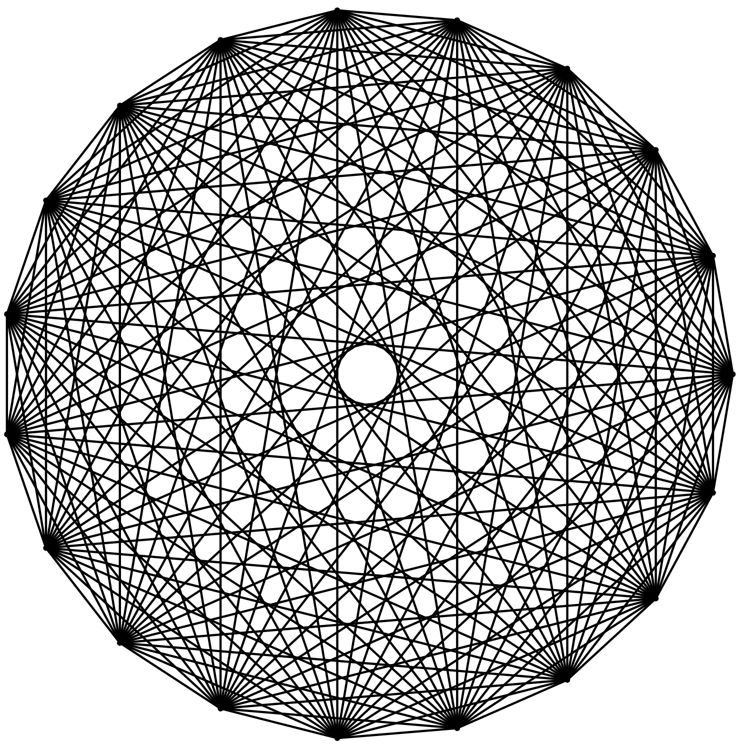
*

ISSN 1350-8539

M500 178



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. Send SAE for details to Norma Rosier, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail. Camera-ready copy can be accepted if it follows the general format of the magazine.

History of the Calendar

David Singmaster

'So teach us to number our days'—Psalm 90:12

This is based on extracts from my various chronologies, particularly my *Medieval Chronology—From The Greeks To The Renaissance*, which includes much of this material as relating to medieval developments. However, I have since expanded many of those entries and added a good deal more. I have done a separate chronology of time [M500 **171** 1]; the division is basically that the calendar is concerned with the length of the year and the month and the arrangement of days into weeks, months and years, while time is concerned with the length of the day, subdividing it into convenient bits and measuring these bits. There is some overlap.

The date of Easter is a bit too technical to deal with here and frankly I find it uninteresting—I will only mention a few points about it.

As with all attempts to summarize ancient history, one finds considerable variance between sources as to dates, origins, even spelling. I have tried to pick the most authoritative statements, but the more one reads, the more divergences one finds. For example, two recent books disagree on what Dionysius Exiguus says about the birth year of Christ. Comments and amendments are always welcome.

Notes

1 year = 365.242216 or 365.242197 or 365.24219870 or 365.2422166 days;
or 365 days 5 hr 48 min 46 sec = 365.2421990741 days = 31,556,926 sec;
or 365.242199 days (according to the Astronomer Royal, *c*.1950);
or 365.242195 days (according to NPL, *c*.1998);
or 365.24219 days (according to the exhibition on Time at Greenwich, 2000).

The year is slowing down by about 6×10^{-8} day/yr = 0.005184 sec/yr.

The length of the year depends slightly on the demarcation point and on the position of the equinoxes, which go through a 21,000 year period. The average of the lengths is the tropical year and is what is given above.

1 lunar month (synodic) = 29.530598 days = 29 days 12 hr 44 min 3.67 sec;
or 29 days 12 hr 44 min = 29.5305555... days;
or 29.530588 days = 29 days 12 hr 44 min 2.8 sec;
or 29.530588715 days = 29 days 12 hr 44 min 2.864976 sec.

1 year = 12.368267 lunar months.

These values are increasing by one or two parts per million each century due to tidal friction slowing down the earth.

The Saros is essentially the period of the precession of the plane of the moon's orbit with respect to the earth's orbital plane. It gives the basic period between eclipses. It is almost exactly 223 lunar months, making 6585.323 days or about 18 years 11 days.

The year has started at different times

1 January. Later Romans. Normans. Medieval Germany and Spain. Christians after 1582.

13 January. Orthodox churches who did not adopt the Gregorian calendar.

The first new moon after the sun enters Aquarius or the second new moon after the winter solstice. China.

1 February. Pre-Christian Irish.

1 March. Early Romans (?—see winter solstice). Medieval Venice.

21 March, the vernal equinox. Zoroastrians and Parsees—but cf. 632. Baha'is. 1920s Persia.

25 March. (The Feast of the Annunciation (or Incarnation)). Some medieval Christians, following Dionysius Exiguus, but Florentine Style started on 25 March of the year, while Pisan Style started on 25 March of the previous year. Cf. late 10C. Britain until 1752 (the English quarter-days are still based on this: 25 March, Lady Day; 24 June, Midsummer; 29 September, Michaelmas; 25 December, Christmas—with quarters of length 90, 91, 97, 87 days).

Easter. France, Germany & the Netherlands from 9C. to *c*.1582.

1 April. Christian Europe before 1564–?.

12 April. Hindus.

The summer solstice. Some Zoroastrians and some modern Parsees. Russia (in 1500–1725). Medieval Christians.

1 July. The Greek Olympic year.

The heliacal rising of Sirius = the flood of the Nile, in July (or late June). Egypt.

11 August (Feast of St Tiburce). Medieval Denmark.

1 September. Ancient Macedonians.

1, 24 or 29 September. Bede (675–735).

1 Tishri. Jews. (This is always 163 days after Passover.)

The autumnal equinox. French revolutionary calendar. Some modern Parsees.

New moon around October/November. Hindus, who call this Diwali, the Festival of Light.

The winter solstice. Early Romans (? see 1 March).

25 December. Anglo-Saxons.

References

Various encyclopedia articles.

Phil Bagnall, *Calendars: A date with time*, Focus (Dec 1994) 66–70.

John H. Conway & Richard K. Guy, *The Book of Numbers*, Copernicus, Springer, Berlin 1996, pp 176–178.

John Dunn & Colin Martin, *John Dunn's Answers Please*, BBC, 1994, revised and expanded, Penguin 1995, pp 70–71. (Details of French Revolutionary calendar.)

Harold Watkins, *Time Counts: The Story of the Calendar*, Neville Spearman, London 1954. (The best book that I have seen. See it for the many proposed rationalizations in the 19/20C.)

P. W. Wilson, *The Romance of the Calendar*, George Allen & Unwin, London 1937. (This has lots of interesting material, indeed more than I can include here, but some information may be out of date.)

Undated information

Most oriental cultures have a cycle of the same twelve animals: mouse (or rat); ox; tiger (or panther); hare; dragon (or crocodile); snake; horse; sheep (or goat); monkey; chicken; dog; pig (or boar). These are used to denote a cycle of twelve years, the twelve months, the twelve double hours of the day. It is claimed that this cycle originated in 1C. Egypt.

The **Chinese** calendar was lunar with intercalary months to keep in approximate synchronization with the seasons. Two months were added every five years, or better, 7 months in 19 years (the Metonic cycle). The method was simple—if the vernal equinox did not occur in the second month, then the second month was repeated. New Year was the first new moon after the sun enters Aquarius (or the second new moon after the winter solstice). The year had 24 named half-months in China and in early Japan. The months had 29 or 30 days, but in irregular order. The years are named by two cycles of 10 and 12 years, giving an overall cycle of 60 years, claimed to have started in -2637. The 12 year astrological cycle is based on the 12 year period of Jupiter. There was a week of seven days, grouped in four-week cycles.

Watkins says that the 60 year cycle first appears in the Han and each year had two characters—one from the ten (or Heaven) stems, that is, the five elements, each counted twice, the other from the series of twelve (or Earth) twigs (ti-chih), each of which has an animal name, from the cycle of twelve animals given above.

The **Japanese** had several ways of counting years. One was by eras—an era could be proclaimed by the Emperor for almost any reason, such as retirement or marriage. They also counted more straightforwardly by years within the reign of the Emperor, with an interrupted reign extended to the following New Year to avoid partial years. They also had a 60-year cycle based on the zodiac (or list of animal names) and their five elements (earth, fire, water, wood, metal—though each of these had two forms: natural and processed). Their year basically had 12 months, but the Emperor could add days to some months. To correct the year to the lunar year, an intercalary month was added every third year, but its length had to be adjusted to compensate for the extra days already added by the Emperor. Watkins says the Japanese adopted the Chinese calendar in 604 and used it until Japan adopted the Gregorian calendar in 1873.

The **Aztecs** had a 260 day religious cycle which overlapped with a year of 18 20-day months plus five extra unlucky days in a 52-year cycle. The Mayans also used this 52 year cycle.

The **Incas** used a calendar of 12 lunar months with intercalations based on observing a gnomon. They had weeks, but the length may have been seven, nine or ten days.

The **Baha'is** have a year of 19 months of 19 days each.

The History

−**20000**. Evidence of counting; 55 marks on a radius bone of a wolf found in Moravia—possibly related to two months.

−**10000**. Grotte du Tai bone with over 1000 notches showing months with indications of solar year events.

−**9000**. Ishango bone, with grouped notches. (Recently redated to −25000 / −20000 and thought to be a 6 month lunar calendar.)

−**4214** (or −**4236** or −**4241**). Conjectural start of the Egyptian calendar—see *c.*−3000.

−**4004**. At 'the entrance of the night preceding' Sunday 23 October, i.e. about 18:00 on 22 October: First day of creation of the world, according to James Ussher, Archbishop of Armagh, in 1650. (The relevant text is quoted in Robert L. Weber; *Science with a Smile*; Institute of Physics, Bristol 1992, pp 430–431. This says it appeared in Ussher's *Annals of the World*, vol. ix, in 1658 and the dates were incorporated into an authorized version of the Bible in 1701. John Lightfoot, Vice-Chancellor of Cambridge University, stated that the world began on 23 October, at nine o'clock in the morning.)

−**4000**. The Masons adopted this as the beginning of their calendar, claiming it was the beginning of Egyptian culture. However, Freemasonry is a 17C. development and all its previous history is modern mythology.

−**3761**. 7 October: Date of creation used as the starting point of the Jewish calendar. It is basically lunar with 12 months, alternating 29 and 30 days. But it has leap years of 13 months of 30 days to try to keep the calendar in phase with the seasons. These occur 7 times in every 19 years—giving a Metonic cycle. It basically derives from the Babylonian calendar; even the month names are Babylonian and hence derived from the Sumerians—cf. −5C. Only four of the earlier month names survive in the Old Testament. See *c.*342 for the definition of the current Jewish system. The date of −3761 is a bit dubious due to major changes made in translating the Hebrew scriptures into Greek in −3C.—the Hebrew dates the call of Abraham as year of the world 2083 while the translation gives year 3549! However, the Old Testament makes no mention of the extra months and it is conjectured that they were simply added whenever the crops would clearly not be ripe in time for the First-Fruits festival. The Talmud forbids the study of the heavenly bodies as this might lead to star worship and divination, and the astronomy in the Old Testament is singularly little and vague.

−3101. Beginning of the earliest Hindu calendar era, the Kali-Yuga.

***c.*−3000?** The Egyptian official calendar, legendarily due to Thoth, had 365 days with 12 months of 30 days or 36 decades of 10 days plus 5 days of holidays (epagomenal days). This caused it to cycle through the seasons every 1461 (or 1460) years (the Sothic cycle). A remark by the Hellenistic philosopher Censorinus in 238 that the calendar was in phase with the seasons (the heliacal rising of Sirius) in 139 has led to conjectures that the calendar started in −4214 (or −4241 or −4236), but this is too early to be reasonable. Also, the Egyptians recognized the discrepancy and had inserted intercalary days, but at rather random times. They had three seasons of four months: sowing, growing and harvesting, but the months seem to have just been called first, second, . . . , until *c.*−6C. Their year started at the autumnal equinox, *c.*22 September. The Coptic (cf. 284), Abyssinian and Armenian calendars retain this general structure, which was also used by the Persians.

−2397. Traditional beginning of Chinese chronology.

***c.*−2000**. Zodiac developed in Babylonia in order to keep track of the seasons.

−776. First Olympiad, used as the beginning of the Greek calendar developed by Eratosthenes, *c.*−240. The Athenian Festival Calendar had months of alternately 29 and 30 days, with occasional intercalary months, but this was not done systematically and the months were often out of phase with the seasons. The year started on 1 July.

−752. Legendary date of foundation of Rome and beginning of the Roman calendar, legendarily established by Romulus. However, Roman scholars already disagreed over this date, variously putting it at −751 or −754, and I have also seen −752 and −753 and even −735 (probably a misprint for −753). Consequently numerical years were never widely used by the Romans. The calendar had a year of only 304 days in ten months, based on the growing season, with March as the first month. The rest of the days were not counted—or else the months drifted through the year? Romulus's successor, the semi-legendary King Numa Pompilius (*c.*−650 or *c.*−700 or reigned −716/−673) added January at the beginning and February at the end (or both at the beginning or both at the end—the last leaves the year beginning in the spring?), extending the year to 354 days with months alternating between 30 and 29 days. An extra day was added, because odd numbers were luckier, giving a year of 355 days, compared to 12 lunar months, which is 354.37 days. (To avoid an even number of days in a month, four 30-day months were given an extra day, while December and February were reduced to 29 and 28 days—however, it's not clear when this was done and the Romans tended to fiddle the calendar.) To correct for the obvious inaccuracy, Numa decreed an intercalary month of Mercedonius to be added between 23 & 24 February of every third year (or an intercalary month of 22 days every second year or of alternately 22 and 23 days every second year; another source says it was 27 or 28 days every other year, but then the last five days of February were dropped—sometimes the extra month was omitted when things got out of phase), making a year of 377 or 378 days. The last of the methods gives 1465 days in four years, making a year of 366.25 days, which was later recognized as too long and supposed to be corrected every 24 years. As can be seen, there is considerable uncertainty—one source says no one really knows how many days were in this month. Just to confuse matters, the name Mercedonius only occurs in Plutarch's *Lives*, which were written in Greek—the Romans apparently never had a definite name for it beyond 'mensis intercalaris'. In −452 or −153, February was shifted from 12th to 2nd month, so the beginning of the year was 1 January. The intercalations were done so poorly that the calendar was out by 117 days in −190 and 75 days in −168. Apparently Roman officials often added months in order to extend their time in office. Caesar reformed the calendar in −46 (q.v.).

−669/−630. Ashurbanipal (= Assurbanipal, also known as Sardanapalus). Dominates Babylon and Elam. He was a scholar and created a library at Ninevah, which was found in the 1840s. Reformed the calendar, introducing rest days on the 7, 14, 19, 21, and 28 of each month.

−613. 6 August: The Mayan calendar might begin on this date, but the earliest date found so far is −98. They have 12 months, alternating between 29 and 30 days. They have a 52-year cycle, like the Aztecs, cf. Undated information, p. 4.

−594. Solon of Athens establishes a calendar with months alternating between 29 and 30 days.

***c.*−624/*c.*−548**. Thales of Miletus, first mathematician known by name, began the study of astronomy. Said to have brought knowledge of the 365-day year from Egypt to Greece. He is credited with predicting the eclipse of 23 (or 28) May −585. Herodotus tells of this eclipse while describing a battle of the Lydians and Medes.

***c.*−543**. Death of Buddha. Beginning of Buddhist calendar.

−587/−538. Babylonian Captivity of the Jews. The Jews essentially adopt the Babylonian calendar—only four of the earlier month names are preserved in the Old Testament. The Babylonian months are: Nisa (or Abib), Iyyai (or Zif), Sivan, Tammuz, Ab, Ellul, Tisri (or Ethanin), Markeshvan (or Bul), Chiseleu, Tebeth, Shebal, Adar. Two months could have extra days to keep the months in phase with the moon. When necessary to keep in phase with the sun, an extra month of We-Adar was intercalated, but the process was a secret to the Sanhedrin or Great Council in Jerusalem—cf. 330/365—though the method was essentially the Metonic cycle adopted in Babylonia in *c.*−383. Another source says the Jewish calendar was not established until *c.*342, q.v. The Jews use Ashurbanipal's seven-day week (cf. −669/−630) but the Jewish day starts at sunset while the Babylonian day starts at dawn. (Others think the Jewish and Babylonian seven-day weeks were independent.) The Jewish religious and civil years start on different days giving two new year's days! The more common New Year's Day is 1 Tisri (Tishri).

−6C. Cleostratus of Tenedos develops a cycle of 2922 days in 99 months in eight years, giving 29.51515 days per lunar month and 365.25 days per year. This was a regular intercalation of three 30-day months every eight 354-day years, a period called an Octæteris. Based on a similar Babylonian cycle.

−527. Death of Vardhamana, founder of the Jainists. Beginning of Jain calendar.

***c.*−500**. Nabu-rimanu or Naburianos determines year as 365 days 6 hr 15 min 41 sec = 365.26089 days. This seems to be the first careful determination of the length of the year. The Saros is determined as 223 lunar months, making $6585^1/3$ days.

***c.*−480**. The Babylonian calendar was lunar, with irregular intercalary months. About this time, it began to be studied systematically and the Metonic cycle is known by −383.

Mid −5C. Greeks determine the year is 365+ days long.

−432. Meton (and Euctemon and Phaeinus) discover the 19-year cycle of 235 months of 6940 days. His year is $365^1/4 + {}^1/76$ days = 365 days 6 hr 19 min = 365.26319 days. This has a regular intercalation of seven months (of 29 or 30 days) in 19 years. Nineteen solar years are 6939.61 days and 235 lunar months are 6939.69 days. This is the basis of the Greek astronomical calendar which started on 27 June −432 (supposed to be a summer solstice), but doesn't seem to have been used for any ordinary purposes. In the −4C., Callippus made a correction by going to a 76-year cycle, but Conway and Guy show that one does not get a better approximation until a 334 year cycle of 4131 months.

−423. Aristophanes' *The Clouds* has the moon complaining that the days do not conform to her phases.

−383. Cidenas (= Kidinnu), in Babylon, finds or uses Meton's cycle (see −432) of 235 months in 19 years and reforms the calendar. This calendar is adopted by the Jews and is still used by them—but cf. −587/−538. Cidenas may have observed the precession of the equinoxes.

***c*.−390/*c*.−340**. Eudoxus of Cnidos (perhaps *c*.−408/*c*.−355). Said to be the first to apply mathematics to astronomy and the first Greek to build an observatory, at Cnidos. Said to have brought knowledge of the $365^1/4$-day year from Egypt to Greece.

−4C. Callippus adjusts the Metonic cycle (see −432) by omitting the last day of every fourth cycle, giving 27759 days in 76 years, making the year equal to $365^1/4$ days.

***c*.−304**. Cneius Flavius works out the Roman calendar rules and publishes them, breaking the secrecy of the College of Pontiffs in Rome.

***c*.−287/−212**. Archimedes. He is said to have built a celestial model showing movements of the planets, i.e. an orrery, and to have made a clock driven by a weight rather than by water.

***c*.−275/−194**. Eratosthenes: director of the library at Alexandria from *c*.−245, measures the earth. He gets a circumference of 28,727 miles. He suggests the calendar adopted by Julius Caesar in −46. He also initiates the first classical calendar with years numbered, based on the four-yearly Olympiads which began in −776, denoted the first year of the first Olympiad. This system persisted well into Byzantine times.

−238. Ptolemy III Euergetes decrees leap years to be included in the calendar. This was carved as the trilingual Decree of Canopus. This was generally ignored.

***c*.−180/*c*.−125**. Hipparchus. He modifies Callippus' year by again multiplying by 4 and dropping a day, getting 111035 days in 304 years, giving a year of $365.24671 = 365^{75}/304$ days. (Other sources say 365.247222 days (or $365.246667 = 365^{74}/300$ days)). He introduces hours of equal length in the day and the night, but these are only used in scientific work.

−159. First waterclock in Rome.

−153. From this time, Roman years start on 1 January, while Greek years continue to start in midsummer. (This may have begun earlier, possibly −452, cf. −752.)

−104. Chinese calendar is reformed by Lo Hsia Hung.

***c*.−80**. Antikythera mechanism, a Greek calendrical computer, with many gear wheels.

−57. 23 February: Beginning of the Vikrama calendar or Samvat (or Samwat) era, established by Vikramaditra, King of Ujjain—however, he seems to be mythical. (−56?)

−**46**. Julius Caesar, advised by Sosigenes of Alexandria, based on Eratosthenes' suggestion or perhaps based on Ptolemy III Euergetes, reforms the calendar to a year of 365.25 days (off by 11 min 14 sec per year). The year −46 (= Roman year 708) has 445 days (another source says 460 days) and is known as the 'last year of confusion'. This was done by adding 23 days to February (i.e. intercalating 'Mercedonius') and two extra months (Undecimber & Duodecimber, having 33 and 34 days) between November and December. (The confusion was partly due to Caesar—he had been Pontifex Maximus, and hence in charge of the calendar, since −52 and he had only inserted one intercalary month, which should have been done every two years.) The vernal equinox is intended to be on 25 March. The beginning of the year is moved from 1 March to 1 January (or confirmed at 1 January). March, April, May, June are renamed to the Latin forms from which the English words are derived (other sources say these names go back to Romulus or Numa, −8C.?). Eight months are lengthened and one shortened to shift from the 355-day lunar calendar to the 365-day solar calendar. Basically the odd months have 31 days and the even months have 30 days, but February has 29 days in normal years and 30 days in leap years. Leap year day is between 25 & 26 February—so there were two 25ths. Since the 25th was *sexto calendas Martias* (the sixth day before 1 March), a leap year was known as a bissextile year. However, other sources say the repeated day was 23 February or 24 February, which was the terminalia or end of the Roman year. Some sources say Caesar lengthened Quintilius and named it for himself, but Quintilius already had 31 days and was later named Julius by the Senate in −45 (or −46), or after his death in −44.

The priests misinterpreted the rules for leap years—because they counted to four inclusively and hence inserted leap years every third year! This was sorted out by Augustus who found that −8 started three days late. He cancelled the next three leap years, so the next leap year was in +8, so the Julian system ran consistently from +5. (These dates are confused—I think I forgot to allow for the absence of year 0 and −8 should be −9 or +8 should be +9?) He may or may not have adjusted the lengths of some months. Whoever did it, a day was removed from February to lengthen August, so the bissextile day became the 24th of February.

The seven-day week was taken over from the Jews in the 4C., q.v. A source says that an eight-day week, called a nundinum (which means nine days, undoubtedly due to counting inclusively), was used before then, with days called A, B, . . . , H. When the week was changed to 7 days, the letters A, B, . . . , G were used and the letter of the first Sunday became the dominical letter of the year as used in calculating the date of Easter. In order for this calculation to work in a leap year after the shortening of February, 24 and 25 February had to be counted as one day. Another source says the weeks had alternated between 7 and 8 days. This continued until the middle ages—cf. 1236. From about 1300 until 1752, the beginning of the year had shifted to 25 March—cf. 298, 552.

−44. Death of Julius Caesar.

−8. Augustus observes that this year starts 3 days late and cancels the next three leap years. See −46, +8. At some time Augustus changed the name of Sextilius to Augustus—the motion of the Senate is preserved. Further, a day is added to Augustus by changing the number of days in later months and stealing a day from February.

0. This year does not exist! See 552.

+1. Date of the birth of Christ according to Dionysius Exiguus in 552. Unfortunately this does not agree with the dates of the death of Herod in −4 and the census of Quirinus in +6. General consensus is that Christ was born about the year −6.

Duncan Steel [The Y2K bug in all our calendars; *The Guardian* Science (23 September 1999) S2–S3] says Dionysius was more concerned with the year of the Annunciation, which he took to be on the Spring Equinox in 1 BC (but most sources say 1 AD?), which he dated as 25 March (Lady Day or the Feast of the Annunciation). Taking a round nine months pregnancy, Christ would be born on 25 December, 1 BC. The apparent contradiction is due to orthodox Jewish belief that a boy's life does not begin at birth, but from circumcision, which is traditionally on its 8th day of life. For Christ, this would be on 1 January 1 AD, and 1 January is the Feast of the Circumcision in the liturgical calendar. However, this led to two ways of counting years—from 25 March, called *stilo Annunciationis*, and from 1 January, called *stilo circumcisionis*.

5. First correct Julian year; cf. −46, −8, +8.

8. Leap years restarted; cf. −46, −8, +5.

***c*.8?** Lin Hsin (= Liu Hsin?), Imperial Librarian, writes on the calendar. He does not know of the precession of the equinoxes and introduces a huge period of 23,639,040 years.

14. Death of Augustus.

33. 3 April: Date of the Crucifixion; the Bible refers to the moon turning blood-red, which is a sign of a full eclipse of the moon. This can be dated.

78. 3 March: Beginning of the Saka calendar or Salivahana (or Shaliwaham) era, used for most later calendrical dates.

284. 24 August: beginning of the 'Diocletian era', the common era for reckoning for the next few centuries.

284. 29 August: beginning of Coptic calendar, using the Egyptian 360 + 5 day year.

298. Diocletian orders a census to be taken every 15 years and a 15-year 'cycle of indiction' becomes used for various fiscal and legal purposes, usually starting on 1 January. The year of indiction is thus $\equiv$ year + 3 (mod 15), with 0 being the last year of a cycle; cf. 1582.

312. Constantine embraces Christianity and wins the Emperorship.

313. Edict of Milan tolerates Christianity in the Empire.

Early 4C. Seven-day week introduced, officially under Theodosius. Watkins, says this was adopted by the Council of Nicaea in 325. (This is based on the first chapter of Genesis, which was rewritten after the Babylonian Captivity in order to justify the Jewish seven-day week.) The Roman eight-day week was used before this time.

324. Constantinople founded. Constantine makes Christianity the state religion and decrees the Christian sabbath to be Sunday instead of the former Saturday which had been used due to its earlier Jewish use.

325. Council of Nicaea decrees that the Vernal Equinox should fall on 21 March and that Easter should be the first Sunday after the full moon following the Vernal Equinox (assumed to be 21 March). In the *Book of Common Prayer*, we find: 'Easter-Day is always the First Sunday after the Full Moon which happens upon, or next after the Twenty-first Day of March; and if the Full Moon happens upon a Sunday, Easter-Day is the Sunday after.' The second clause is to prevent Easter from actually falling upon Passover. See 330/365. The Bishop of Alexandria was commissioned to announce the date each year, but due to the slowness of communication, it was essential to provide a formula or tables so each community could do the calculation and get the same date. However the process started with an 8-year cycle (based on Cleostratus (−6C)?), amended to an Alexandrine cycle of 19 years (obviously based on Meton), then a Roman cycle of 84 years and finally a Victorian cycle of 532 years—cf. 457.

Watkins says the seven-day week was adopted by the Council.

336. A calendar of holy days first lists 25 December as birthday of Christ.

***c*.342**. Under the Jewish patriarch Hillel II (Patriarch 320–365), the Jewish calendar is fixed on the 19-year Metonic cycle with 12 years of 354 days and seven 'embolismic' years with an extra month in years 3, 6, 8, 11, 14, 17 and 19 of the cycle.

330/365. The date of Easter is of great interest and the Sanhedrin are forced to reveal the rules for intercalary months in the Jewish calendar—cf. −587/−538. Being basically lunar, it has 5 months of 30 days, 5 months of 29 days and two months which can vary between 29 and 30 days, giving a year of 353, 354 or 355 days. An intercalary month of 30 days was added 7 times in 19 years—in years 3, 6, 8, 11, 14, 17 and 19 of the cycle—but these also have two variable months, so they can have 383, 384 or 385 days. Passover was 14 Nisan and Christ was crucified on a Passover which was a Friday. This leads to a basic division—the Quarto-decimans held that the Crucifixion must be commemorated on Passover, the 14th of the lunar month at the full moon, while the majority wanted it to be on a Friday. This was one of the major topics considered at the Council of Nicaea in 325.

440. A source says this was the first year that Christmas was celebrated on 25 December.

457. Victorius develops his 'Victorian' cycle of 532 years for the date of Easter and it is promulgated by the Pope. However, the Irish and British Churches continue with the previous Roman cycle of 84 years—cf. 664, 710, 768.

552. (or **525** or **532** or ***c*.531**?) Dionysius Exiguus suggests the Christian Era, i.e. the division between BC & AD, but omits a year 0 and determines or adopts the birth of Christ as 25 December 1 AD, with the Annunciation and Incarnation on 25 March 1 AD, which would be the first day of the Christian Era. (One source claims these were in 1 BC.) Gibbon says the usage does not become common until the 10C.—see also 748, 635/735. The terms BC and AD (see 1219) are much later. The Spanish had a Christian era starting in +38—this was abolished by the Council of Tarragona in 1180, but continued in Portugal until the early 15C. The Greek world adopted the Christian Era in the 15C. Dionysius starts his years on 25 March, but, it wasn't entirely clear which year started on this day and there were two later styles. The Florentine Style had the year starting on 25 March of the year, but Pisan Style had the year starting on 25 March of the previous year; cf. +1 and 675–735.

The lack of the year 0 often leads to confusion—in 1930, the 2000th birthday of Virgil was celebrated a year early!

552. Tuesday, 9 July: beginning of the Armenian Era.

604. Japanese adopt Chinese calendar, called Genk-reki. A Buddhist bronze, Kwanroka, had brought Chinese calendrical texts to Japan in 602 and Yakoshiso Tamafuru had studied them and drew up the new calendar. It was modified in 673, 856 and 861, then continued until the Gregorian calendar was adopted in 1873.

622. Prior to Mohammed (or Umar I?), the Arabic calendar has 12 months, alternately of 29 and 30 days with an intercalary month inserted when necessary, approximately every third year; cf. 622 and 640, below.

622. 15 July: Mohammed's Hegira from Medina. The Moslem calendar dates from 16 July of this year, the first moon after the Hegira, which was a Friday, hence the Moslem holy Day is on Friday. This calendar was established by Umar I in 640, q.v. It is lunar with 12 months, alternately of 30 and 29 days, except that the last month has an extra 'intercalary' day in 11 years out of 30—cf. 622 above. So the year is 354.36666 days long and this gives 10631 days in 360 lunar months, or 29.53055556 days to the lunar month, compared with 10631.012 (or 10631.015) days, corresponding to a lunar month of 29.530589 (or 29.530597) days, which is an agreement to one day per 2400 years or about one part in a million. The intercalary years are years 2, 5, 7, 10, 13, 16, 18, 21, 24, 26, 29 of the 30 year cycle. The Moslem months are: Muharram, Saphar, Rabia I, Rabia II, Jomada (or Jamada) I, Jomada II, Rajab, Sha'ban (or Shaaban), Ramadan, Shawall (or Shawwal), Dulka'da (or Dulkaada), Dulheggia. [Martin Stern, Dates for Ramadan, *Mathematics Review* **1:2** (November 1990), 18–20. He gives algorithms for finding the Gregorian date of a Moslem date, etc. See also Watkins.] Leap year is called Kabishah. However, for agricultural purposes, a solar calendar was often used, for example, as devised by Omar Khayyam in 1079, q.v. Modern Moslems also use the Gregorian calendar, subtracting 622 from the year AD to produce the year AH solar. Thus 1976 AD was 1354 AH solar (but 1396 AH lunar). The solar months have different names.

632. 16 June: The last Sassanian king, Yazdigerd (or Yazdagird), starts a new calendar using 365 days to the year. Although the Sassanians soon collapsed, the 'Persian years' remained in some use in Islamic and Byzantine astronomy. This is the beginning of the Parsee or Zoroastrian calendar. This has twelve 30-day months with five extra days. The ancient Persians intercalated a month every 120 years. The Parsees, who are the Zoroastrians who fled Persia when it collapsed, omitted the intercalation, while the Zoroastrians who remained in Persia (later called Iranis) remembered to do it, but only once! When the two groups came together again, there was a bitter controversy over whether the year started in September or August which was only resolved when it was found that both were wrong and it should start on the Vernal equinox.

640. Caliph Umar I (Caliph in 634–644) interprets Mohammed's phrase: 'The number of months in the sight of God is 12' as requiring a year of 12 lunar months, so the intercalary month is suppressed—cf. 622. He establishes the Moslem calendar, starting it from the first moon after the Hegira. Also the Koran IV:37 says: 'Postponement (of a sacred month) [meaning intercalation] is only an excess of disbelief whereby those who disbelieve are misled ....' The Arabic month names are derived from ancient Syriac names. Persia uses a Jalali calendar (cf. *c.*1048/1131?) with month names from Mazdaean angelology. Afghanistan uses the Persian calendar with Zodiacal month names.

664. Synod of Whitby adopts Victorian cycle for Easter, though Ireland and Wales remain attached to the older Roman cycle; cf. 457, 710, 768.

675–735. Venerable Bede (now St Bede) points out the error in the Julian Calendar and suggests adding three days per 400 years. He describes dating using the Christian era of Dionysius, but Gibbon says it did not become popular until the 10C. Steel (cf. +1) says he started his years in September, leading to later confusion—cf. late 10C. Bede variously used 1, 24 and 29 September corresponding to a tax cycle, the autumnal equinox and Michaelmas. He expresses BC dates as ‘ante incarnationem Domini Nostri’, apparently the first usage of the concept.

710. Naitan, King of the Picts, adopts the Victorian cycle for Easter. Bede describes this and is believed to have written the letter to Naitan which gave the instructions for the Victorian cycle; cf. 664.

748. Earliest extant document dated by the Christian Era of Dionysius.

768. Welsh church adopts Victorian cycle for Easter; cf. 457, 664.

9C. France, Germany and Netherlands begin using Easter as New Year’s Day. This leads to the year having 11, 12 or 13 months, with some years having repeated days and others having omitted days! I have seen an assertion that the Anglo-Saxons started their year on 25 December.

Late 10C. Steel (cf. +1) says New Year shifted to 25 March due to increasing veneration of the Virgin Mary, but that the New Year was moved ahead from Bede’s September date, effectively losing a year! However, not all places adopted this count—Florence did, but Pisa moved its New Year back. Hence the date we would now consider to be 25 March 1001 would be considered the beginning of 1000 in Florence, but the beginning of 1001 in Pisa. England used the (incorrect) Florentine style until 1752, when the New Year was moved back to 1 January, but this still means we are one year ahead, i.e. the year 2000 ought to be numbered 2001!

973/1048. Abu’l-Rayhan al-Biruni. He determines the year as 365.240278 days.

1079. Omar Khayyam and others produce the Jalalian or Malik-Shah calendar, cf. *c*.1048–1131?.

***c*.1080**. I have seen a claim that William the Conqueror changed the beginning of the year to 1 January from the Anglo-Saxon 25 December.

***c*.1048/1131?** Omar Khayyam. Astronomer Royal to Sultan Jalal ad Din Malik Shah. In 1074, he was appointed with seven other astronomers to produce a revised calendar. In 1079 they produce the Jalalian or Malik-Shah calendar with eight leap years in a 33-year cycle, giving a year of 365.24242 days, commencing on 15 March 1079 (= 10 Ramadan 471 AH)—cf. 622 & 640. The error is one day in about 4363 years (using the current year length of 365.242195 days). ‘*Ah, but my calculations, people say, / Have squared the year to human compass, eh? / If so by striking out / Unborn tomorrow and dead yesterday.*’ Watkins says the details are not definitely known and various sources say the ratio of intercalary years to cycle length was 15/62, 17/70, 31/128. These have errors of one day in about 3853, 1510 and 133333 years respectively. This calendar was in use for a short time, then Persia reverted to the Moslem calendar. However, one modern source claims the Persians are still using the 8/33 calendar—?

1219. Earliest known document to use the phrase ‘Anno Domini’.

1236. Henry III’s statute De Anno Bissextili states that ‘the day of the leap year and the day before should be holden for one day’, so 24 and 25

February should be one day. This still affects the liturgical calendar—in a leap year, all the saints' days from 24 February on are shifted ahead one day.

1263. Roger Bacon. In a letter to Pope Clement IV in 1263 (or *c*.1267), he points out the error in the Julian calendar and suggests calendar reform, as carried out in 1582.

1292. *Alfonsine Tables*—astronomical tables computed in Toledo for King Alfonso. (Or 1272 or 1277–?) Gives year as 365.242546 days.

Mid 14C. Civil adoption of 24 equal hours for the day, beginning in Italy.

1436. Council of Basle. Nicholas of Cusa proposes reform of the Julian calendar.

1474. Regiomontanus publishes the first almanac. He is called to Rome to assist Sixtus IV with calendar reform, but dies of the plague in 1476.

***c*.1555**. Earliest known English version of the months rhyme: *Thirty days hath September*, ...

1564. Charles IX of France decrees the year begins on 1 January. Prior to this, the year started on 1 April. (I have seen this only in a letter from Gerald Warren to *The Guardian* Notes & Queries column (11 June 1997) 17, which cites: Margo Westrheim, *Calendars of the World*, Oneworld 1993.)

1577. Pope Gregory initiates study of calendar reform and circulates a draft proposal.

1582. Clavius completes Aloysius Lilius's (Luigi Lilio's (or Giglio's)) proposal to reform to the Gregorian Calendar. The year is taken as $365^{97}/400 = 365.2425$ days (= 365 days 5 hr 49 min 12 sec) (off by 12 days (= 2.88 hr) per 400 years). One author claims that Copernicus's data were the basis for the reform and other authors claim that John Dee contributed. Gregory issues his Papal Bull in 1581 ordering that in 1582, the 10 days (5–14 October) will be omitted. This corrects back to 325, the date of the Council of Nicaea, rather than to the time of Caesar, because the Council of Nicaea fixed the date of Easter. The date of the Easter full moon was also corrected. The vernal equinox should now average as 20 March. The beginning of the year is moved from 25 March to 1 January. Switzerland, Flanders and the Catholic Netherlands adopted it in 1583, with the Catholic German states adopting in 1584. (Apparently Switzerland adopted it gradually, beginning in 1582, but not finishing until 1812—presumably this was due to religious differences among the various cantons.) Poland adopts in 1586; Hungary in 1587. Tuscany does not change until *c*.1751. Various Protestant countries refuse to accept the reform. Denmark, (presumably Protestant) Netherlands and Protestant German states change in 1700. (Watkins says the Protestant German states adopted it in 1699, at the urging of Leibniz.) Scotland changes (or at least changes the beginning of the year to 1 January) in 1600, but England does not adopt it until

1752, when 11 days have to be omitted. The orthodox countries, including Russia, do not change until the early 20C. and Vietnam changed in 1967, having to skip 13 days. Some Orthodox countries may still be using the Julian calendar—certainly some Orthodox churches are. Clavius exposits the ideas in a book in 1603. I had the belief that the Gregorian scheme went further and specified behaviour for years like 4000, but this does not appear to be the case.

1582. Joseph Justus Scaliger devises the Julian Day calendar—named after his father Julius Caesar Scaliger. This is simply the number of days elapsed since noon on 1 January -4713. He notes that the three calendrical cycles—28 year solar cycle; 19 year lunar cycle; 15 year cycle of indiction (see 298)—together give a cycle of 7980 years and bases his calendar on this compound cycle with an arbitrary starting point intended to be before any significant historical event. This system is used by modern astronomers, with the convention that the day begins at noon so the whole night falls on one date. (Where is noon taken? Greenwich?)

1585. Queen Elizabeth I and John Dee suggest a calendar reform, but the Bill was not adopted.

1587. Hungary adopts Gregorian calendar.

1600. Scotland adopts Gregorian calendar.

1649–1660. The Puritans adopt a new calendar after the execution of Charles I, with years called 'the first year of freedom', ...

1650. James Ussher, Archbishop of Armagh, publishes his famous assertion that Creation had taken place 'upon the entrance of the night preceding' Sunday 23 October -4004.

1700. Protestant German states, the Netherlands (but another source says they changed in 1583), Denmark adopt Gregorian calendar.

1701. Bishop Lloyd uses Ussher's 1650 chronology to produce dates for Biblical events, which are added to the Authorized or King James Version in 1701.

1700/1753. Sweden converts to Gregorian calendar by dropping the leap year in 1700, but returned to the Julian calendar in 1712 by adding a 30 February, then converting to Gregorian in 1753. I recall that they omitted 29 February for either 44 or 40 years? The idea of converting in this way had been proposed by John Greaves, a professor of astronomy at Oxford, in 1645.

1745. A letter from Hirassa ap-Iccim (presumably a pseudonym, but unidentified) of Maryland, in the *Gentlemen's Magazine*, is the first known proposal of a perpetual calendar of 13 28-day months with an odd day. The odd days are not weekdays, so each year (and hence each month) would start on the same day of the week. It also advocates a fixed date for Easter.

1752. Britain adopts Gregorian calendar—see 1582. Eleven days dropped: 3–13 September, so Wednesday 2 September is followed by Thursday 14 September. The additional day in a leap year is defined to be 29 February. Beginning of year changes from 25 March to 1 January, but the fiscal year end moves by 11 days to 5 April. (I have long been confused by this—5 April is actually the end of the fiscal year, so this would seem to count 25 March as the end of the year. I have seen a claim that only 10 days were dropped: 4–13 September, which would account for the discrepancy. However, I have seen another claim that the financial year used to end on 31 March to allow a week for making up the Treasury accounts and that the financial year still ends on 31 March. See below for a fuller explanation.) There was considerable popular resistance and even some rioting! However, Watkins was unable to find any record of such riots and contemporary records simply refer to irritation with the change. Hogarth's painting *The Election Entertainment* has a placard on the floor saying 'Give us our Eleven Days'—this is clearer in the engraved version. The Glastonbury Thorn actually bloomed its fullest about or slightly before the new Christmas Day!

The change was primarily due to the efforts of the Earl of Chesterfield, assisted by the Earl of Macclesfield (astronomer, elected PRS later in the year, who largely wrote the 'Bill for Regulating the Commencement of the Year and for Correcting the Calendar Now in Use' which Chesterfield introduced and Macclesfield seconded in the Lords on 25 February 1751 (OS)) and the Astronomer Royal, James Bradley. (Watkins says most of the bill was drawn up by one Davall, a barrister of the Middle Temple, who was an eminent amateur astronomer, assisted by Bradley and Martin ffolkes, the current PRS.) The oratorical powers of Chesterfield and the scientific powers of Macclesfield were so convincing that the bill passed through both houses without opposition! The Royal Assent was given on 25 May, referring to the Bill 'for correcting the Style, and regulating the Calendar now in use'—Watkins says this was the first use of the word 'style' with this spelling. When Bradley was dying in 1762, 'many people ascribed his sufferings to a judgement from heaven for having taken part in the "impious undertaking"'.

One source says both 1 January and 25 March were used as the beginning of the year up to this time; 1 January was the beginning of the 'historical year' while 25 March was the beginning of the civil year.

In *c.*1993, the question of the end of the UK tax year was raised in the Answers column of *The Sunday Times*. This is collected in: *The Sunday Times Book of Answers*, ed. Christopher Lloyd, Times Books, London 1993, pp 126–127. (The column started in January 1993, but 70% of the book material did not appear in the paper.) Jo-Ann Buck sent in a letter she had received from Inland Revenue and I quote it as it appears to be definitive.

> The government's financial year originally ended on Michaelmas Day, September 29th. In 1752, on the change from the Julian to the Gregorian calendar, the calendar year 'lost' a total of 11 days. However, the financial year was not shortened and thereafter ended on October 10th, the equivalent of the former September 29th after adding back the lost 11 days. The Quarter days for public accounting were also changed by 11 days so that the Christmas Quarter day moved forward to January 5th, Lady Day from March 25th to April 5th and Midsummer from June 24th to July 5th.
>
> In 1799, the government's accounting period was altered to end on January 5th to bring it into line with the Trade and Navigation accounts and the then current commercial practice. This was the position up to 1832. Estimates of future expenditure and the Budget proposals were always presented to parliament at much the same time of year as they are today which meant that parliament could not consider the main financial proposals for the year until some time after the year had begun. In order to correct this position, Lord Althorp introduced his budget for the year 1832 to run for the 15 months from 6th January 1832 to April 5th 1833 and thereafter the budget financial year ended on April 5th. Income tax, which had been abolished since 1817, was re-introduced by Sir Robert Peel in 1842 and the income tax year was based on the Budget year ending on April 5th.

This implies that the previous financial periods ended on the quarter days, rather than began on them as I had assumed because the year began on one.

A rather dubious source says that the Julian year ended on 25 March, but there was general festival from then through 1 April and the April Fool's Day is a relic of the old system. (This would make April Fool's Day a uniquely English custom.)

18C (1752?). The date of Easter, given in the *Book of Common Prayer*, as quoted at 325, is set by an Act during the reign of George II. Easter can fall on any of 35 days. The date of Easter was deliberately confused due to anti-Catholic and anti-Semitic feeling and the dates are now taken from the Catholic calculation.

18C? The Society of Friends (Quakers) use numerical names for the days and months—First Day, Second Month, etc.—to avoid the pagan references.

1793. As part of the Revolution and the adoption of the metric system, France adopts a year of twelve 30-day months with 5 or 6 holidays at the end. This was devised by a committee headed by Charles Gilbert Romme, president of the Committee of Public Instruction. Laplace, Monge, Lagrange were the scientific members of the committee and Fabre d'Eglantine, the dramatic poet, represented the social implications. The calendar started on 22 September, the day after the date when the Republic was proclaimed, which was conveniently adjacent to the Autumnal Equinox. The months were named by d'Eglantine: Vendémiaire (grape harvest or vintage), Brumaire (fog), Frimaire (sleet of frost), Nivôse (snow), Pluviôse (rain), Ventôse (wind), Germinal (seed), Floréal (blossom of flower), Prairial (pasture), Messidor (harvest), Thermidor (or Fervidor) (heat), Fructidor (fruit). The five extra days were at the end of Fructidor and were also named by d'Eglantine: Virtue, Genius, Labour, Reason (or Opinion) and Reward (or Recompense), with leap-year day being Revolution Day. Though introduced on 5 October 1795, the calendar was antedated to 22 September 1792 (another source says it was introduced on 24 November 1793). There was a 10-day week, called a decade, with day names: Primidi, Duodi, Tridi, Quartidi, Quintidi, Sextidi, Septidi, Octidi, Nonidi, Décadi. A four-year period was called a Franciade. The day was to be divided into 10 hours of 100 minutes, each of 100 seconds, but this idea was postponed and then abandoned in 1795. (They also introduced angles with 100 centigrades in a right angle.)

The system was very unpopular because there were only 36 (+ 5) holidays in the year and it led to conflict between the Decadists and the Dominicans. Robespierre avoided making a decision and permitted both sets of holidays to be observed, leading to about 84 holidays per year! The 10-day week was abolished by Napoleon in 1802 and the Gregorian calendar was re-adopted on 1 January 1806 (another source says 19 January 1805). Watkins says Napoleon used this as a bargaining point in getting the Pope to recognize him. The Paris Commune reinstated the Revolutionary calendar briefly in 1871.

1849. Auguste Comte produces his Positivist Calendar of 13 28-day months with a Year-End Day and a Leap Year Day when needed.

1867. Alaska bought by the US. It has to drop 12 days to convert to the Gregorian calendar. But because it had been colonized by Russia, it also had to have one eight-day week to conform with the rest of the New World. Effectively, it had crossed the Date Line.

1883/1884. The Rome and Washington Conferences for the Purpose of Fixing a Prime Meridian and a Universal Day propose and adopt the Greenwich Meridian and the basic idea of time zones, which implies the acceptance of the International Date Line. The vote was 22 to 1 with 2 abstentions. It was estimated that 90 per cent of sea charts in use were already based on Greenwich. The Philippines, having been colonized from the New World, had to skip a day to conform with its Asian neighbours, but I don't know when this happened. Alaska had already made changes in 1867, q.v. **1884**. Astronomical Society of France offers prizes for best improved calendar. First prize is awarded to Gustav Armelin, a French astronomer and his plan is promulgated by the Society. It has 12 months in four groups of three having 31, 30, 30 days with a Year End Day and a Leap Year Day when needed. This gives equal quarter-years. Armelin proposes a fixed Easter on 7 April.

1902. 29 April, 10:40. One billion minutes have elapsed since the beginning of year 1.

19/20C. Gregorian calendar adopted by Japan (1873 or 1872), St Kilda (in the Outer Hebrides, 1912), China (1912 or 1911 or 1949?), Bulgaria (1916), Turkey (1916 or 1917 or 1927), Russia (1918), Yugoslavia and Roumania (1919 or 1923), Greece (1923), Greek Orthodox Church (1923). The Russian Orthodox Church has still not adopted it and is now 13 days behind. Ethiopia may still be using their calendar based on the ancient Egyptian calendar—cf. −3000 and was seven days off from the Gregorian calendar in 1937. Perhaps some other countries have not yet adopted it. Since Russia adopted it after the Revolution, it turns out that the October Revolution was actually in November and its anniversary was celebrated in November!

In Greece, the change was approved by the Bishops in 1923, but three Bishops later abjured the new calendar and returned to Old Style. They were tried by an ecclesiastical court in 1935 and demoted to monks and sentenced to spend five years in confinement. During the trial, demonstrators had to be dispersed with fire hoses.

1920s. Rizi Shah Pahlevi introduces a new calendar called Sale Hejra Shamsi in Persia which combines features of the Moslem and Gregorian calendars. It has 365 days, with the first six months having 31 days, the next five months having 30 days and the 12th month having 29 days (or 30 days in leap years). The year begins at the vernal equinox.

1923. Eastern Orthodox adopt Gregorian calendar. They propose a slight modification to omit seven leap years in a 900 year cycle, giving a year length of 365.24222 ... days, though they seem not to have adopted this.

1925. 28 April: The Convocation of Canterbury agrees that the Anglican Church has no dogmatic objection to Easter being on a fixed date, but they would not adopt it unless it was adopted by other Christians.

1928. The Easter Act in Britain sets Easter as the first Sunday after the second Saturday in April. This will come into effect by Order of His Majesty in Council 'Provided ... that ... regard shall be had to any opinion officially expressed by any Church or other Christian body.' On 14 February 1929, the Convocation of Canterbury said they would prefer the first Sunday in April.

1929. The Soviet Union adopted a five-day week in 1929, with four days work and one day of rest. The days were coloured: yellow, orange, red, purple, green. Rest days were to be staggered so that factories would run continuously. The week sequence restarted each month and the 31sts were treated as extra days. The months were then all made 30 days long. In

1932, they changed to a six-day week, with 30-day months. All of this was accompanied by confusions, since some people retained the seven-day week and some special industries continued on a five-day week. They returned to the seven-day week on 27 June 1940.

1930. Old Lunar Calendar made illegal in China.

1935. August: riot in Roumania when police attempt to enforce the Gregorian calendar on an Orthodox priest—several persons killed.

1937. P. W. Wilson [*The Romance of the Calendar*, George Allen & Unwin, London 1937, p. 250] says: 'If the five-day week in Russia has proved a failure, as many people declare, ...'.

1972. 1 January: Atomic clocks replace astronomical observations as the basis of time keeping, producing UTC (Universal Coordinated Time). As of early 1998, 240 atomic clocks were involved, in 35 laboratories in 24 countries, coordinated by the International Earth Rotation Service at the Observatory in Paris. (There are seven clocks at the National Physical Laboratory in outer London.) The clocks mostly use vibrations of caesium-133 atoms. To standardize time, the average length of day in 1820 was used. The earth has since slowed down a bit and this accumulates to a second every 400 or 500 days, leading to the occasional introduction of 'leap seconds', which have been added at the end of December or June. Twenty-one leap seconds were added during 1972–1997, but they were not regularly spaced as the earth's motion is not uniform. (Another source says 32 leap seconds were introduced in 1972–1998 (one being in 1998), but that is surely a misprint for 22.) All observatories, radio stations, etc. adopt UTC from *c*.1986, but I have read that GMT is still the legal time! In late 1999, there was debate about whether leap seconds are really necessary and whether it would be better to abolish them. This would have minimal effects on us, but missiles and satellites which navigate by the stars might be disrupted.

1967. Vietnam adopts Gregorian calendar.

1998. 31 December: A leap second is added to 1998.

2000. 1 January: The 'Year 2000' Timebomb! Many computers will crash! The End of the World as We Know It! Fortunately, very little of this happened.

2000. 29 February: This day exists, but some computers may fail to recognize it. No problems were reported.

Errata In 'Two theorems with some applications' by David L. Brown, M500 **177**, p. 6, THEOREM 1 should read 'The radius of the circumcircle of an equilateral triangle is twice the radius of its incircle.' In 'Problem 177.3 – Mahatma's triangle' by John Bull, M500 **177**, p. 16, the acute angle BAC in Figure 2 should be $2x$, not 20.

Hats

Nick Pollock

A number of people go to a party all wearing hats. The party is very good—so good that when it's over each one reels home wearing the wrong hat. What is the probability of this happening?

The problem

If there are n people at the party and none is too drunk to put on a hat when they leave, there are $n!$ ways in which the n hats can be worn by the n people. If the permutations are expressed in cycle notation, then $N_0(n)$, the number of permutations in which no one is wearing the right hat, is the number of permutations in which there are no singleton orbits. The problem seems to be rather difficult to deal with directly in this form.

Recursive solution

Let $N_k(n)$ be the number of permutations in which exactly k of the n people wear their own hats. Introduce another person, T_{n+1}, wearing his own hat. Thus T_{n+1} can create a situation in which none of the $n+1$ is wearing the right hat in two ways.

1. If none of the n others is wearing the right hat, T_{n+1} can swap hats with any of them. This gives $nN_0(n)$ possibilities.

2. If just one of the others is wearing the right hat, T_{n+1} must swap with that person. This gives $N_1(n)$ possibilities.

So

$$N_0(n+1) \;=\; nN_0(n) + N_1(n). \qquad (1)$$

But what is $N_1(n)$? Imagine T_{n+1} joining n people all of whom are wearing the wrong hat. T_{n+1} can create a situation in which exactly one of the $n+1$ is wearing the right hat in two ways:

1. T_{n+1} can do nothing. This gives $N_0(n)$ possibilities.

2. T_{n+1} can swap hats with any i who is wearing js hat, and then swap hats again with j who is now the only person wearing the right hat. This gives $nN_0(n)$ possibilities.

So

$$N_1(n+1) \;=\; N_0(n) + nN_0(n) \;=\; (n+1)N_0(n). \qquad (2)$$

Combining (1) and (2) gives

$$N_0(n+1) \;=\; n(N_0(n) + N_0(n-1)), \qquad (3)$$

for $n > 1$. Rearranging slightly,

$$\begin{aligned} N_0(n) &= (n-1)(N_0(n-1) + N_0(n-2)) \\ N_0(n) - nN_0(n-1) &= -N_0(n-1) + (n-1)N_0(n-2) \\ &= (-1)(N_0(n-1) - (n-1)N_0(n-2)). \end{aligned}$$

Using this last result repeatedly gives

$$\begin{aligned} N_0(n) - nN_0(n-1) &= (-1)^2(N_0(n-2) - (n-2)N_0(n-3)) \\ &= \dots \\ &= (-1)^{n-2}(N_0(2) - 2N_0(1)). \end{aligned}$$

Now $N_0(1) = 0$ and $N_0(2) = 1$, so for $n > 1$

$$N_0(n) - nN_0(n-1) = (-1)^n. \tag{4}$$

We're interested in the probability $N_0(n)/n!$; so using

$$\frac{N_0(n)}{n!} - \frac{N_0(n-1)}{(n-1)!} = \frac{(-1)^n}{n!}$$

we get

$$\begin{aligned} \frac{N_0(2)}{2!} - \frac{N_0(1)}{1!} &= \frac{(-1)^2}{2!} \\ \frac{N_0(3)}{3!} - \frac{N_0(2)}{2!} &= \frac{(-1)^3}{3!} \\ &\dots \\ \frac{N_0(n-1)}{(n-1)!} - \frac{N_0(n-2)}{(n-2)!} &= \frac{(-1)^{n-1}}{(n-1)!} \\ \frac{N_0(n)}{n!} - \frac{N_0(n-1)}{(n-1)!} &= \frac{(-1)^n}{n!}. \end{aligned}$$

Adding all these equations together gives

$$\begin{aligned} \frac{N_0(n)}{n!} - \frac{N_0(1)}{1!} &= \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \cdots + \frac{(-1)^n}{n!} \\ \frac{N_0(n)}{n!} &= \sum_{i=2}^{n} \frac{(-1)^i}{i!}. \end{aligned}$$

So the probability, $P(n)$, that everyone is wearing the wrong hat is given by

$$P(n) \;=\; \sum_{i=2}^{n} \frac{(-1)^i}{i!}$$

and $P(n) \to 1/e$ as $n \to \infty$.

Computer investigation

BASIC, C, etc. are not very good at big integers, but Scheme works very nicely. You can get a free version for Win9x that runs these programs from www.bushcomp.cwc.net.

We need a factorial function

```
(define (fact n)
  (if (= n 0)
    1
    (* n (fact (- n 1)))))
```

and the formula for $N_0(n)$

```
(define (N_0 n)
   (cond
     ((= n 1) 0)
     ((= n 2) 1)
     (else (* (- n 1)(+ (N_0 (- n 1))(N_0 (- n 2)))))))
```

then the required probability is given by

```
(define (prob n)
  (/ (N_0 n) (fact n)))
```

This is what my computer does:

```
> (prob 4)
0.375
> (prob 20)
0.367879441171442
>
```

This is accurate to 15 places of decimals. If you want to try larger values of n you will need a slightly more sophisticated version of N_0 with memoizing, which you can find in hats.scm on my website.

Acknowledgements

Thanks to Barry Lewis, Hugh Steers and Jonathan Weinreich for their contributions.

Solution 176.1 – Two cyclists

ADF

> Two cyclists travelled towards each other, one at 10 m.p.h., the other at 20 m.p.h. When the riders were 180 miles apart, a fly left the handlebar of one cycle and travelled towards the other cyclist. When it reached the latter, it instantly reversed direction and flew back to the first cyclist, and continued winging back and forth until the two cyclists met. If the fly's speed was 100 m.p.h., what was the total distance that the fly had covered?

As many readers pointed out, one way to solve the problem is to observe that total time the fly spends travelling between the two cyclists is the same as the time it takes for the cyclists to meet. That's 6 hours.

Alternatively, one can consider each leg of the fly's journey separately.

First leg: The fly meets the second cyclist after t_1 hours, where $100t_1 + 20t_1 = 180$; $t_1 = 3/2$. After this time the distance between the cyclists has shrunk to d_1 miles, where $d_1 = 180 - 30t_1 = 135$.

Second leg: The fly meets the first cyclist after a further t_2 hours, where $100t_2 + 10t_2 = d_1$; $t_2 = 27/22$. After this time the distance between the cyclists has shrunk to d_2 miles, where $d_2 = d_1 - 30t_2 = 180 \cdot 6/11$.

The third leg is like the first except that the starting distance has shrunk by a factor of $6/11$. So $t_3 = 6/11 t_1 = 6/11 \cdot 3/2$. Similarly $t_4 = 6/11 t_2 = 6/11 \cdot 27/22$. And so on.

Therefore the total time of the fly's journey is given by

$$t_1 + t_3 + \cdots + t_2 + t_4 + \cdots \;=\; \sum_{n=0}^{\infty} \left(\frac{3}{2} \left(\frac{6}{11} \right)^n + \frac{27}{22} \left(\frac{6}{11} \right)^n \right) \;=\; 6.$$

We like the second solution. Although it hasn't the elegant simplicity of the first, we think it is instructive. Combining the two together could give us a novel method of evaluating a geometric series.

Of course, both solutions make assumptions. Indeed, **Brian O'Donnell** wonders what species of fly can remain intact while attempting to change its velocity by 200 m.p.h. infinitely often in a time that is negligible compared with 6 hours.

Solution 176.3 – Tricubic

Find the positive real root of $x^9 + 768x^6 = 768156$.

Tony Forbes

The answer is π. Well, no. A root of a polynomial with integer coefficients must be algebraic, but, as **Elsie Page** points out,

It's really fundamental
That π is transcendental.

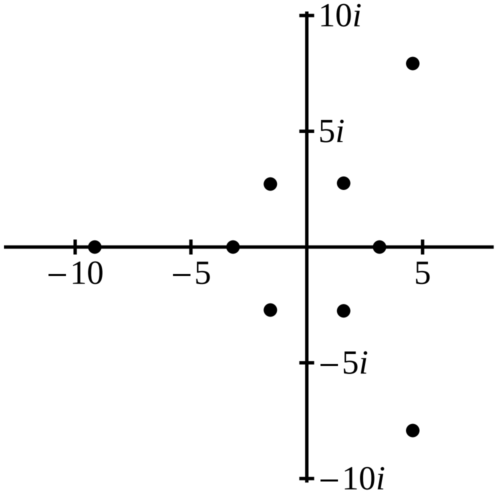
However, the difference is small enough to excite the attention of the circle-squaring community. In fact

$$\begin{aligned} x &= 3.141592653590\ldots, \\ \pi &= 3.141592653589\ldots. \end{aligned}$$

Paul Terry agrees.

A root of this thing equals π
But, as the question doesn't ask why,
I won't bother showing
Just how I was knowing
That this is the answer . . . Goodbye!

Martyn Lawrence, **Keith Drever** and **Jim McIlroy** used the 'Polyroots' function of *Mathcad* to obtain the same answer. Martyn's solution included this symmetric pattern showing all nine roots in the complex plane.



Peter Fletcher started with $x_1 = 3.5$ and iterated the formula

$$x_{n+1} \;=\; x_n - \frac{x_n^9 + 768x_n^6 - 768156}{9x_n^8 + 4608x_n^5}.$$

Since the equation is really a cubic in x^3 it must have an algebraic solution. As it happens, *Mathematica* has the formula for the roots of a general cubic 'built in', and after a certain amount of tidying up I was able to obtain this interesting approximation:

$$\pi \;\approx\; \left(256\left(z + \frac{1}{z} - 1\right)\right)^{1/3},$$

where

$$z \;=\; \frac{1}{256}\left(-16393138 + 14\cdot\sqrt{65000016447}\, i\right)^{1/3}.$$

You must choose whichever cube roots work. Note that $|z| = 1$, hence $z + 1/z$ is real.

Problem 178.1 – Lottery guarantee

Tony Forbes

What is the smallest number of National Lottery tickets you need to purchase if you want to guarantee winning at least £10? In other words: What is the size of the smallest set

$$T \subset \{\{a,b,c,d,e,f\} : 1 \le a < b < c < d < e < f \le 49\}$$

such that for any six numbers, u, v, w, x, y, z, $1 \le u < v < w < x < y < z \le 49$, at least three occur in at least one member of T?

Beware of this simple argument: 'There are 18424 ways of choosing three numbers from 49, and 20 ways of choosing three numbers from six. Therefore at least 922 (18424/20 rounded upwards) tickets are required.'

We even printed something along these lines in M500 **161**. Alas, the author must have caught me at a weak moment! Although the error is not immediately obvious, the reasoning is flawed. It does not take into account that *six* numbers are drawn in the National Lottery. (The 'bonus ball' is irrelevant.) Although you might be unlucky enough to have them all on the same ticket, the six numbers in the draw are actually capable of generating many potential prizes, and a clever purchasing strategy can take advantage of this. The correct answer is a lot smaller than 922.

Solution 176.5 – Construct a square

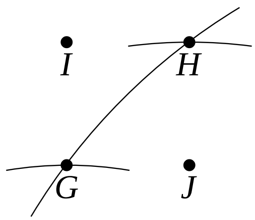
> Given a unit length line segment, construct a square of side one unit, using only a pair of compasses.

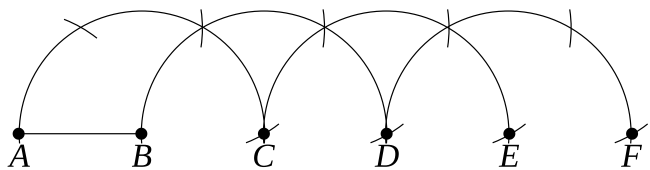
R. M. Boardman

The solution is based on a 3 : 4 : 5 triangle and on repeated use of a simple lemma.

> *Given a circle, its centre and a point P on the circumference, construct a point diametrically opposite to P.*

It is well known that if you strike six successive arcs around a circle, with a radius equal to that of the circle, you get back to the initial point and that the six points are the vertices of a regular hexagon. Hence three such strikes give a point diametrically opposite.





Label the ends of the unit line segment A and B. See diagram, above.

Draw a circle with centre B, radius 1. Construct C opposite A.

Draw a circle with centre C, radius 1. Construct D opposite B.

Draw a circle with centre D, radius 1. Construct E opposite C.

Draw a circle with centre E, radius 1. Construct F opposite D.

Draw a circle S with centre E, radius 5. Draw a circle with centre A, radius 3, intersecting the previous circle at a point G above A. The points $\{G, A, E\}$ form a $3:4:5$ triangle with its right-angle at A.

Draw a circle with centre B, radius 4, intersecting the circle S at H, above B. The points $\{E, B, H\}$ form a $3:4:5$ triangle with the right-angle at B.

The length GH is $\sqrt{2}$, the diagonal of a unit square. Set the compasses to radius 1 to construct the other two corners, I, J, with intersecting arcs from G and H.

ADF writes—Three other solutions were received. We disqualified two immediately for the shameless use of a ruler. We really did mean it when we specified 'only a pair of compasses.' The third was more subtle. **Stuart Cresswell** required the mid-point of an arc for a part of his construction and he thought he could get away with an infinite sequence of arcs converging to the desired point. However, it is a rule of the game that the construction should take only a finite amount of time.

There is an interesting dual to the problem, below.

Problem 178.2 – Construct another square

Given a unit circle and its centre, construct a square of side one unit, using only a ruler.

Problem 178.3 – Square-free integers

An integer is *square-free* if it is the product of distinct primes. Are there infinitely many positive integers n such that both n and $n+1$ are square-free?

'Single men are about 1.8 times as likely to die as comparable married men; single women are about 1.5 times as likely to die as comparable married women.'—*The Guardian*. [Spotted by **Peter Fletcher**.]

'Which nautical unit of speed is equal to 1.5 mph?'—*The Weakest Link*, BBC2. [**JRH**—1.5 miles is 0.5 leagues, which we think is some kind of natural unit. Tennyson: *Half a league, half a league, / Half a league onward.*]

When is $\infty + \infty$ not equal to 2∞?

Martin Cooke

In my 'Do you know your infinity times tables?' of issue 176, I claimed that $\infty + \infty = X$ (where X is an 'extended set' and so is not equal to ∞ but contains it as an element) in order to create my algebraic tables, a claim which I will now try to justify by looking at three underlying concepts:– extended sets, the potential incompletability of $\mathbb{N}$, and the concept of lines not being made of points. Basically, however, subtraction is an implicit question which is derived from addition (e.g. $2 - 4$ is the question 'what, when added to 4, yields 2?') so $\infty + \infty = \infty - \infty$ could be thought of as the extended set of all the (relevant) numbers which, when added to ∞, yield ∞, which is why $\infty + \infty = X$.

Extended sets are best introduced by an example. Consider a wardrobe of clothes—these can be subdivided in many ways (e.g. by colour, by fabric types for laundry purposes, by clothing types for storage) which are not naturally in the form of nested subsets, but are context-dependent (what are regarded as similarities in one context may be dissimilarities in another) and hence conceptually independent (not unlike the famous rabbit/duck illusion). The underlying set of garments is not a good place to start from, as it presumes that they are known and are indeed fundamental, whereas the general situation would start from noun-concepts which develop alongside our number-concepts (e.g. red, delicates, socks); but a more pertinent example of this may be the 'internal' structure of ∞: I have used ∞ as a symbol for 0^{-1} but others may wish to regard several 'numbers' as mapping to 0 under reciprocation. If you like to think of infinite straight lines as 'really' straight then you may want $\infty = (+\infty, -\infty)$. Replacing ∞ by this pair in my tables gives the same pattern, because, for example, $(+\infty) + (+\infty) = (+\infty)$, but $(+\infty) + (-\infty) = X$, as above, so $(+\infty, -\infty) + (+\infty, -\infty) = X$, combining the four elements of the table. That is, ∞ is not then a 'set' of two elements, it is these two elements which are involved in the table, and no presumptions are being made about other possible substructures for ∞, which may depend on context (e.g. if you are doing projective geometry you will want the infinite circle concept). The actual nature of the line is guessable, but it is the nature of its labelled parts, of these labels, which is of interest in maths. However, the circle idea indicates another way of looking at $\infty + \infty = \infty$, since the addition can be seen as taking one from the origin to ∞ and then back to the region of the origin, carrying the vagueness of ∞ (which could not correspond to only one point) back to this region, whence the answer is X.

As regards the incompletability of a sequence, the question of whether all the natural numbers exist (in whatever way the first few do) is already thousands of years old, and is still an issue in the philosophy of maths, despite their totality being assumed in maths itself (i.e. the set $\mathbb{N}$ is taken to exist, although the ontological identity of 1 with $\{0\}$ is less popular, and can confuse even professors!) so I shall only *describe* a 'Platonistic' version of incompletability here (the standard philosophical position against the totality of $\mathbb{N}$ being 'Intuitionistic'). $\mathbb{N}$ is fundamentally a sequence (this being its ordinal aspect) produced iteratively (from l, by the rule $n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N}$ and hence is best modelled temporally, as a process, whereas the geometrical (Euclidean, infinite) line is abstracted from a space which is already 'all there.' If the points of this line which can be labelled by $\mathbb{N}$ only exist as such when they are labelled by an incompletable $\mathbb{N}$, then the collection of all *these* points is also inherently incompletable, since this labelling is an incompletably endless sequential process (points which are to be labelled are already in the line and hence are already Platonistically labelled, since I am not assuming any shortage of time for the labelling here). That is, the spatial nature of the geometric line does not necessarily imply such a nature for $\mathbb{N}$, despite it being possible to regard $\mathbb{N}$ as a collection of points on this line. Although such a Platonistic incompletability (i.ethis being regarded as an inherent part of the nature of the numbers which we learn about in infancy) does not imply less maths (unlike the Intuitionistic version, which does not allow $\mathbb{R}$'s structure, for example) and is the common belief in history and amongst children, and may be quite common among practising mathematicians, it is certainly not standard, and a rigorous defence of it would be unmathematically lengthy and conceptually messy. It can be stated quite simply though:– The numbers 1, 2, 3, ... go on and on in a sequence and cannot reach infinity, since if they did then infinity would be some finite number plus 1. But if they stay finite then they remain increasable and so never reach every finite number (if there are always more then they aren't all reached—this doesn't imply that there is some finite which is not reached!) so they are incompletable. Any assumption of them existing 'all there,' on a line or axiomatically as a set, can't bypass their sequential definition, their inherent and implicit nature, what they individually are.

Saying that lines are not made of points should not be read as meaning that geometric points do not 'exist' everywhere on a geometric line, but that no collection of labelled or specifiable points can entirely compose such a line. For example, assuming incompletability can yield a version of $\mathbb{R}$ which is uncountable and exhaustively subdivides any finite part of the line, but which doesn't discriminate between points which are only infinitesimally distinct (and also implies the existence of such geometric infinitesimals), and only extends incompletably along the actually infinite geometric line. There is therefore 'room' on this geometric (but not on the real number) line for a part actually at ∞, and many points on it (I imagine $+\infty$ of them, quite a large cardinal!), most of them unlabellable. It is also implied that line sections are always closed (although sets of points can be open in the usual sense of excluding the end point), so that ∞ could be regarded as closing the line quite consistently. Hence I envisage a line which is made entirely of geometric points, but as a line, not made of points which could be usefully considered as distinct, specifiable points. In practice this would not change much, as we deal mostly with labelled points, and may actually only add to the maths, clarifying the ontology of (but not disallowing) the maths we already have.

In passing, note that totalities are not obviously well-behaved (in a set-theoretical way) for the noun concepts either. E.g. any line dividing Red from not-Red must separate colours which are indistinctly tinged (and similarly with Socks and Delicates) even though there's clearly no problem with using the word Red to refer to countable objects.

Solution 176.2 – Population

> In a given population, 2/3 of the men are married and 3/5 of the women are married. What fraction of the population are married?

The answer is the harmonic mean of the men and the women. After all, isn't that what marriage is all about? Thus

$$\frac{1}{\frac{1}{2}\left(\frac{1}{m}+\frac{1}{w}\right)},$$

where $m = 2/3$ and $w = 3/5$.

Ralph Hancock wrote a BASIC program to determine the smallest number of people, 19, for which the given proportions can hold. However, **Martin Cooke** observes that the population can be as small as eight in a society where the marriage rules are sufficiently flexible.

Problem 178.4 – Palindromic birthdays

Tony Huntington

My birthdate is palindromic: 25/6/52. What is the probability of that? How does the probability vary from year to year? Is there any pattern or rule for the occurrence of palindromic dates?

A mathematician should have both a mistress and a wife. He can say to his mistress, "I am going to stay with my wife for a short while." He can then tell his wife, "I am leaving you temporarily to go and live with my mistress." Now he is free to visit the university library and do some mathematics.

Fractal geometry

Barbara Lee

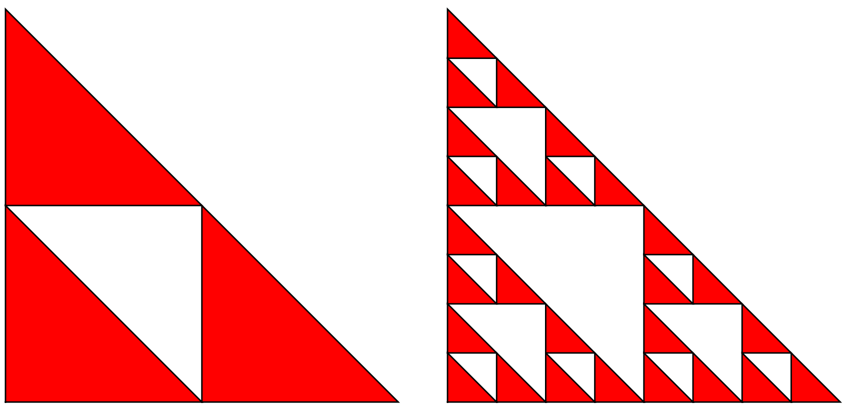
The Faculty of Mathematics does not run a course on fractal geometry as such. However, much of the underlying maths can be found in units D1 and D3 of M337 and in algebra, groups and number theory.

Most books tend to avoid actually defining *fractal* so we may as well say that fractal geometry consists of images plotted from mathematical functions that have been subject to iteration, the final pattern being made up of repetitions of the first iteration. Fractal geometry models physical structures from the natural world in the Euclidean plane as well as sets such as the Julia and Mandelbrot sets in the complex plane.

There has been a tendency to think of these fractal images as 'pretty pictures' which is rather unfortunate because the maths is interesting and often produces elaborate and unusual pictures, especially if some of the iterations are plotted in different colours. Ferns, leaves, flowers and coastlines are very attractive in several colours.

One of the most common and simple examples is the Sierpiński triangle, which is best plotted as a right-angled isosceles triangle. (See diagram, below.) Starting with a plain black triangle the first iteration removes the centre triangle, leaving three black ones. The centre triangle is removed from each of these three black ones and so on, the nth iteration resulting in 3^n black triangles.

Function codes, algorithms and programs for plotting fractal images can be found in books on fractal geometry.



A reply to Arthur Quigley's letter in 176

> [Problem 174.4: You start with £32 and bet on the toss of a coin. On each turn you stake half your capital, and your opponent matches your stake. You play six times and win half of the plays. What is your capital now?]

Martin Cooke

I feel qualified to explain the result (solution 174.4) , as my first thoughts upon seeing the problem were similarly that 'you' would end up with your original £32, and that therefore the maths had been misapplied. I see now that this problem had something of the flavour of a paradox. To start with, it became clear that the opponent must have more than £32, since otherwise you couldn't win more than once initially. Basically, you win more from winning than you lose by losing, and balancing the wins and losses over all possible outcomes means that the high potential gains from mostly winning the tosses are balanced by a net loss for the given 50–50 outcome. For example, for two tosses the remaining capital could be £8 (two losses), £24 (twice) or £72, giving an average of £32, as desired.

This problem reminded me of the once-famous Petersburg paradox [l] dating from the 1720s and concerning two people A and B betting on coin tosses. If a head appears in a sequence of tosses for the first time on the nth toss then A gives £2^{n-1} to B. The problem is to find out how much B should pay A in order to entice A to play. The mathematical expectation (obtained much as Tony Huntington obtained his solution to 174.4 in M500 **176**) is infinite whereas common sense would advise most people to play for £10. At the time, some people explained the paradox away by pointing out that A's resources aren't infinite, which is true but hardly relevant, although it does illustrate a truth of paradoxes, that a 'solution' is often just what you think of before deciding to think of something else! What makes a problem a paradox is that all your assumptions are attractive to you, but people often have different underlying assumptions.

Another maths paradox involving expectations concerns the setting of a 'surprise' exam [2]. It is of continuing interest in the journals, like those of Zeno (although that is a different story—see my article in **176**), and goes something like this: A teacher tells the class that an exam is to be set the following week (Monday–Friday) but that they won't know for sure which day it will be until they get to the classroom. The problem is to find the last day when she could set the exam, using induction. Friday, for example, does not seem possible since on Thursday evening the exam would seem to be certainly on Friday. The paradox arises when we apply this reasoning (moving from Friday being the last available day to Thursday being the last available) inductively, to rule out all the days—countering the common sense idea that if the exam was set on Tuesday, say, then it would not be expected with certainty (least of all by someone who had ruled out every day!). My 'solution' to this is that since you prove that the test can't be on a Friday to start with then it would indeed be a surprise if it was set on Friday, so that the teacher could set it for the Friday and do so surprisingly. That is, the reasoning that gets the starting point of the induction is itself only half of a piece of reasoning which is viciously circular. There is no standard solution to this paradox, it being deserving of the name, and this is not the more mathematical solution you would find in [2], but what made me happy to leave it there (and return to Zeno) is the thought that if the teacher told the class that the exam would be the following week but not that it would be a surprise, but wanted it to be a surprise anyway, then she couldn't have set it for the Friday.

1. *A History of Mathematics*, Carl B. Boyer, 2nd ed., Wiley, New York 1991.

2. How To Set A Surprise Exam, Ned Hall, *Mind*, Oct. 1999.

Solution 175.5 – abc

Suppose that $a+b+c = ab+ac+bc = 0$. Prove that if n is a positive integer then

$$a^n + b^n + c^n = \begin{cases} 3(abc)^{n/3} & \text{if } n \text{ is a multiple of 3} \\ 0 & \text{otherwise} \end{cases}.$$

John Reade

Surely the way to do this problem is to construct a cubic equation with a, b, c as roots. The conditions $a+b+c = ab+ac+bc = 0$ imply that the coefficients of x^2 and x are zero. Therefore the equation must take the form

$$x^3 - \alpha^3 = 0$$

(if α is real). So a, b, c are α, $\alpha\omega$, $\alpha\omega^2$, where $\omega = e^{2\pi i/3}$ is the primitive complex cube root of unity. Thus

$$\begin{aligned} a^0 + b^0 + c^0 &= 1+1+1 = 3, \\ a+b+c &= 0 \qquad \text{(given)}, \\ a^2+b^2+c^2 &= \alpha^2 + \alpha^2\omega^2 + \alpha^2\omega = \alpha^2(1+\omega^2+\omega) = 0, \\ a^3+b^3+c^3 &= \alpha^3+\alpha^3+\alpha^3 = 3\alpha^3 = 3abc, \end{aligned}$$

etc.

'Many of our arithmetic senior citizens have been advised by their doctors that swimming is the best exercise to keep them mobile.'—*Stockport Recorder*. [Spotted by **Peter Fletcher**.]

Letters to the Editors

Cylinder

Dear Jeremy,

There are several ways of looking at this [Problem 171.1: What is the probability that a cylinder thrown in a random fashion will land on one of its ends? See M500 **171** 9, **173** 19, **174** 16, **176** 22].

Consider a normal six-sided cubic die. Mark two opposite faces as the 'ends'. It has a 1 in 3 chance of landing on an 'end', and a 2 in 3 chance of landing on a 'side'. Now lathe off the edges bordering the 'sides', and you have a cylinder with length equal to its diameter. I don't see why this should be much more likely to land on its end.

The area of each end is πr^2 (so two ends: $2\pi r^2$), and the area of the cylinder's side is $2\pi r \cdot 2r$. So the side has twice the area of the ends put together (just like the die, above). As the side and the ends are equidistant from the centre of mass of the cylinder, the cylinder has the same energy regardless of orientation (side or end). This implies that it's more likely to finish on its side.

If you put the cylinder on end and knock it over, it will fall on its side, and may bounce further. Assuming the bouncing is random, it has a 50 per cent chance of finishing on end (as Gordon Alabaster [M500 **174** 16] stated).

If you put the cylinder on its side and knock it, it will either fall on its end, or roll, depending on the direction you knock it. Assuming you knock it hard enough, the above 50 per cent chance of its finishing up on its end is probably multiplied by $\sin x$ where x is the angle between the rolling angle and the angle of hit. Gordon Alabaster doesn't seem to have considered this option.

I don't know what aspect ratio will give a 50 per cent chance of the cylinder landing on end, but the length is less than the diameter. I'd guess $1/\sqrt{2}$. I suspect it depends on the roughness of the surface.

Jim Davies

---

1 + 1 = 2

Dear Mr Forbes,

Now that the exams are over I have had the opportunity to look in detail at issue **175** and was surprised to find on page 40 reference to the Russell and Whitehead proof that $1 + 1 = 2$. I was even more surprised to learn that this was part of a wider purpose of establishing a rigorous foundation for mathematics. I was under the impression that Gödel's Incompleteness Theorem, published some eighteen years after Russell's and Whitehead's efforts, showed that such a rigorous foundation could not be proved.

As a mathematical novice I should be grateful if anyone could explain in terms simple enough for me to understand them how these apparently contradictory theorems can be reconciled, or have I just got the wrong end of the stick?

Yours sincerely,

Malcolm Fowler

Gerald Whitrow

Dear Tony

Eddie Kent's short piece on Gerald Whitrow [M500 **176** 19] brought back very special memories of my days as an undergraduate at Imperial College in the late 50s. There were 32 of us in my year studying straight mathematics and all our courses were compulsory. All were examined except M30, *History and Philosophy of the Mathematical Sciences*, a two-hours-a-week course in the second semester of the third year. It was taught by Gerald Whitrow and was the only course with full attendance. His lectures were inspirational and the syllabus included 'the history of dynamics from Aristotle to Einstein and the associated mathematical developments; the significance of the subject in the history of scientific method.' He had no lecture notes and we took none, we just listened and were enthralled. Whenever I read historical mathematics I recall the start he gave me.

Bryan Orman

---

32 pounds

Dear Tony,

I would like to reply to a point made by Arthur Quigley in his letter about the 32 pounds problem [M500 **176** 26]. He says that the situation is the same for the opponent; that he will also be left with £13.50, so where is the missing money?

The crucial difference is that the opponent always matches your stake. You are the one who stakes half your capital. The opponent matches that stake regardless of how much he has, so the situation cannot be the same for both players, and there is no missing money.

Regards,

Gail Volans

Kiwi fruit

Continuing the mini-saga of the kiwi fruit, I note that the Sainsbury's chit in 176 uses decimal notation correctly, yet, amazingly, after over 30 years of decimalization many of our contributors still use £13.50p, for example.

I think I am correct is saying that a number with a decimal point is all in the same units. Please: £13.50 or 1350p.

Brian O'Donnell

Factorial squares

The 'factorial squares' appears to be an interesting problem, but seems to me to fizzle out quickly. I've seen this before, and it was called 'Brown numbers' on that occasion. Maybe someone can enlighten me as to why.

I couldn't find any solutions to $n! + 1$ a square, searching up to integer lengths of 256. There are only a few trivial solutions to $n! - 1$ a square (0, 1, 2).

A slightly more interesting version is $n! \pm 1$ a prime. Here are the solutions I found: $0! + 1 = 2$ $1! + 1 = 2$, $2! + 1 = 3$, $3! + 1 = 7$, $11! + 1 = 39916801$, $\ldots$; $3! - 1 = 5$, $4! - 1 = 23$, $6! - 1 = 719$, $7! - 1 = 5039$, $12! - 1 = 479001599$, $\ldots$.

Regards,

Dave Ellis

Dear Editor,

Greetings. My thoughts on Problem 176.4. To quote from A. H. Beiler's *Recreations in the Theory of Numbers* (Dover 1966), p. 161: 'What, in general, are the solutions of $n! + 1 = x^2$ besides the three values $n = 4, 5, 7$? The equation has been investigated up to $n = 1020$ and no other solutions have been found. If any such squares exist, they must be enormous numbers; even 100! has 158 digits, and 1020! has over 2600.'

As for $n! - 1$, the only possible cases are 0, 1 and 2; $n = 3$ does not work and, for higher values of n, $n! - 1 \equiv 3 \pmod 4$ and hence cannot be a square.

Michael Adamson

ADF—In a few idle moments I extended the search for squares all the way up to $51000000! + 1$. (Exercise for reader: How?) No more solutions.

The search for primes of the form $n! \pm 1$ seems to be a favourite sport amongst number theorists. See www.utm.edu/research/primes/largest.html for the most recent achievements.

Problem 178.5 – Reward a friend

Tony Forbes

In Chris Tarrant's popular television programme *Who Wants to be a Millionaire?*, what is a fair reward for a person who helps the contestant when the 'phone a friend' lifeline is used?

The game involves a single contestant, C, and proceeds in stages.

At each stage the host asks C a general-knowledge multiple-choice question with four options. To help find the (unique) correct answer, C may invoke one or more of three 'lifelines': (i) C may ask the host to identify two wrong answers; (ii) C may request a frequency distribution showing how the studio audience answered the question; (iii) C may phone a friend. If a lifeline is used, it becomes unavailable for the remainder of the game. After viewing the question and perhaps invoking one or more lifelines C chooses whether or not to attempt an answer.

The game begins with question 1. If C does not answer question n, the game ends and C receives $R(n-1)$. If C gives a wrong answer to question n, the game ends and C receives $W(n)$. If C answers question n correctly, the game continues with question $n+1$ unless $n = 15$ in which case C receives $R(15)$—hence the name of the show—and the game ends. The functions R and W are defined by

$$R(n) = \begin{cases} £100 \cdot n & \text{if } 0 \le n \le 3 \\ £500 \cdot 2^{n-4} & \text{if } 4 \le n \le 11 \\ £125000 \cdot 2^{n-12} & \text{if } 12 \le n \le 15, \end{cases} \qquad W(n) = \begin{cases} R(0) & \text{if } 1 \le n \le 5 \\ R(5) & \text{if } 6 \le n \le 10 \\ R(10) & \text{if } 11 \le n \le 15. \end{cases}$$

Just desserts

Eddie Kent

Having dealt in recent months with various prizes on offer for excellence, it is probably no bad thing that I mention the most important one of all: the Ig Nobel Prize. This is given each year to an individual whose achievements 'cannot or should not be reproduced'. For instance the year 2000 Ig Nobel Prize for Physics was awarded for an experiment that levitated a frog using magnets. Have a look at www.improbable.com/ig/ig-2000-winners.html.

While you're about it you might as well glance at the site's poor relation: www.nobel.no/indexen.html.

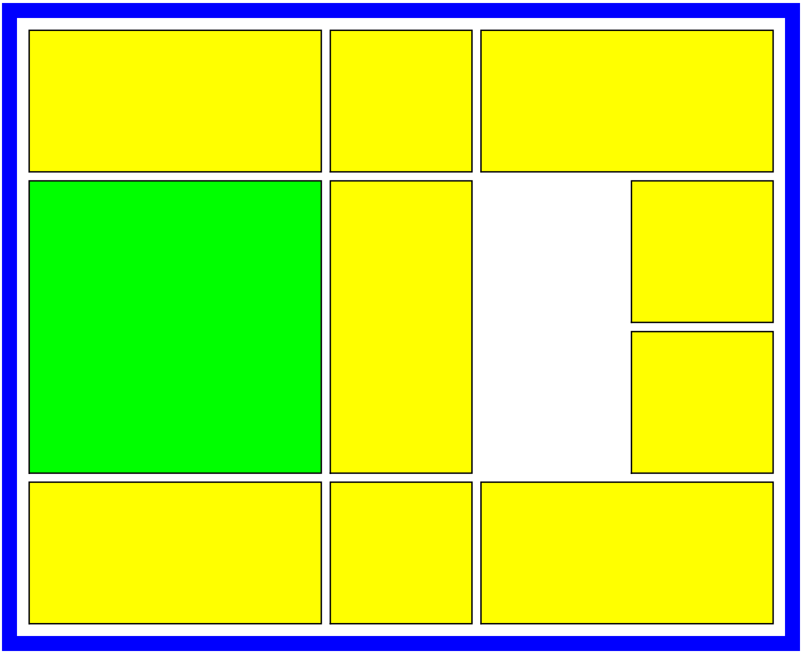
In common with the Nobel Prize, mathematics seems to be once again ig-excluded. There must be some examples of mathematical reasoning that would fit the criteria, so how about suggesting a candidate or two for an OutField Medal (Brickfield? Cornfield?). I do not think a half-page proof of FLT, RH or 4CT would ever have qualified.

Problem 178.6 – Ten blocks

Colin Davies

This is like Sam Loyd's well-known '15 Puzzle'. Ten blocks of various sizes slide about in the obvious manner.

Move the large square from the middle-left of the array to the middle-right.



Problem 178.7 – Series

Barry Lewis

Prove that

$$\frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \dots \;=\; \frac{x}{1+x^2} + \frac{x^2}{1+x^4} + \frac{x^3}{1+x^6} + \dots .$$

Twenty-five years ago

From M500 29 and 30

John Reade—Error analysis of $e - (1+1/n)^n$. [We think this helps with Problem 177.4.] The error is roughly equal to $e/2n$ for large n, which bears out Peter Weir's guess (M500 **28** 5) that the number of correct decimal places in $(1+1/n)^n$ is the same as the number of digits of n. To see this, consider the expansion

$$\begin{aligned}(1+x)^{1/x} &= e^{1/x\log(1+x)} \\ &= \exp\left(\frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right)\right) = \exp\left(1 - \frac{x}{2} + \frac{x^2}{3} - \dots\right) \\ &= e\, e^{-x/2} e^{-x^2/3} \dots \\ &= e\left(1 - \frac{x}{2} + \frac{1}{2!}\left(\frac{x}{2}\right)^2 - \dots\right)\left(1 + \frac{x^2}{3} + \frac{1}{2!}\left(\frac{x^2}{3}\right)^2 + \dots\right)\dots \\ &= e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 - \dots\right),\end{aligned}$$

valid for $|x| < 1$. Writing $x = \dfrac{1}{n}$, we have $\left(1 + \dfrac{1}{n}\right)^n = 1 - \dfrac{1}{2n} + \dfrac{11}{24n^2} - \dots$, which shows that $e - \left(1 + \dfrac{1}{n}\right)^n = \dfrac{e}{2n} + O\left(\dfrac{1}{n^2}\right)$ for large n.

Jeremy Humphries—

Said a man to his offspring, "Indeed, Ron,
It is obvious that what you need, Ron,
Is a twelve-sided hat."
(For the boy he begat
Had a head like a dodecahedron.)

Marion Stubbs—Last month I told a fair number of people that Peter Weir had coped with 111 enquirers from *Sesame* in one week. It turned out that he was using Roman numerals, merely clocking up each enquiry as it arrived, and forgot to convert the total to 3 before posting.

Tom Dale and Max Bramer—What's the next term: O, T, T, F, F, S, S, E, N, ? [Is there a last term? Cf. M500 **177** 27.]

Contents

M500 178 – February 2001