n-1}$; induction using double angle formulae is possible in theory but very messy.

Problem 175.1 – Nested roots

Barry Lewis

Prove that

$$\sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{\ldots}}}} \;=\; \cfrac{x^2-1}{\cfrac{x^2-1}{\cfrac{x^2-1}{\cdots}+2}+2}+2.$$

Solution 170.3 - Reciprocals

When is it true that $\dfrac{1}{a+b} = \dfrac{1}{a} + \dfrac{1}{b}$, $\dfrac{1}{a+b+c} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$, $\dfrac{1}{a+b+c+d} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} + \dfrac{1}{d}$, ...?

Ken Greatrix

Further to Brian O'Donnell's solution in M500 **172**, page 14, I made a different approach and then found a mistake—but only after reading Brian's analysis. We have

$$\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b} \;\Rightarrow\; a^2 + ab + b^2 = 0 \;\Rightarrow\; b = a\left(\frac{-1}{2} \pm \frac{i\sqrt{3}}{2}\right),$$

$$\frac{1}{a+b+c} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \;\Rightarrow\; a^2b + a^2c + ab^2 + b^2c + ac^2 + bc^2 + 2abc = 0.$$

I thought this might be a little difficult to solve so I went on a different tack; I noticed that

$$\frac{1}{a+b+c} \;=\; \frac{1}{(a+b)+c} \;=\; \frac{1}{a+b} + \frac{1}{c} \;=\; \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

and this assumed that $c = b(-1/2 \pm i\sqrt{3}/2)$ and later that $d = c(-1/2 \pm i\sqrt{3}/2) = a$. So I had three repeating solutions, placed 120° apart on the complex plane. (I was later to discover the error in my solution, but not until reading Brian's account.)

My first reaction was that Brian was wrong! (It can't be me—I'm always right!) I even found two reasons:–

1. If $z_{n-1} + z_n = 0$, these are opposite (180° apart) on the complex plane.

2. If $1/(x+y) = 1/x + 1/y$ and $x = a+b$, $y = c+d$ then $a+b$ and $c+d$ cannot be zero.

This is now getting very silly—surely the editor would have spotted these errors.

Brian and I agree for $n = 1$ and $n = 2$ (although I use algebraic notation, Brian uses complex). For $n = 3$

$$\frac{1}{a+b+c} \;=\; \frac{1}{a+b} + \frac{1}{c}$$

$$\Rightarrow\; c \;=\; (a+b)\left(\frac{-1}{2} \pm \frac{i\sqrt{3}}{2}\right) = a\left(1 - \frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right)\left(\frac{-1}{2} \pm \frac{i\sqrt{3}}{2}\right) = -a.$$

So now I see my first mistake. I had made an assumption without justifying it. I now agree when $n = 3$. For $n = 4$, similarly,

$$\frac{1}{a+b+c+d} = \frac{1}{a+b+c} + \frac{1}{d} = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \frac{1}{d}$$

$$\Rightarrow \; d = (a+b+c)\left(\frac{-1}{2} \pm \frac{i\sqrt{3}}{2}\right) = a\left(\frac{-1}{2} \mp \frac{i\sqrt{3}}{2}\right),$$

the conjugate of b.

Continuing in this fashion I get $e = -b$, $f = a$, $g = -d$, $h = b$, $i = -a$, $j = d$, $k = -b$, $l = a$, $\ldots$.

I now have six solutions which are closely linked to each other and although I have 'zero pairs', they are not as independent as Brian's.

Then I realised another mistake; I have assumed that each expression should be dependent on the previous one. It does not seem to be a condition which is in the original problem but is it implied in the notation?

So I started again at $n = 3$.

$$\frac{1}{a+b+c} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \; \Rightarrow a^2b + a^2c + ab^2 + b^2c + ac^2 + bc^2 + 2abc = 0.$$

Had I not shied away from its apparent level of complexity I would have noticed that this factorizes to $(b+c)(a+b)(a+c) = 0$, so either $b + c = 0$ (again, this agrees), or $a = -b$ or $-c$ (again, the 'zero pairing').

By lifting the dependency restriction we now have for any a and b, $c = -a$ or $-b$.

Continuing with $n = 4$,

$$\frac{1}{a+b+c+d} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$$

$$\Rightarrow \; a^2(bc+bd+cd) + a(b+c+d)(bc+bd+cd) + bcd(b+c+d) = 0.$$

Since I can't simplify this any further I now assume (without justification—that's dangerous!) that for any b, c and d we can calculate a.

Thus for any $n > 3$, choose $n - 1$ values (real or complex) and calculate the nth.

The moral of this story is 'Don't make assumptions that you can't justify.' The keen-eyed among you will see that I've done this in a few places—in the spirit of M500-fun. I expect someone to put me right.

Book Review

The Calendar by D.E. Duncan (Fourth Estate, London, 1998)

Reviewed by Peter L. Griffiths

Fourth Estate continues its theme of popularizing the sciences with *The Calendar* by the American author D.E. Duncan. The author starts by describing the problems confronting Roger Bacon (*c*.1267) in trying to explain the need for calendar reform to Pope Clement IV. He then shows (pages 17–19) that the lack of synchronization between the movements of the sun and moon was something of concern to the earliest civilisations. However, it should be pointed out that the early Babylonians were interested in the signs of the Zodiac not primarily for astrological reasons but because of the regularity of the movement of those recognizable constellations during the night and indeed throughout the year. One might offer an alternative to Duncan's ideas (pages 19–20) about the significance of the hour by pointing out that at the start of the night there would be six signs of the Zodiac displayed across the sky, and at the end of the night the other six would be visible. Divide the time of the night by this figure of twelve, and the result is one hour. The mathematical logic is not impeccable, but it is credible. The division of the hour into 60 minutes, and the minute into 60 seconds, would have been inspired by the old Babylonian sexagesimal system.

The New Testament. On pages 67–68, the author gives various fanciful reasons for the virtual omission of satisfactory dates from the New Testament. It should however be pointed out that unlike the works of Josephus (A.D. 37 – *c*.l00), which contain numerous dates, the New Testament contains only one year date (in Luke 3:1); this reflects the different motives of the writers. Josephus' purpose was to record the facts of history as best he could, whereas the motive of the writers of the New Testament was to create a new religion, particularly for the Jews, involving a replacement of Passover by a new festival commemorating the crucifixion and resurrection of Jesus Christ. The result is that in the New Testament there are many references to the time of the Passover but none to its actual year.

Stating the year. The main system for stating the year was to specify the year of the reign of the country's ruler. One variation on this was the Roman *ab urbe condita*, meaning 'from the foundation of the city'. The other variation was the Christian system, invented by Dionysius Exiguus (*c*.531 A.D.). Duncan indicates (page l00) that adopting the date of Jesus' Incarnation was the main theological motive for the change to Dionysius' system from the existing one based on the accession of Diocletian in 284. Dionysius Exiguus recognized 25 December as the birthday of Jesus Christ, but theologically he was more interested in the date of the incarnation, which could have occurred at any time within the previous nine months. To be on the safe side he decided that 25 March should be recognized as the date of the incarnation, and hence 25 March of 1 A.D. would become the first day of the Christian era, and 25 March would be the first day of the ecclesiastical year. This would be in contrast to the Jewish calendar, whereby the New Year was always 163 days after Passover.

Duncan seems to contradict himself regarding the start of the Julian calendar. On page 46 he mentions 1 January, but on page 111 it becomes 25 December. The significance of 25 December was not recognized until much later, about 336 A.D., when it was included as the feast of the birth of Jesus Christ in an almanac of the Church's holy days. (See the entry 'Christmas' in *The Oxford Dictionary of the Christian Church*, edited by Cross and Livingstone.) An important historical fact quite rightly mentioned on pages 311 and 317 is that in most Protestant countries 25 March was the First day of the year until 1752, when, along with the loss of the eleven days, it was replaced by 1 January. As the author rightly says on the back cover, the loss of the eleven days gave rise to 5 April as being the end of the British tax year, but he omits to mention that the actual difference between 25 March and April 6 (the first day of the new tax year) is 12 days, not 11.

On pages 101–2 the allegation is made that the first mention of 'Before Christ' occurs in 1627. In fact, the original Latin version of Bede's *Ecclesiastical History of the English People* expresses dates before 1 A.D. as 'ante incarnationem Domini Nostri', so that the concept if not the actual wording should be attributed to Bede. On page 274 the diagram of the earth's tilt in its elliptical orbit round the sun is unconventional but not incorrect. The direction of the tilt really depends on whether the view of the earth from the sun in spring or autumn is preferred. The author seems to prefer the autumn view; readers should have been told this. On page 143, Duncan professes to know nothing about the obscure saints Linus and Cletus, but as any Catholic reference book would have told him, these are the names of the second and third Popes. Linus is mentioned in Timothy 4:21.

On pages v–vii, under the heading 'Calendar Index', Duncan gives a useful outline of the history, showing that the ancient Egyptians succeeded in measuring the solar year as $365^1/4$ days. One of the major problems since then has been the mathematical reconciliation of the solar and lunar calendars. Diogenes Exiguus' Christian system of stating the year has been generally accepted, even by most non-Christians. This however has not prevented disputes within Christendom over the dating of Easter, the start of the New Year, and the introduction of the Gregorian calendar—initially by the Roman Catholic states in 1582 and later by most of the rest of the world, as vividly described by the author in Chapter 14.

In spite of its shortcomings, this book is well structured, and after a thorough revision with the assistance of a knowledgeable referee could become a useful and entertaining work.

References

F. L. Cross and E. A. Livingstone, eds, *Oxford Dictionary of the Christian Church*, OUP, 1974.

Peter L. Griffiths. *Who Wrote the New Testament and Why?*, Minerva Press, 1994.

Problem 175.2 – Reciprocal inequalities

Barry Lewis & Nick Pollock

Suppose that a_0 and a_1 are positive numbers such that $a_0 + a_1 = 1$. It is fairly easy to prove that

$$\left(a_0 + \frac{1}{a_0}\right) + \left(a_1 + \frac{1}{a_1}\right) \;\geq\; 5.$$

There are two generalizations to this result:

i. the case when there are n such numbers, $a_0 + a_1 + \cdots + a_{n-1} = 1$, determine the function $f(n)$ so that

$$\left(a_0 + \frac{1}{a_0}\right) + \left(a_1 + \frac{1}{a_1}\right) + \cdots + \left(a_{n-1} + \frac{1}{a_{n-1}}\right) \;\geq\; f(n),$$

ii. the case when there are n such numbers and s is a positive integer, determine the function $g(n,s)$ so that

$$\left(a_0 + \frac{1}{a_0}\right)^s + \left(a_1 + \frac{1}{a_1}\right)^s + \cdots + \left(a_{n-1} + \frac{1}{a_{n-1}}\right)^s \;\geq\; g(n,s).$$

Problem 175.3 – Series squared

Barry Lewis

Prove that

$$\left(\frac{2^2-2}{2!} + \frac{3^2-2}{3!} + \frac{4^2-2}{4!} + \cdots\right)^2 = \frac{2^3-5}{2!} + \frac{3^3-5}{3!} + \frac{4^3-5}{4!} + \cdots.$$

'It's going to cost us x amount of hundreds of thousands of pounds.' — Insurance man on *You and Yours*, R4. [Spotted by JRH.]

Philip Howard told a story about an article he wrote for a foreign (but English-speaking) newspaper. It had an enthusiastic sub editor who disliked the slanginess of 'press gang' and substituted 'party of journalists' throughout. [EK]

'Parmar has served four times as many aces as Sampras so far; Parmar four, Sampras zero.'—Commentator, Stella Artois Tournament, BBC2. [JRH]

Constant integral

ADF

I found this on the Internet: Evaluate the definite integral

$$I(x) \;=\; \int_0^{\pi/2} \frac{dt}{1+(\tan t)^x},$$

where x is an arbitrary real number.

Larry Mead of the University of Southern Mississippi posted it on the newsgroup sci.math.symbolic as a challenge to computer algebra systems.

He says: 'Human intelligence will solve this problem after a bit of thought and the use of symmetry. Machines are not yet up to this type of problem. It bothers me how so many students of physics and mathematics now will simply throw a problem onto the computer without first giving the problem adequate thought.'

Being human, I gave the problem a bit of thought. However, intelligence must have been in short supply because I didn't get very far with it. So I put it to Mathematica (version 2.2) and, sure enough, Mathematica didn't get very far with it either.

Then I hit upon the idea of trying out a few special cases: $x = 0, 1, 2, \ldots$; Mathematica duly obliged by delivering the same answer, $\pi/4$, every time.

Obviously $I(x) = \pi/4$ for all x. But why?

One way of getting there without resorting to a picture is to make the substitution $t \to \pi/2 - t$, $dt \to -dt$. Since $\tan(\pi/2 - t) = \cot t$, we have

$$I(x) \;=\; -\int_{\pi/2}^{0} \frac{dt}{1+(\tan(\pi/2-t))^x} \;=\; \int_0^{\pi/2} \frac{dt}{1+(\cot t)^x}.$$

Furthermore, $\cot t = 1/(\tan t)$ and therefore

$$\frac{1}{1+(\tan t)^x} + \frac{1}{1+(\cot t)^x} \;=\; 1.$$

Hence

$$2I(x) \;=\; \frac{\pi}{2}.$$

Klyx: a typesetter for assignments

John Hudson

The increasing availability of Linux gives maths students the chance to use the same typesetting program for their assignments as their tutors use to write the units. Klyx is a graphical user interface for Lyx, a user friendly front end to LaTeX, a compiler for the typesetting program TeX.

Lyx, and therefore Klyx, is unlike any other program you are likely to encounter. Everything is determined by style choices. There is no ruler and there are no margins or tabs. The text wraps at the end of the screen regardless of the document style so that you can see clearly what you have typed; there is no need for you to bother about line spacing or page lengths, hyphenation, widows or orphans. All these are taken care of by the program so that you can get on with the business of writing your masterpiece.

You can choose up to a dozen document styles, including 'letter', 'book', 'article', 'paper', 'slide' and 'AMS (American Mathematical Society) paper'. Once you have done this you have a particular choice of paragraph styles; 'title', 'chapter' and 'author' are available in 'book' style but not in 'letter' style. If you choose 'AMS paper', you get a long list of paragraph styles including 'theorem', 'proof' and 'lemma'.

However, the feature which will save mathematicians most time is *Mathed*, the integral maths editor, which allows you to enter mathematical expressions in any document within the constraints of a computer keyboard while printing them out perfectly. For example, to write e^{x^2}, you click on the Mathed icon and enter

e ⟨up-arrow⟩ x ⟨up-arrow⟩ 2 ⟨down-arrow⟩ ⟨down-arrow⟩

Though the 'x' and the '2' will appear the same size as the 'e' on the screen so that you can read what you have entered, when it is printed, the 'x' will appear as a small superscript and the '2' as a smaller super-superscript and, if necessary, the line spacing will have been increased to accommodate the expression.

If you have finished the expression, you press ⟨space⟩; otherwise, you carry on with the expression because ⟨space⟩ is used to tell Klyx to return to normal text entry. This can be difficult to get used to if you have become accustomed to spacing long maths expressions but you need not fear. Klyx will take complete care of this; if you want to see what your masterpiece is going to look like before it is printed, you can ask Klyx to show you a page preview of what you have done.

Integrals and sums are handled by entering the integral or sum sign and then using the up and down arrows to enter the limits in any order; Klyx, or rather LaTeX, knows that limits follow an integral and go under a sum. Similarly, you can enter indices and powers of the same expression in any order; they will be printed correctly. Everything you need to create a mathematical expression, including matrices and complex nested structures, is available in Klyx.

If you want to include a table or figure, you can do that though you will need to create the figure in another program. Xfig is normally supplied to allow you to do that but any program that will export EPS files will do. You can fix where they will go or attach them to a float and specify whether you want them as near to a piece of text as possible, at the top or bottom of a page or on a separate page. Klyx will work out the best position based on your preferences when it comes to printing it out.

The Klyx graphical user interface was produced by the German KDE group, whose desktop environment is rapidly becoming the *de facto* standard for Linux. The desktop environment is supplied with the Rehat, S.u.S.E. and Caldera distributions among others which all cost less than £40. If Klyx does not come with your distribution, you can download the whole package over the Internet.

At the moment installing Linux is not a job for complete beginners but it is not that difficult either and free advice is readily available over the Internet. You need at least 32MB of RAM to get a basic level of performance, the more the better because, in graphical mode, Linux is memory-hungry though sparing on the processor. It is best installed on hardware which is at least a year old as development of Linux drivers for new hardware lags behind development of Windows drivers for the same hardware. You may even find that, because it uses the processor more efficiently, it will rejuvenate some old kit on which Windows would only grind along.

Linux will coexist on the same hardware as Windows; so, if you have 800 to 1500MB of free hard disc space, you can happily load both and swap between them. With the more recent distributions, you can copy files directly between Linux and Windows because Linux can 'see' Windows even though Windows pretends it doesn't exist.

Whether or not you are attracted by the idea of using Klyx, if you are doing maths and computing you should seriously consider installing Linux since it comes with nearly everything a computer science student could ever need. So far Mathcad isn't available under Linux, one reason for choosing a dual Windows/Linux system, but I'm sure that will come in time if some enterprising Linux programmer doesn't beat them to it.

Problem 175.4 – The first prime

The prime numbers are arranged alphabetically. Which is the first?

West

JRH

I was just listening to a programme about Mali, made by the BBC. One of the main topics was female circumcision, how it is still prevalent, albeit declining in the face of pressure and education (from the likes of the British and enlightened Malis). A prominent Muslim gent in the capital, Bamako, was interviewed, and he acknowledged that the practice was probably on the way out, and would gradually disappear 'without unwanted interference from the West'. Now this reference to the West was a snipe at the meddlesome British, who were coming to Mali and making proselytizing programmes like this, and not minding their own business.

I was intrigued to realize that in Mali they think of us as the West. Mali extends from 4 degrees E to 12 degrees W. Bamako lies 8 degrees west of London.

Is this (that we are 'the West') a general conception in Africa? Of course, everywhere, except the two poles, is both east and west of everywhere else in reality, but the natural feeling that there is a west part and an east part of the world comes from the standard flat map with 0 degrees in the middle and 180 degrees at each side. I suppose the feeling is that America and Europe constitute the West, and the rest of the world is somewhere else.

But how far does this feeling extend? What about South America? Some parts of that are fairly 'third world'. Do they think of the USA (and even Europe) as the West? Where do Australians think they are, and do they think of America as east or west of them? How about Russians and Americans facing each other across the Bering Strait?

Squires

Martin Cooke

Some time ago, a port on the West Coast was industrializing, and to its north and south were two large estates, the country homes of two rich squires, who built factories and docks in the port and poor urban housing for their workers there too. The population of the countryside was thereby reduced, and a less progressive squire, living to the east of the port, found himself with cash flow difficulties. So he opened his mansion to the public and also built a zoo. One day, however, his difficulties deranged him, and, letting all the zoo animals go, he climbed onto the back of his hippo and rode it towards the port.

The two rich squires saw him crashing through their workers' houses yet were unable to stop him, as he had the law on his side—the law which states: *The squire on the hippopotamus is evil to the slums of the squires on the other two sides.*

Goldbach's conjecture

Eddie Kent

Do you want to be a millionaire? It's easy—all you have to do is prove Goldbach's conjecture and so long as you get the proof into a reputable journal within two years, Faber will give you a million dollars. OK, so it's not pounds, but you can't have everything.

Christian Goldbach (1690–1764), tutor to Peter the Great, corresponded with the Bernoullis and Leonhard Euler. In 1742 he wrote to Euler and either he or Euler said (authorities differ) 'That every even number is the sum of two primes, I con- sider an entirely certain theorem in spite of that I am not able to demonstrate it.'

So far nor has anyone else, though in 1998 it was checked on a Cray up to 400,000,000,000,000. [ADF has taken the verification slightly further: 400,000,000,000,002 = 5 + 399,999,999,999,997.] An earlier breakthrough came in 1966 when the Chinese mathematician Chen Jing-Run showed that every sufficiently large even number is the sum of a prime and a number which is either prime or the product of two primes. No one has taken that one any further.

If anyone makes the conjecture into a theorem, thinks Anjana Ahuja, it will be one of only 20 men in the world, so her list cannot include Carol Vorderman, even though she is 'comfortable with numbers' and is the highest paid television presenter in Britain. Unfortunately she wasn't very good at maths at Cambridge. Not like Alan Baker, Ahuja's hot tip for the prize and a Fields Medallist.

Ian Stewart disagrees. In his opinion number theory is populated by solitary geniuses who can sometimes stun the Establishment. 'It could well be a loner who gets this,' he says.

Of course, though quite genuine this is still a publicity stunt. The idea is to promote yet another mathematics book aimed at the best seller lists. This time it is *Uncle Petros and Goldbach's Conjecture*, by Apostolos Doxiadis, published by Faber at £9.99.

Doxiadis read mathematics at Columbia University and now writes novels and plays in Greece. *Uncle Petros and Goldbach's Conjecture* is about a man who devotes his life to the conjecture. The book has already been translated into 15 languages. Of the prize, Doxiadis says 'I know that Andrew Wiles took seven years to do his proof of Fermat's last theorem. But if you had laid a bet even one day before Wiles announced his proof, and said you thought it would be solved within a few years, you would have been called crazy.'

If you want to know more about the conjecture and the results so far, go towww.utm.edu/research/primes/glossary/GoldbachConjecture.html.You can find Doxiadis at www.apostolosdoxiadis.com/. Finally, don't worry about the money. Faber are insured.

Big River

Jeremy Humphries

'Well I met her accidentally in St Paul, Minnesota ...' Thus sings Johnny Cash, who follows his unrequited love down the Mississippi ('And the tears that I cry for that woman are gonna flood you, Big River ...'), through Davenport, St Louis, Memphis, Baton Rouge and New Orleans, to the Gulf of Mexico.

But ... is he really going down the river? Or is he, in a sense, going up? That is, does the Mississippi flow uphill?

The earth is an oblate sphere, an ellipsoid, having a polar radius of 3950 miles and an equatorial radius of 3963.5 miles, so there is a difference of 13.5 miles. St Paul lies 45 degrees north, and the Mississippi Delta, south of New Orleans, lies 29 degrees north. Let us slice the earth in half through the poles, and apply the geometry of the ellipse. The parametric equations for a point (x, y) on the ellipse are

$$x = a \cos t, \quad y = b \sin t$$

where, in the case of the earth, with the origin at its centre, a is the equatorial radius, b is the polar radius, and t is the angle of latitude. A little calculation tells us that the radius at the Delta is 3960.3 miles and the radius at St Paul is 3956.7 miles. The difference is 3.6 miles. Therefore, in our model, when the Mississippi flows into the Gulf of Mexico it is 3.6 miles further from the centre of the earth than it is when it flows through St Paul. Can this be true?

Maybe Minnesota is very high up. Well, no. My atlas gives the highest point of Minnesota as Eagle Mountain in the Superior National Forest, at 2300 feet. Even if St Paul was on top of Eagle Mountain, which of course it isn't, that would give us only half a mile. We've still got three miles to account for.

I asked an expert about this (http://www.askanexpert.com/) but I don't know if I made the point clearly enough, as my expert replied that rivers do not flow uphill.

He explained that water flows when there is a difference of hydraulic head, hydraulic head being the sum of pressure head, velocity head and elevation head. Although the details of hydraulic flow are complicated, rivers have to flow from high head to low head, and the only head that really matters with rivers on the earth's surface is elevation head, namely geographic elevation of the surface of the river. Therefore, rivers have to flow from higher elevation to lower elevation. They cannot do otherwise without violating laws of physics.

That's what he said. He went on to explain what is meant by sea level, and consequently what is meant by elevation. Mean sea level, at any point on earth (latitude and longitude), is determined from a standard mathematical surface, called the reference geoid. The earth is imagined as being covered with water, with no currents, waves, and wind, and that is what the reference geoid attempts to model. The US Geological Survey states that the geoid is the basis for all elevation or topographic measurement. The geoid is 'the figure of the earth considered as a sea-level surface extended continuously through the continents. It is a theoretically continuous surface that is perpendicular at every point to the direction of gravity (the plumb line). It is the surface of reference for astronomical observations and geodetic levelling.'

Hmm. Doesn't an object have a centre of gravity, which is a single point? If you have a load of straight lines (gravity vectors) in 3-space which all meet in a single point, and each of these lines is perpendicular to a continuous surface at the point where the line and the surface intersect, then isn't the surface a sphere? But the earth is an ellipsoid. I think that the geoid is indeed a surface that is everywhere perpendicular to the plumb line, but the plumb line doesn't show the direction of gravity. Rather, it shows the direction of the resultant when the gravitational force combines with the force due to the rotation of the earth.

So, we had the three heads, pressure, velocity and elevation, which cause water to flow. Do we have to add a fourth, a sort of centrifugal head? And, if some great rivers do indeed flow against gravity, as I have speculated, is it the rotation of the earth which drives them? I am really mystified by this business. Can anyone tell me what's going on?

Ref. *The Collins / Rand McNally Road Atlas of the USA, Canada and Mexico.* If you want to get your kicks on Route 66; if you want to leave your home in Norfolk, Virginia, California on your mind; if Highway 61 runs right by your baby's door; etc., etc.; if you want a new perspective on the music, films and literature which have defined a large part of the culture of the world you grew up in, then get yourself this wonderful book. It will give you endless hours of pleasure, especially if you happen to walk into your library at the moment they put it on the 'for sale' shelf, and you get it for one pound, as your fortunate editor did.

Problem 175.5 – abc

Barry Lewis

Suppose that $a + b + c = ab + ac + bc = 0$. Prove that if n is a positive integer then

$$a^n + b^n + c^n = \begin{cases} 3(abc)^{n/3} & \text{if } n \text{ is a multiple of 3} \\ 0 & \text{otherwise.} \end{cases}$$

Unique factorization

Sebastian Hayes

If we define a 'prime' as a number which does not have any factors except itself and a unit within a particular set, then it is easy to construct sets where factorization into 'primes' is not unique.

Hilbert did this using the set of positive integers which are 1 (mod 4):

$$H = \{1, 5, 9, 13, 17, \ldots\}.$$

An H-prime is defined as a number in H other than 1 whose only divisors within H are 1 and itself. The H-primes are thus

$$\{5, 9, 13, 17, 21, 29, 33, 37, 41, 49, 53, 57, 61, 69, 73, 77, \ldots\}.$$

Now 693 is in the set since it gives remainder 1 on division by 4. But $693 = 21 \cdot 33$, where 2l and 33 are both H-primes, and $693 = 9 \cdot 77$ where 9 and 77 are both H-primes.

Factorization is thus not unique and Euclid VII, Prop. 30, 'If two numbers by multiplying each other make some number, and any prime number measure the product, it will also measure one of the original numbers', is not true in H.

The conditions are: we require two pairs of (ordinary) integers, n, m, and p, q such that one number at least in each pair is not 1 (mod 4) while the products nm and pq are 1 (mod 4). If n, m, p, $q \equiv -1 \pmod 4$, these conditions are met. Then $nm = N$ is an H-prime, and so is $pq = M$. The number NM thus has alternative prime factors. It is not hard to show that there will be alternative prime factors in any residue set 1 (mod s) where $s > 2$ since we can use this construction. I leave you to investigate further should the issue interest you.

Factorization is unique in the ring of so-called Gaussian integers, the complex numbers $x + iy$, where x and y are ordinary integers.

Other rings that have been studied are the integer rings $\{x + \sqrt{d}y\}$, where d is a square-free integer, positive or negative. In such a ring ordinary integers can have new factors, e.g. $(1+\sqrt{-5})(1-\sqrt{-5}) = 6$. It can be shown that the two factors $1+\sqrt{-5}$ and $1-\sqrt{-5}$ are irreducible in $\{x+\sqrt{-5}y\}$, as are 2 and 3, the familiar divisors of 6. To complete the proof we also need to show that $1 + \sqrt{-5}$, say, is not an associate of 2 or 3 in $\{x + \sqrt{-5}y\}$.

It is not always the case that a prime in $\mathbb{Z}$ is prime in the extended ring. The integer 2, for example, is not prime in $\{x + \sqrt{2}y\}$ since $2 = (0 + \sqrt{2})^2$.

Letters to the Editors

Y2K and all that

Grace Hopper actually found a moth spreadeagled on her computer contacts. She called it a bug because that is American English for a moth. Presumably the whole style of advertising would have been different if we had been dealing with the Millennium Moth.

The cost of adding two bytes to a date field when you were dealing with millions of date entries in the 1980s would have been far greater than the cost of adding these two bytes in the 1990s when memory prices had dropped so dramatically. So for many commercial companies in the 1980s two byte date fields were a sound and economical strategy.

The Y2K bug relates only to hardware chips that use a two byte field and have been fitted to PCs and some electronic equipment.

Whether you are affected by the Millennium bug depends on your operating system and software. For example, CP/M and the current Macintosh operating systems are unaffected but some of their software is—in the ways suggested by ADF.

The next panic will be the Unix software clock in twenty years time. CP/Ms have until 2150 (*sic*) before their software clock runs out.

John Hudson

Galileo

Dear Tony,

Colin Davies is a bit perplexed by my reference to Galileo and timing rolling bodies. Galileo was interested in this because he wanted to determine how a falling body fell, but it goes too fast to be timed. He rightly realised that a body rolling down an incline is like a falling body subject to less gravity, hence takes longer to descend. Rolling down a rather shallow incline gives reasonable times. According to Stillman Drake ['The role of music in Galileo's experiments', *Scientific American* 232:6 (June 1975) 98–104], Galileo timed the rolling, not by use of any clock or pendulum, but by simple counting. He came from a musical family and his sense of counting time was more accurate than any mechanical device available. He used an incline with a slope of about 1.7°. It is remarkable that this simple process allowed him to determine the laws of falling bodies, of which he had no *a priori* knowledge.

Regards,

David Singmaster

Lottery odds

In M500 **173** Richard Hill wrote asking for some help. Here is my contribution.

In each lottery draw, six main numbers are drawn from 49 numbers and then one bonus number is drawn from the remaining 43. It is assumed that all numbers are drawn randomly.

So, there are ${}^{49}C_6$ ways of drawing the six main numbers and for each of these combinations there are 43 ways of drawing the bonus number.

If you choose K numbers ($K = 6, 7, \ldots, 49$) and these K numbers contain N (and only N) of the six main numbers drawn, then they also contain $6-N$ other numbers which are not in the six main numbers drawn. The number of ways of doing this is ${}^{K}C_N \cdot {}^{49-K}C_{6-N}$ provided the second term is defined. Which it is for all N if $5 < K < 44$. If $K > 43$ then N must be greater than or equal to $K - 43$.

For example, if $K = 49$ the second term is only defined if $N = 6$, which is equivalent to saying that if you choose all 49 numbers the only possibility is that you match all six of the numbers drawn.

So, the probability of your choosing exactly N correct numbers is

$$A = \frac{{}^{K}C_N \cdot {}^{49-K}C_{6-N}}{{}^{49}C_6}.$$

When $N = 5$ there are two possibilities.

(1) The $K-5$ remaining numbers contain the bonus number drawn.

(2) They do not contain it.

Since the bonus number can be drawn in 43 different ways and you have $K-5$ numbers left over which may match the bonus number. The probability of (1) is $B = (K-5)A/43$ and the probability of (2) is $A - B$.

Paul Terry

A lottery is a taxation
Upon all fools in Creation
And Heav'n be praised
It is easily rais'd.

—Henry Fielding (author of *Tom Jones*). [Sent by Malcolm Maclenan.]

'I won the Lottery! Just imagine ... That's a chance of 1 in 3 million!'—Man about to be struck by a small asteriod; *Spectator* cartoon. [Sent by Basil Holder.]

π

Dear Jeremy,

I read in *Geology Today* that (in Kansas, where the creationists abound) a group called FLAT (Families for Learning Accurate Theories) has argued that if Genesis is to be taken literally, why not the rest of the Bible? Revelation refers to Earth's four corners. So FLAT argues that all references to round earth be removed from the science education standards. Also, the value of π, which the Bible gives as exactly 3, 'should be left to local school boards, and not mandated by the state.'

Colin Davies

Mathfest.com

Dear Tony,

You may be interested to know that I recently reserved the Internet domain name mathfest.com. I have it in mind to set up a Web site that contains a collection of interesting maths snippets, with contributions invited from the general public. I wonder if M500 might be interested in participating in this venture. I have already done some experimentation concerning on-screen display of maths symbols and constructions through HTML, and it all appears to be feasible.

Yours sincerely,

John Bull

Kiwi fruit

EK

In issue 174 ADF tells of kiwi fruit on sale at 15p each or ten for £1. He wonders what would be a fair price for nine. Well of course the simple solution would be to buy ten and distribute the excess one amongst the poor and needy. A harder question is what should one pay for eleven?

It happened to me—I was in a well-known music shop and saw a basket of CDs. They were priced at £4.50 each or three for £10. I picked three, then as I was moving away I saw another I wanted. At the counter I was asked for £14.50. I tried to explain—well, what? In the end I gave up and left with nothing, not even what I had originally gone in for.

It just seemed odd to me that if I'd taken three they would have been £3.33 each, whereas for four they wanted £3.62 each. Nothing wrong with that, I suppose, but it doesn't inspire a lot of confidence.

Twenty-five years ago

From M500 25 & 26

Eddie Kent—Fermat showed that if m is odd and composite then $2^m + 1$ cannot be prime, because

$$2^{(2k+1)d} + 1 \;=\; (2^d + 1)\left((2^d)^{2k} - (2^d)^{2k-1} + \cdots - 2d + 1\right).$$

If m contains no odd factors, it is of the form 2^n and Fermat remarked that in that case $2^m + 1$ is prime. Such numbers are called Fermat numbers: $F_n = 2^{2^n} + 1$. $F_0 = 3$, $F_1 = 5$, …; F_5 has ten digits.

About 100 years later Euler showed that 641 divides F_5.

Proof: $641 = 625 + 16 = 5^4 + 2^4 = 5 \cdot 2^7 + 1$; thus we have, modulo 641, $5 \cdot 2^7 \equiv -1$, $5 \cdot 2^8 \equiv -2$, $5^4 \cdot 2^32 \equiv 2^2$, $-16 \cdot 2^8 \equiv 16$; whence $2^{32} \equiv -1$, or $2^{32} + 1 \equiv 0 \pmod{641}$.

ADF—This story gets repeated so often that it is in serious danger of being taken literally! You can find it in just about every elementary number theory text book. Well, think about it. We are saying that the primality of $2^{32} + 1$ was a significant undecided problem for about 100 years. It was eventually solved by the most prolific mathematician of all time, Leonhard Euler, who cracked it by showing that 4294967297 is divisible by 641. Simple long-division could have been used to establish this fact but it was obviously an inappropriate tool for a mathematician of such standing; hence the ingenious proof.

I am trying hard, but I cannot think of another example of a trivial result that has been attributed to a famous mathematician. Can you?

One possibility that comes to mind is a theorem proved by Bertrand Russell and Alfred North Whitehead in 1913, namely that

$$1 + 1 = 2.$$

Russell and Whitehead's proof runs to 757 pages; it is Proposition *110.643 in their *Principia Mathematica*. The proof of *110.643 itself consists of three lines of apparently meaningless symbols on page 83 of volume II; to understand it you need to be familiar with everything that has gone before, including all 674 pages of volume I.

However, the statement '$1+1 = 2$' is not usually associated with Russell and Whitehead. It just happened to be a small part of their grand design—to develop the Theory of Types, and thereby provide a rigorous foundation for mathematics.

Jeremy Humphries—At the last day school at Cambridge none of us knew how the Wronskian got its name. George F. Simmons (*Differential Equations with Applications and Historical Notes*, McGraw-Hill, 1972) tells us:

> Hoëné Wronski (1778–1853) was an impecunious Pole of erratic personality who spent most of his life in France. The Wronskian determinant was his sole contribution to mathematics. He was the only Polish mathematician of the nineteenth century whose name is remembered today, which is a little surprising in view of the many eminent men in this field whom Poland has given to the twentieth century.

The *Simmons* is a very nice book, with a lot of biographical notes ranging from a few lines to many pages.

Another book which I recommend is *Elementary Number Theory* by Underwood Dudley, W. H. Freeman & Co., 1969. Dudley says that the longest arithmetic progression known consisting entirely of primes has 12 terms:

$$30030n - 6887 \quad \text{for } n = 1,\ 2,\ \ldots,\ 12.$$

Here's a problem based on one in Dudley's book:

> A man sold n cows for n dollars each. With the proceeds he bought an odd number of sheep for 10 dollars each and a pig for less than 10 dollars. How much did the pig cost?

This is so simple that we will not set it but we will ask, can you prove the answer?

M500 Winter Week-end

Norma Rosier

The **twentieth M500 Society WINTER WEEK-END** will be held at **Nottingham University** from **Friday 5 to Sunday 7 January, 2001**.

This is an annual residential weekend to dispel the withdrawal symptoms due to courses finishing in October and not starting again until February. It is an opportunity to get together with friends, old and new, and do some interesting mathematics. Ian Harrison is running it and the theme will be announced later. It promises to be as much fun as ever!

Cost: approximately £120 for M500 members, £125 for non-members. This includes accommodation and all meals from dinner on Friday to lunch on Sunday. Please send a stamped, addressed envelope for booking form to Norma Rosier. Details will be sent out after September 17th, once the theme and cost are known.
