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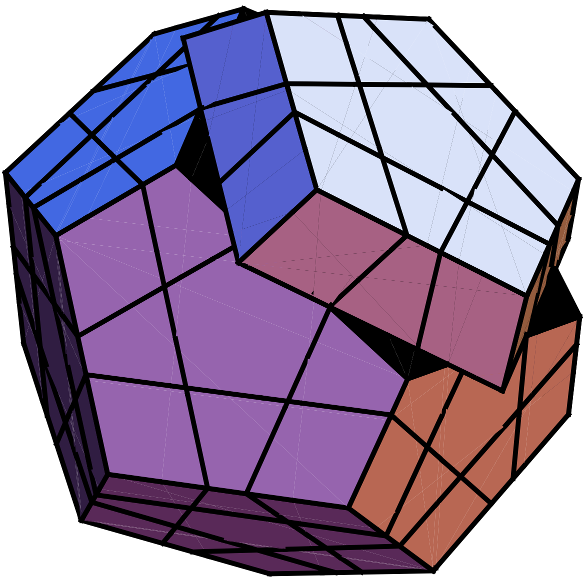
M500 184



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

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Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail. Camera-ready copy can be accepted if it follows the general format of the magazine.

Cups and downs

Paul Garcia

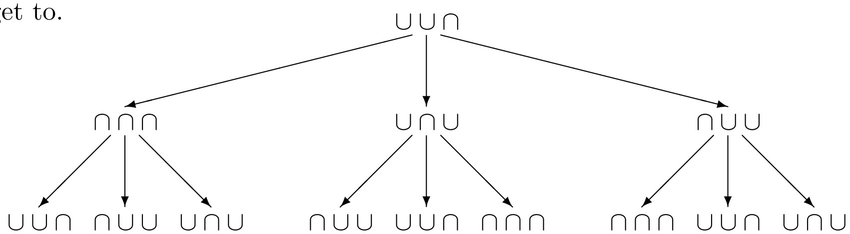
The problem is this: you have three cups in a row, two the right way up and one the wrong way up:

$$\bigcup \quad \bigcup \quad \bigcap$$

Can you get all the cups the right way up if you are allowed to invert the cups only two at a time? Some possible sequences of moves are:

$$\cup\cup\cap \to \cap\cap\cap \to \cap\cup\cup \to \cup\cap\cup$$
$$\cup\cup\cap \to \cap\cup\cup \to \cup\cap\cup \to \cup\cup\cap$$
$$\cup\cup\cap \to \cup\cap\cup \to \cap\cap\cap \to \cap\cup\cup$$

After a few goes, you might get the feeling that it can't be done. You might try drawing a tree diagram to show all the possible positions you can get to.



It is clear after any three moves, no new positions are generated, reinforcing the view that the puzzle is insoluble. But why is this so? Well, there are only three possible 'moves'; cups 1 & 2, 2 & 3 or 1 & 3. Moreover, if a cup is moved an even number of times, it will finish the same way up as when it started. To get a cup to finish the other way up from its starting position, it will have to be turned an odd number of times. So we need a sequence of moves which turns cups 1 & 2 an even number of times whilst turning cup 3 an odd number of times

Suppose we code the moves like this: moving cups 1 & 2 is 110, cups 2 & 3 is 011 and cups 1 & 3 is 101.

A sequence of moves then looks like this.

$$\cup\cup\cap \xrightarrow{110} \cap\cap\cap \xrightarrow{011} \cap\cup\cup \xrightarrow{101} \cup\cup\cap \xrightarrow{110} \cap\cap\cap$$

Counting the 1s in each column, we see that cup l gets moved three times, cup 2 three times and cup 3 two times.

This is clearly not a solution to the problem. But what would a solution look like? All we are interested in is whether a cup moves an odd or even number of times, so all we need to do is add up the 1s in each column and then divide by 2. If the remainder is 0, the cup has moved an even number of times. If the remainder is 1, the cup has moved an odd number of times. This is addition modulo 2.

So if we had a solution to the problem, and did this addition, the result would be 001. Now the question becomes, can we make 001 by adding combinations of 011, 110 and 101 modulo 2? If we call 110 move A, 011 move B and 101 move C, we can draw an addition table like this (where I means the move 000):

$+$	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

No new positions are generated. No matter which pair of moves we combine, the result is always the same as one of the other moves (or results in move I, which effectively does nothing). So the system is closed, and doesn't include 001. This confirms what we already suspected from the tree diagram earlier. It also means that changing the position of the upside cup won't help. If we start with $\cup\cap\cup$, then we need to make 010 which is also unattainable. Similarly, $\cap\cup\cup$ requires 100, which also can't be done.

Now, a natural question to ask is, what if we alter the problem a bit. Suppose we have 4 cups and invert them 3 at a time. Starting with $\cup\cup\cup\cap$, can we get $\cup\cup\cup\cup$?

Using the method for coding moves developed above, we see there are four moves, plus the 'do nothing' move, I:

$$I = 0000, \quad A = 1110, \quad B = 1011, \quad C = 1101, \quad D = 0111.$$

Let us combine these in pairs and see if we get any new moves (our target is 0001):

$$\begin{array}{cc} A & 1110 \\ B & 1011 \\ \hline U & 0101 \end{array} \qquad \begin{array}{cc} A & 1110 \\ C & 1101 \\ \hline V & 0011 \end{array} \qquad \begin{array}{cc} A & 1110 \\ D & 0111 \\ \hline W & 1001 \end{array} \qquad \begin{array}{cc} B & 1011 \\ C & 1101 \\ \hline X & 0110 \end{array} \qquad \begin{array}{cc} B & 1011 \\ D & 0111 \\ \hline Y & 1100 \end{array} \qquad \begin{array}{cc} C & 1101 \\ D & 0111 \\ \hline Z & 1010 \end{array}$$

This has generated six new moves, U, V, W, X, Y and Z. Do these form a closed set?

Looking at all the possible sums, we get the following.

$$\begin{array}{cc} U & 0101\\ V & 0011\\ \hline X & 0110\end{array}\quad \begin{array}{cc} U & 0101\\ W & 1001\\ \hline Y & 1100\end{array}\quad \begin{array}{cc} U & 0101\\ X & 0110\\ \hline V & 0011\end{array}\quad \begin{array}{cc} U & 0101\\ Y & 1100\\ \hline W & 1001\end{array}\quad \begin{array}{cc} U & 0101\\ Z & 1010\\ \hline T & 1111\end{array}$$

$$\begin{array}{cc} V & 0011\\ W & 1001\\ \hline Z & 1010\end{array}\quad \begin{array}{cc} V & 0011\\ X & 0110\\ \hline U & 0101\end{array}\quad \begin{array}{cc} V & 0011\\ Y & 1100\\ \hline T & 1111\end{array}\quad \begin{array}{cc} V & 0011\\ Z & 1010\\ \hline W & 1001\end{array}$$

$$\begin{array}{cc} W & 1001\\ X & 0110\\ \hline T & 1111\end{array}\quad \begin{array}{cc} W & 1001\\ Y & 1100\\ \hline U & 0101\end{array}\quad \begin{array}{cc} W & 1001\\ Z & 1010\\ \hline V & 0011\end{array}$$

$$\begin{array}{cc} X & 0110\\ Y & 1100\\ \hline Z & 1010\end{array}\quad \begin{array}{cc} X & 0110\\ Z & 1010\\ \hline Y & 1100\end{array}\quad \begin{array}{cc} Y & 1100\\ Z & 1010\\ \hline X & 0110\end{array}$$

Three of the combinations, $U+Z$, $V+Y$ and $W+X$ give a different result, which I have called T; all the other combinations of these six moves give another one out of the six.

Now by inspection it seems clear that we can get 0001 (the target) by combining T with A:

$$\begin{array}{cc} T & 1111\\ A & 1110\\ \hline & 0001\end{array}.$$

But T is a combination of U, V, W, X, Y and Z, which are themselves combinations of A, B, C and D. So we can decompose $T+A$ into a sequence of moves to solve the puzzle, like this.

$$\begin{tikzcd} & T \arrow[dl] \arrow[d] \arrow[dr] & \\ U+Z \arrow[d] & U+Z \arrow[d] & U+Z \arrow[d] \\ (A+B)+(C+D) & (A+C)+(B+D) & (A+D)+(B+C) \end{tikzcd}$$

$$\begin{array}{cc} A & 1110\\ B & 1011\\ \hline U & 0101\end{array}\quad \begin{array}{cc} A & 1110\\ C & 1101\\ \hline V & 0011\end{array}\quad \begin{array}{cc} A & 1110\\ D & 0111\\ \hline W & 1001\end{array}\quad \begin{array}{cc} B & 1011\\ C & 1101\\ \hline X & 0110\end{array}\quad \begin{array}{cc} B & 1011\\ D & 0111\\ \hline Y & 1100\end{array}\quad \begin{array}{cc} C & 1101\\ D & 0111\\ \hline Z & 1010\end{array}$$

So we get three sequences of five moves each of which will solve the puzzle: $ABCDA$, $ACBDA$, $ADBCA$.

If we look at combinations of T with all our other moves, we get the following.

$$
\begin{array}{rl} B & 1011 \\ T & 1111 \\ \hline & 0100 \end{array} \quad
\begin{array}{rl} C & 1101 \\ T & 1111 \\ \hline & 0011 \end{array} \quad
\begin{array}{rl} D & 0111 \\ T & 1111 \\ \hline & 1000 \end{array} \quad
\begin{array}{rl} U & 0101 \\ T & 1111 \\ \hline Z & 1010 \end{array} \quad
\begin{array}{rl} V & 0011 \\ T & 1111 \\ \hline Y & 1100 \end{array} \quad
\begin{array}{rl} W & 1001 \\ T & 1111 \\ \hline X & 0110 \end{array}
$$

$$
\begin{array}{rl} X & 0110 \\ T & 1111 \\ \hline W & 1001 \end{array} \quad
\begin{array}{rl} Y & 1100 \\ T & 1111 \\ \hline V & 0011 \end{array} \quad
\begin{array}{rl} Z & 1010 \\ T & 1111 \\ \hline U & 0101 \end{array}
$$

There are two things we can tell from this. Firstly, notice that T with any of U, V, W, X, Y or Z just gives another one out of that set. Now the moves U to Z can also be regarded as inverting the cups two at a time. Just as in our first problem, the set of moves I, T, U, V, W, X, Y, and Z is closed and doesn't include our target of 0001. We can summarise this in a table:

$+$	I	U	V	W	X	Y	Z	T
I	I	U	V	W	X	Y	Z	T
U	U	I	X	Y	V	W	T	Z
V	V	X	I	Z	U	T	W	Y
W	W	Y	Z	I	T	U	V	X
X	X	V	U	T	I	Z	Y	W
Y	Y	W	T	U	Z	I	X	V
Z	Z	T	W	V	Y	X	I	U
T	T	Z	Y	X	W	V	U	I

The second thing we notice is that using T and B, C or D, we can generate 0100, 0010 and 1000. This means that we can still solve the puzzle, wherever the upside-down cup is.

Let us see if we can get from $\cap \cup \cup\cup$ to $\cup \cup \cup\cup$. Our target is thus 1000. From the results above, we see that $T + D$ is the solution, and by decomposing this as before we get three five move solutions:

$$\{ABCDD\}, \quad \{ACBDD\}, \quad \{ADBCD\}.$$

But wait! The first two of these terminate in DD. Now any move combined with itself just gets you back to where you started, and so is the same as the 'do nothing' move, I. So we can delete the DD from the end, since it does nothing, and get a shorter, three move solution, either ABC or ACB. It seems that our first effort was not the best possible. We need to look at all possible sequences of three moves. We can do this by looking at all the possible sums of A, B, C and D with U, V, W, X, Y and Z. This gives us this table, where $1 = 1000$, $2 = 0100$, $3 = 0010$ and $4 = 0001$

$+$	I	U	V	W	X	Y	Z
A	A	B	C	D	1	3	2
B	B	A	1	3	C	D	4
C	C	1	A	2	B	4	D
D	D	3	2	A	4	B	C

So we can read off solutions to the original problem, where the target is 0001 (= 4). They are $B + Z, C + Y$ and $D + X$, which reduce to the moves B, C, D in any order.

Using all this information, we can solve all these problems.

to get ∪∪∪∪ from	∪∪∪∩	BCD in any order
	∪∪∩∪	ABD in any order
	∪∩∪∪	ACD in any order
	∩∪∪∪	ABC in any order
	∪∪∩∩	AC
	∪∩∩∪	BC
	∩∩∪∪	BD
	∪∩∪∩	AB
	∩∩∪∪	BD
	∩∪∩∪	CD
	∩∪∪∩	AD
	∩∩∩∩	$ABCD = T$

With four cups being moved three at a time, we now have the following results:

number of cups facing down	number of moves to get ∪∪∪∪
0	0
1	3
2	2
3	1
4	4

We now know that, with three cups moving two at a time, the puzzle can't be solved, and with four cups moving two at a time it still can't be solved, but with four cups moving three at a time, it can be solved. So let us make a wild conjecture:

CONJECTURE If the number of cups to be moved is even, the puzzle can't be solved. If the number of cups to be inverted each time is odd, the puzzle can be solved in d moves, where d is the number of cups facing up at the start.

Let us look at five cups. Suppose we can move two at a time. Then the possible moves are: 00011, 00101, 01001, 10001, 00110, 01010, 10010, 01100, 10100, 11000.

Drawing up an addition table looks like a daunting task. So we'll try to work out what will happen in general terms. If we pick two different moves, and add them modulo 2, then either both 1s will 'miss' each other and the result will have four 1s in it, or two of the 1s will cancel out to give a result with two 1s in it.

So either we get another 'two-at-a-time' move, or we get the equivalent of a 'four-a-time move'. Now the possible 'four-at-a-time' moves are: 11110, 11101, 11011, 10111, 01111.

If we add one of these moves to one of the 2-moves, then either both of the 1s will cancel, and we get another 2-move, or one pair of 1s will cancel, and the 0 in the 4-move will match with the other 1 in the 2-move. The result will have four 1s in it (try it if you don't believe me!).

If we add two 4-moves together, then all the 1s will cancel except where the two 0s are, so the result will have two 1s in it and be a 2-move.

We can put this in a table.

+	2-move	4-move
2-move	2- or 4-move	2- or 4-move
4-move	2- or 4-move	4-move

Now, our target moves are 00001, 00010, 00100, 01000 or 10000. We can call these 1-moves. Clearly, we are not going to get a 1-move by combining 2- and 4-moves, so the puzzle won't have a solution. That leaves moving three at a time, or 3-moves. They are: 00111, 01110, 11100, 10011, 10110, 11001, 11010, 10101.

Combining two 3-moves, by the same reasoning as above, will give either a 2-move or a 4-move. Combining a 3-move with a 2-move or a 4-move can easily create a 1-move, and solve the puzzle.

It also looks as if only three moves are required for a solution. Let's try an example: start with $\cup\cup\cup\cup\cap$, so our target is 00001. Three moves will work: 00111, 11100, 11010: 00001.

Can we see why this system works? Well, if we are turning an even number of cups on each move, we will have a move of the form $00\ldots0111\ldots11$ where there are $2n$ 1s. Now an even number can only be split into a sum of either two odd or two even numbers. When we add two $2n$-moves, there are two possibilities: either an even number of 1s will pair up and cancel, leaving an even number of 1s in the result, or an odd number of 1s will pair up and cancel, in which case each move will contribute an odd number of 1s to the result. Since two odd numbers add to make an even number, we have another even move. So adding two even moves always gives another even move, making odd moves like 0000. . . 0001 unattainable.

If we are turning an odd number of cups each time, a $(2n + 1)$-move will have an odd number of 1s in it. When we add two such moves, then either an odd number of 1s pair up, leaving an even number of 1s in the result, or an even number of 1s match, leaving an odd contribution from both moves, again giving an even move as a result. But now we can add an odd move to an even move. If an odd number of 1s pair up, the result will have an even contribution from the odd move, and an odd contribution from the even move, giving an odd result. If an even number of 1s pair up, the contributions are reversed, but the result remains odd. So targets like 000. . . 0001 are attainable.

This all follows really from the well-known facts that

$$\text{odd} + \text{odd} = \text{even} + \text{even} = \text{even},$$
$$\text{odd} + \text{even} = \text{even} + \text{odd} = \text{odd}.$$

Suppose we have six cups turning 5 at a time. I can select two moves which will match in four places, leaving an effective 2-move. This I can combine with a suitable 5-move to create a 3-move. By carefully selecting two more 5-moves to make an appropriate 2-move, I can reduce the 3-move to a 1-move (the target), and all in 5 moves.

For example.

move	position
111101	∩∪∪∪∪∪
111011	∪∩∩∩∪∩
110111	∩∪∪∩∩∪
101111	∩∩∩∩∩∪
111110	∪∪∪∪∪∩

So we have proved this theorem.

Theorem It is the case that m cups moved n at a time can be solved in at least n moves if n is odd, but not at all if n is even, where only one cup is facing down at the start.

Perhaps you can find the result for m cups moved n at a time where p are facing down at the start.

Problem 184.1 – Twelve boxes

Jeremy Humphries

There are twelve closed boxes. They are numbered 1, 2, ..., 12. On each turn you throw a pair of dice—suppose the result is (a, b), $1 \leq a, b \leq 6$. You must then open closed boxes whose numbers add up to $a + b$. If this is impossible, the game stops and you lose. If you manage to open all the boxes, the game stops and you win. If neither, the game continues.

Clearly the game must stop after a finite number of throws. *What's the probability of winning?*

[**ADF** writes—In case the above is not clear, here's how a typical game might go.

We start with boxes 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12 closed. 1) A double-six is thrown. We appreciate this good fortune and take the opportunity to open the box numbered 12. 2) A double-six is thrown. Open 1 and 11. We are not allowed to waste this move by 'opening' the already open box 12. 3) A double-six is thrown. Open boxes 2 and 10. 4) A double-six is thrown. Open 5 and 7. 5) A double-six is thrown. Open 4 and 8. 6) A double-six is thrown. Open 3 and 9.

At this point we have opened all the boxes except the one numbered 6. A double-three, or any pair of numbers that add to 6, such as (1,5), or (2,4), would win the game because box 6 can be opened. Alas! It is not to be. A final double-six appears and there is no set of closed boxes adding up to 12. We lose.]

Problem 184.2 – Monk

John Bull

A monk sets out at 6 am from a monastery at the base of a mountain to visit a shrine at the top. He follows a long and tortuous path to arrive at the shrine at 6 pm. Every so often, as the mood takes him, he stops to rest, so his progression along the path is not uniform. When he arrives at the shrine he is so tired he decides to rest for a couple of days before returning. A few days later he sets out at 6 am, follows the same path back down the mountain and arrives back at the monastery at 6 pm.

Show that on his outward and return journeys there is a point somewhere along the path where he will set foot at the same time of day.

'In physics we try to explain in simple terms something that nobody knew before. In poetry it is the exact opposite.'—Paul Dirac.

196

Eddie Kent

Tell me if I am wasting my time and I might stop. But still, it is intriguing. I mentioned the palindrome problem recently to [the Editor of M500], and the existence of the number 196, but I don't suppose he noticed.

Take a number, reverse it. Add the two. Sooner or later you get a palindrome.

I found myself thinking not about the problem itself, but how one could use Excel to do the calculations. Excel is packed with functions, but there is nothing that I know that will allow you to reverse a number—say 123 $\to$ 321. I finally managed by cobbling various simple ones together, and then it was easy to do vast amounts of sums in an instant, just by selecting and dragging. So I set up a page to do 5000 iterations (Excel has a limit on the number of decimal places—this looks about it).

Sure enough any number I picked at random worked, and 196 didn't. But then I began to notice something very odd. I haven't got far, it's too exciting to get worked up and then find it is all well known. I have not done a careful check, but a significantly large number of examples I did pursue eventually produced one of two (or perhaps three) palindromes. This is where I stopped—work to do. But shall I carry on; carefully?

8813200023188: 187, 37, 38, 46, 47, 49, 55, 56, 58, 28, ...

47337877873374: 48, 57, 60, 3, ...

The gaps and ordering are random. One could do a list with numbers of iterations, and even complete lists, though some of these are trivial. In fact I believe there is another popular number, beginning with 3. By the way, 123 is one of the boring ones—all fairly obvious and appearing within a dozen iterations.

ADF writes—I put 196 to the computer and left it running for a time. When I next looked it had done 27250 iterations and the last number had about 11313.3 decimal digits. Then I started thinking.

At each iteration the number increases by approximately 0.301 digits. A $2k$-digit number has probability $1/10^k$ of being a palindrome. The same for $2k+1$ digits. So the probability of hitting a palindrome changes by a factor of about

$$\frac{1}{10^{(\log 2)/(\log 100)}} = \frac{1}{\sqrt{2}}.$$

After 27250 iterations the probability is therefore something times $1/10^{4101}$. I stopped the computer.

Problem 184.3 – Lake escape

Keith Drever

A young lady was visiting Circle Lake, a large artificial body of water named for its precisely circular shape. To escape from a man who was pursuing her, she got into a rowboat and rowed to the centre of the lake, where a raft was anchored. The man decided to wait it out on shore. He knew she would have to come ashore eventually. Since he could run four times as fast as she could row, he assumed that it would be a simple matter to catch her as soon as her boat touched the lake's edge.

But the girl gave some thought to her predicament. She knew that once she was on solid ground she could outrun the man; it was only necessary to devise a rowing strategy that would get her to a point on shore before he got there. She soon hit on a simple plan, and it worked successfully.

What was the girl's strategy?

A few days later the young lady was minding her own business at a different lake—this time it was a natural body of water with no particular mathematical shape. Once again the man was making a nuisance of himself. As before, there was a rowing boat (with the same top speed) and a raft to the help the woman make her escape.

Now what does she do?

Problem 184.4 – Three real numbers

Colin Davies

Find three real numbers, a, b, c, such that

$$a + b + c \;=\; ab \;=\; \frac{70 + 26\sqrt{13}}{27}$$

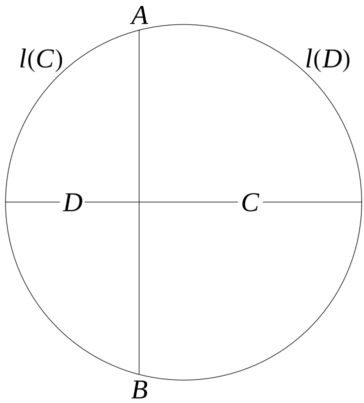
and

$$\frac{a}{b} \;=\; \frac{b}{c}.$$

[A similar problem appeared in *IEE News*; the only difference was that they had $a + b + c = ab = 25$. However I found it too difficult and after a considerable amount of effort I gave up trying to find a solution. So I simplified the problem by changing 25 to $\dfrac{70 + 26\sqrt{13}}{27}$. Honest.—**ADF**]

Solution 180.3 – Fence

> A farmer has an L-shaped barn which measures $80' \times 80'$ with a $30' \times 40'$ rectangle cut out of one corner. He also has a $50'$ roll of fencing which he wants to use to enclose a grazing area next to the barn. What is his largest possible grazing area?



Dick Boardman

There is a simple proof that a circular arc encloses the largest possible area in this case.

Draw a circle and a chord AB through it. Let the sectors have areas C and D and let the respective arc lengths be $l(C)$ and $l(D)$. Suppose there is a shape bounded by the chord AB and a curve of length $l(C)$ whose area is more than C. Then the sector D could be added to it to give a shape whose perimeter is the same as the circle but whose area is greater than the circle. This contradicts the well known theorem a circle is the shape containing the maximum area for a given perimeter. Hence no such shape can exist and the sector bounded by the circular arc contains the maximum area.

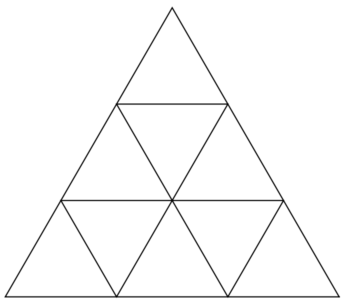
Divide this figure into two by a diameter perpendicular to AB and we have the solution to problem 180.3.

Problem 184.5 – Triangles

Martyn Lawrence

Each side of the large triangle, opposite, has been divided into three, the points joined such that there are thirteen triangles in total; nine of side 1, three of side 2 and one of side 3. If x_n is the total number of triangles, then $x_3 = 13$.

What are x_{15}, x_{16}, …, x_n?



Solution 181.1 – Find the centre

> Given a circle, is there a ruler-only construction to find the centre?

Dick Boardman

This is discussed in the book *What is Mathematics* by Courant and Robbins. This book says :-

> The circle and its centre cannot be dispensed with. For example, if a circle but not its centre is given, it is impossible to construct the latter by the use of a straight edge alone.

The book only gives an outline proof. What follows is my expansion of that proof.

Suppose there is a correspondence between the points of two planes such that straight lines transform into straight lines and the unit circle transforms into a circle.

Then any valid ruler-only construction in one plane is also a valid ruler-only construction in the other plane. In particular, if there were a valid ruler-only construction in one plane to construct the centre of the circle, it would also be a valid construction in the other. From this it follows that the centre of the circle in one plane must transform to the centre in the other plane. However, there is a transformation of this type in which the centre of one circle does not transform into the centre of the other circle. Hence the construction is impossible.

The book does not give an example of such a transformation but it is fairly easy to construct one provided that we use homogeneous co-ordinates.

In homogeneous co-ordinates, one replaces the variable x by X/Z and y by Y/Z. Thus a straight line in non-homogeneous co-ordinates is $y = mx+c$ and becomes in homogeneous co-ordinates $Y = mX + cZ$. A general conic in non-homogeneous co-ordinates is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c \;=\; 0.$$

In homogeneous co-ordinates this becomes

$$aX^2 + 2hXY + bY^2 + 2gXZ + 2hYZ + cZ^2 \;=\; 0.$$

Note that every term is now of the second order in X, Y and Z in a conic and of first order in a line. A point at infinity in homogeneous co-ordinates is a point for which $Z = 0$.

A homogeneous co-ordinate transformation is

$$\begin{aligned} X' &= aX + bY + cZ, \\ Y' &= dX + eY + fZ, \\ Z' &= gX + hY + iZ. \end{aligned}$$

In a homogeneous transformation, points become points, straight lines become straight lines and conics become conics. The intersection of two lines is always the intersection of two lines and the intersection of a line and a conic is always the intersection of a line and a conic.

However, ellipses may become hyperbolas and points at infinity become normal points. Thus a homogeneous transformation is a very general transformation indeed. Distance, angle and the ratio of two sections of a line are not necessarily preserved. The simplest quantity to be preserved is the cross ratio of four points on a line. However, all the things that can be done with a straight edge and a conic are preserved.

If we start with a unit circle and any pattern of lines in non-homogeneous co-ordinates, convert to homogeneous co-ordinates, apply a homogeneous transformation, and convert back to non-homogeneous co-ordinates we get a conic and a pattern of lines. The nine parameters in the transformation are at our disposal. If we choose them so that the coefficient of XY is zero, the coefficient of X^2 is equal to the coefficient of Y^2 and the radius is greater than zero, then the resulting conic is a circle and we have a general transformation of lines and a circle to lines and a circle. However, the centres of the two circles need not coincide.

Hence, no pattern of lines using features preserved by a homogeneous transformation can identify the centre of a circle.

In homogeneous co-ordinates the unit circle is $x^2 + y^2 = z^2$ and its centre is (0,0,1). As an example,

$$x = 3X, \quad y = 5Y - 4Z, \quad z = 4Y - Z$$

transforms the unit circle into

$$9X^2 + 9Y^2 - 32YZ + 15Z^2 = 0,$$

which is a circle whose centre is at $(0, 16/9, 1)$. The original centre (0,0,1) becomes $(0, 4/5, 1)$.

There are two Russian mathematicians; one knows the product and the other the sum of two [integers] between 2 and 100. They meet and the first says to the other, "I know the product of the two numbers, but not the two numbers."

The other replies, "Ah! I know the sum but not the two numbers."

Then the first one says, "I have just found them!"

And the other replies, "Well, me too!"

Question: What are the two numbers?

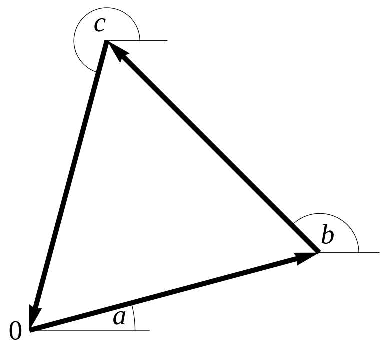
Solution 182.7 – Three cos and three sins

> Show that $\cos 2a + \cos 2b + \cos 2c = \sin 2a + \sin 2b + \sin 2c = 0$, where a, b, c are real numbers which satisfy $\cos a+\cos b+\cos c = 0$ and $\sin a + \sin b + \sin c = 0$.

Miland Joshi

John Bull's problem can be solved in terms of complex numbers. The conditions are equivalent to saying that $e^{ia} + e^{ib} + e^{ic} = 0$.

Treating the complex numbers as vectors, this requires them to form the (unit length) sides of an equilateral triangle, and in that case $e^{ib} = e^{i(a+2\pi/3)}$ and $e^{ic} = e^{i(a+4\pi/3)}$. Thus $e^{ia} + e^{i(a+2\pi/3)} + e^{i(a+4\pi/3)} = 0$ for any a gives the conditions of the problem.



Now

$$
\begin{aligned}
& e^{2ai} + e^{2bi} + e^{2ci} \\
&\quad = e^{2ia} + e^{i(2a+4\pi/3)} + e^{i(2a+8\pi/3)} \\
&\quad = e^{2ia} + e^{i(2a+4\pi/3)} + e^{i(2a+2\pi/3)} \\
&\quad = e^{2ia}(1 + e^{2\pi i/3} + e^{4\pi i/3}) \\
&\quad = 0,
\end{aligned}
$$

since $e^{8\pi i/3} = e^{2\pi i/3}$. The real and imaginary components of $e^{2ia} + e^{2ib} + e^{2ic}$ are respectively

$$\cos 2a + \cos 2b + \cos 2c$$

and

$$\sin 2a + \sin 2b + \sin 2c.$$

Hence the result is proved.

A similar theorem holds for four numbers: Given

$$\cos a + \cos b + \cos c + \cos d \;=\; \sin a + \sin b + \sin c + \sin d \;=\; 0,$$

it follows that

$$\cos 3a + \cos 3b + \cos 3c + \cos 3d \;=\; \sin 3a + \sin 3b + \sin 3c + \sin 3d \;=\; 0.$$

The proof is slightly more complicated because the points e^{ia}, e^{ib}, e^{ic} and e^{id} do not necessarily form a square.

ADF

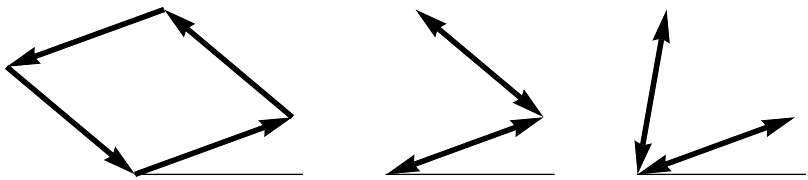
Miland Joshi also suggests a generalization: For which integers p, q is it the case that for any real numbers a_1, a_2, ..., a_p,

$$\sum_{k=1}^{p} \cos a_k \;=\; \sum_{k=1}^{p} \sin a_k \;=\; 0 \;\Rightarrow\; \sum_{k=1}^{p} \cos q a_k \;=\; \sum_{k=1}^{p} \sin q a_k \;=\; 0? \quad (1)$$

We have just seen that this is true for $(p, q) = (3, 2)$ or $(4, 3)$. A complete answer would be interesting.

The proof given above for $p = 3$, $q = 2$ works because if you start from the origin and take three equal steps, you cannot return to the origin except by travelling around an equilateral triangle.

Similarly, four steps require either a rhombus-shaped path or a path where the steps occur in opposite pairs. In each case one can verify that tripling all the angles results in a path that starts and finishes at the origin.



If $p > 4$ the situation can be far more complex (in the ordinary English meaning of that word), with many different options to consider. On the other hand, it might not be too difficult to find a p-step path such that the destination-equals-origin property is not preserved when all the relevant angles are multiplied by q. I leave the details to the reader. Meanwhile, here is a counter-example for $p = 5$, $q = 2$, 3 or 4: $a_1 = a_2 = 0$, $a_3 = 120°$, $a_4 = 180°$, $a_5 = -120°$.

Patrick Lee, **D. M. Tansey** and **Edward Stansfield** sent solutions to Problem 182.7 similar to that of Miland Joshi. **Jim James** showed that it is possible to solve the problem using 'pure' trigonometry (rather than by considering paths in $\mathbb{C}$) but the details are more complicated. Jim also proved that (1) holds for $p = 3$ if and only if q is not divisible by 3.

Of course, the real difficulty was in the title of the problem. The Editors agonized over the plural form of 'cos'. Did we get it right?

Letters to the Editors

Recurring decimals

Dear Tony,

I have taken the liberty of changing the title of this topic, in view of the snippet that appears at the bottom of page 27 of M500 **182** (no pun intended).

Rather than search for an integer with a recurring decimal of five digits, it is possible to calculate the required integer given its recurring rate.

If you have a prime number other than 2 or 5, then its reciprocal is always a recurring decimal. This is because no prime other than 2 or 5 evenly divides any power of 10 and, when performing a long division, the remainder at each stage is less than the prime. Each remainder can only occur a maximum of once. Since, for each prime, p, there are a maximum of $p - 1$ remainders, then the reciprocal of p will recur in a maximum of $p - 1$ digits. This is obvious when someone points it out, but otherwise gets missed!

For composite numbers $p \cdot q$ the recurrence is $(p - 1)(q - 1)$ maximum, unless the recurrence rate is the same for each prime factor. In general, there are several other complications which need not concern us at this stage.

Where you have a $p - 1$ recurring decimal (call this d), you will find that multiplying it successively by the integers 1 to $p - 1$ merely rearranges the digits.

E.g. for $p = 7$, $d = 142857$, $2d = 285714$, $3d = 428571$, etc.

This example is often quoted in maths books and it also works for the numbers 17, 19, and 97. You may wish to try calculating the reciprocal of these numbers with a calculator: Successively multiply its reciprocal by 2, 3, etc., writing down the digits of the result. When you have completed this, you will be able to string together the full reciprocal (allowing for the possible error in the last digit of each result).

However 13 is an exception to this rule and so is 37.

Following on from the above example, $7d = 999999$ so 7 recurs in 6 digits. Since 13 is also a factor of 999999, its reciprocal also recurs in 6 digits. The other factors of 999999 are 37, 11 and 3, but their reciprocals recur in less digits and they will have been noted in a previous stage of this process.

So, rather than search for an integer with five recurring digits, start with the number 99999 and factorize it. Hence $99999 = 3 \cdot 3 \cdot 41 \cdot 271$, which means that reciprocals of these numbers (and their composites) will recur with five digits.

Finally, to answer Susan's puzzle, 41 is the smallest integer whose reciprocal has a run length of five.

Ken Greatrix

Dear Tony,

For a positive integer N, $\gcd(N, 10) = 1$, the reciprocal is always a recurring decimal. The run length is the smallest L such that N is a factor of $10^L - 1$.

For run length 1 we consider the factors of 9, namely 9 itself and 3. We get $1/9 = 0.111\ldots$, $1/3 = 0.333\ldots$.

For run length 2 we consider *new* factors of $99 = 9 \cdot 11$. The only new factor is 11, which gives $1/11 = 0.090909\ldots$.

For run length 3 consider $999 = 9 \cdot 111 = 9 \cdot 3 \cdot 37$. The only new factor is 37, which gives $1/37 = 0.027027027\ldots$.

For run length 4 consider $9999 = 9 \cdot 1111 = 9 \cdot 11 \cdot 101$. The only new factor is 101, which gives $1/101 = 0.009900990099\ldots$.

For run length 5 consider $99999 = 9 \cdot 11111 = 9 \cdot 41 \cdot 271$. The new factors are 41 and 271. We get $1/41 = 0.0243902439\ldots$ and $1/271 = 0.0036900369\ldots$.

So the answer to Susan Cook's question is 41. (Almost but not quite the solution to Life, the Universe and Everything.)

John Reade

Tony,

This question was not numbered and so I wasn't sure whether it was there to be answered or not. I am also not sure whether it is a trick question.

Suppose $1/x = 0.abcdeabcdeabcde\ldots$. Then it is clear that $100000/x = abcde.abcdeabcde\ldots$ and so $99999/x = abcde$. Now x must be a factor of 99999 which is satisfied by 1, 3, 3, 41, 271; x must be one of these or a product of two or more.

So there several values that could be assigned to x to produce a five digit recurring decimal but $x = 1$ is the smallest giving $0.99999\ldots$. This meets the conditions of the question but if it upsets you then the alternative is $x = 41$ and the decimal is $0.02439\ldots$.

Regards,

Ron Potkin

Re: Problem 182.6 – n balls

After an extensive computer simulation I believe the answer to this is $E(n) = (n-1)^2$, a remarkably simple result. But I do not have an analytical solution.

John Bull

Kiwi fruit

On the subject of kiwi fruit [M500 **182** 18], I found myself in the same situation in Marks and Sparks' [Spencer's] food department, trying to buy the last two cans of lager in the shop when they were priced per four cans. I had to argue the case with the manager, but he did let me have the two for half the price for four. However, on the way out I saw a sign on a rack of socks: Buy 2, Get 3. Possibly this was a devious ploy to make you buy 4 and get 6 (though no doubt they would probably then give you 7).

Ralph Hancock

$a = b$

Dear Tony,

The letter in M500 magazine 182 from Keith Drever, entitled '$a = b$', where he gives a 'proof' that a number is equal to a number which is smaller than itself, contains a classic error of mathematical logic.

The final line of the proof involves dividing two equalities, which have been factorized into $a(a - b - c)$ and $b(a - b - c)$, by the factor $(a - b - c)$.

However, as at the start, we are given that $a = b + c$. Then $a - (b + c) = a - b - c = 0$. Therefore to try to divide by the common factor $a - b - c$ is the same as trying to divide by zero, and as this is undefined, then there is an error in the final line of the proof.

I suspect Mr Drever knew this all along and was just teasing the readers.

Yours sincerely,

D. M. Tansey

Dear Tony,

Keith Drever's 'proof' that $a = b$ is invalid; division by zero is not allowed. This paradox first appeared in *A Scrap-book of Elementary Mathematics* by William Frank White (Open Court, Chicago 1908).

Barbara Lee

Dear Tony,

In Keith Drever's letter '$a = b$' on page 26 of M500 **182**, he is actually dividing by 0! But, since 0! = 1, the proof would appear to be quite legitimate.

Patrick Lee

Two envelopes

Seeing as this [A game-show host offers the player two sealed envelopes one of which contains twice as much money as the other. The player chooses one, opens it and then decides whether to keep it or exchange it for the other envelope.] was offered as a philosophical problem, I have no hesitation giving an engineer's response.

I open one envelope and see a £10 note. From this I can deduce that you are running one of two possible games. Either you are offering the game with £5 and £10 notes or you are offering the game with £10 and £20 notes. My decision to swap envelopes has no effect on which game you are running and I don't know whether one game is more likely than the other.

The description in M500 implies that both situations are equally likely. If I open an envelope and see £10 then I know I'm playing either the 5/10 game or the 10/20 game with equal chances of swapping my £10 for a £5 or for a £20. My expectation is £12.50, the average of these two. You are offering me a £12.50 return for a £10 outlay. Of course I am going to swap—you're just giving money away.

Geoff Franklin

Films

Further to 'Films likely to be of interest to mathematicians' [M500 **180** 28], I offer the following:–

Last Tangent in Paris

The Man with the Golden Ratio

The Bridge over the River π

From Here to Infinity

Midpoint Cowboy

The Day of the Fractal

Angles One-Five

The King and i

633 Quadrant

The Gradient

Dual in the Sun

How the West was One (and its sequel: *How the West was Two*)

By the way, did you know that the definition of a 'millihelen' is the unit of beauty required to launch one ship?

Martyn Lawrence

M500 182 crossnumber solution

1	6	3		4	7	1
4		1		3		0
7	6	7	7	3	0	0
		6		0		
4	2	5	4	7	8	9
4		2		4		1
1	1	3		7	0	2

Sundial

Dear Tony,

I write to you in your capacity as Editor for the M500 Society, with a request for help.

Many years ago, I recall reading an article in one of the Engineering Institution's journals, which detailed the construction of a precise sundial. Precise in that it was designed for the latitude and longitude values of its proposed location. Hence it would be an accurate teller of the time, assuming the sun was shining.

I just fancied the idea of building such a device and wondered whether through the M500 magazine someone out there might be able to assist in the details of construction or be able to aim me in the right direction.

Yours sincerely,

Peter Lord

[The construction of a digital sundial was described in the August 1991 issue of *Scientific American*.]

British Mathematics

Dear eds,

I attended 'A Thousand Years of British Mathematics' at the Royal Institution on 21st November, 2001. This was a one-day event from the Gresham College autumn lecture programme. It was held in the original 200 year old lecture theatre, which made a suitable setting for Robin Wilson's splendid period costumes.

Michael Faraday's laboratory and museum, containing most of his experimental, electrical and chemistry equipment, are housed in the basement and are particularly interesting to physics students.

I do not suppose that I was the only member of M500 in the audience but it has since occurred to me that not everyone was aware of this event and perhaps we should make a mention of any maths-related forthcoming events in M500 when possible.

Further information on maths lecture dates is available on Gresham's website, www.gresham.ac.uk.

Barbara Lee

'Most engineering products can be readily adapted to their originally intended purpose.'—Chris Thornycroft.

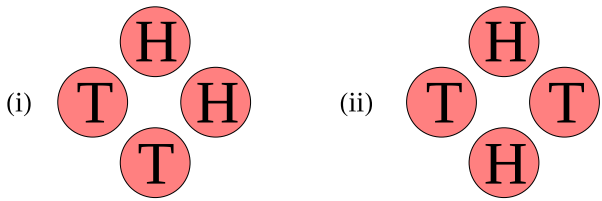
Solution 182.4 – Four coins

> Four coins are arranged in a circle. You cannot see them. You choose one or more numbers between 1 and 4. Your opponent inverts the coins in the positions which you selected. The positions are numbered 1, 2, 3, 4 clockwise around the circle but your opponent is free to choose which coin is in position 1.
>
> If the result is four heads, the game stops and you win. Otherwise the procedure is repeated.
>
> Devise a winning strategy.

ADF

We have had no responses to this one. After a brief look I can offer the following solution. Can anybody improve it? Does the problem generalize to n coins?

First we show how to deal with the situation where you know that there are exactly two heads and two tails. This can occur in one of two configurations:



1. Choose {1, 3} followed by {1, 2, 3, 4}. This deals with (ii) and leaves (i) unchanged. (It is understood that instructions are carried out only if play is still in progress. If the choice {1, 3} results in four heads, you would not continue with the 'followed by {1, 2, 3, 4}' part.)

2. Choose {1, 2} followed by {1, 2, 3, 4}. This either solves (i) or changes it to (ii).

3. Choose {1, 3} followed by {1, 2, 3, 4}. This solves (ii).

Now for the general solution.

1. Carry out the procedure for two heads and two tails.
2. Carry out the procedure for two heads and two tails.
3. Choose 1.
4. Carry out the procedure for two heads and two tails.
5. Carry out the procedure for two heads and two tails.

Construction of the regular dodecahedron

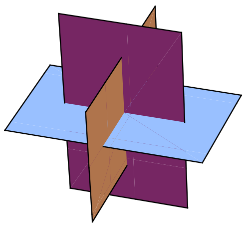
Barbara Lee

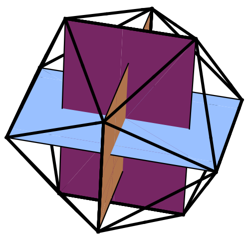
We start with ϕ, the golden ratio,

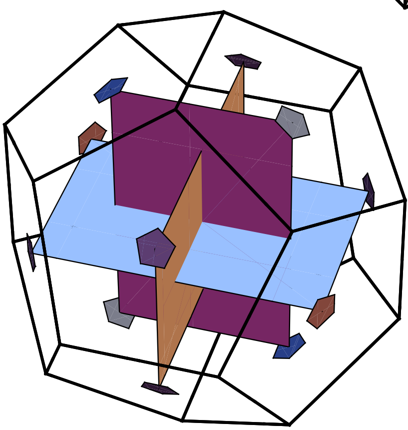
$$\phi = 1.6180339887498948482\ldots \quad = \quad 1/0.6180339887498948482\ldots.$$

Take three golden rectangles of the same size and intersect them at the centre so that each is perpendicular to the other two. Join each of the twelve corners to the nearest equidistant five corners. We now have an icosahedron with the corners of the rectangles meeting the twelve vertices of the icosahedron.

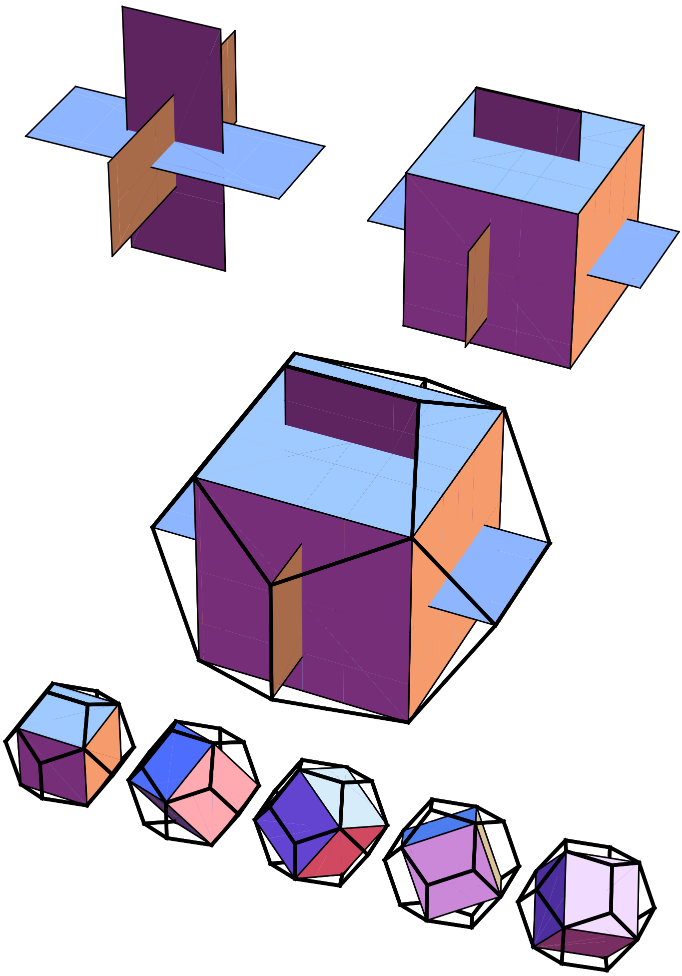
The dodecahedron is the dual of the icosahedron. Its twenty vertices coincide with the centres of the faces of the icosahedron. Join the centre of each face of the icosahedron to the centres of the three adjacent ones above the surface so the faces of the dodecahedron are in the same plane as the corners of the rectangles.







For an alternative construction, start with three intersecting orthogonal $1\times\phi^2$ rectangles and a $\phi\times\phi\times\phi$ cube centred at the origin. Then the twelve vertices of the rectangles together with the eight corners of the cube define the 20 vertices of the dodecahedron.



Solution 182.3 – Russian roulette

> Is there any way to modify the rules of Russian roulette to make it fair for both players?

Tony Forbes

Clearly the rules must be bent but not by too much.

Many people (probably including myself) suggested that a coin-toss to decide who starts would suffice. End of story. However, this solution blatantly goes against the spirit of the game. In order to preserve rouletteness we should forbid such artificial devices and allow just one process of randomization: the spinning of the gun's chamber assembly before each and every firing.

Altering the sequence of turns, too, is unsatisfactory because it causes considerable variation in the probabilities as the game progresses. Take, for example, the sequence beginning $ABBA\ldots$ (We assume that thereafter the As and Bs appear with whatever frequencies are necessary to equalize the probabilities of winning. *Very Interesting Question*: How can this be achieved? **John Bull** points out that if the turns are taken in pairs after the first, $ABBAABBAA\ldots$, the probability of the first player losing is reduced to $31/61 = 0.508\ldots$.)

In my opinion player B would have legitimate cause for complaint if he is required to take two turns in succession when A has had only one. After A has survived his turn with probability $5/6 \approx 0.833$, B's chances of staying in the game and being able to hand the gun back to A are significantly lower at only $25/36 \approx 0.6944$. This is not a trivial matter; remember that in serious games the bullet inflicts a mortal wound on the person who fires it.

There is a neat solution in which the players take turns as usual and the probabilities are constant throughout the duration of the game. It requires a five-chamber revolver as well as the usual six-chamber variety. At the start of the duel the challenger invites his opponent to choose his weapon. The only condition is that whoever ends up with the six-shooter goes first.

The disadvantage of taking the first turn is exactly balanced by the slightly higher probability of a 'successful' firing by the other player. It's no coincidence that 5 happens to be an integer. As you can verify, any pair of numbers n, $n-1$ would work, although in the case $n=1$ there is an absence of a random element for the second player if the first player happens to survive his round.

If a special weapon is unavailable, a modified standard gun should work just as well. Insert a wooden plug into one of the slots. The player who uses the gun agrees that if the striker hits wood, the firing is void and the turn must be taken again.

Addition

Sebastian Hayes

Our arithmetic is based on adding. Subtracting is adding the other way, multiplication is repeated adding and division is multiplication in reverse. So if you can add you're all right. Computers can add, and for a long time couldn't do much else, so they're all right. The author of the once popular *Mathematics for the Million*, Lancelot Hogben, was so convinced that adding was the basis of mathematics and mathematics the basis of so much else, that he suggested beaming out to the universe '$2 + 2 = 4$'. An intelligent species would recognize this and recognize that an intelligent species had sent it.

Although Platonism as a philosophy is long dead in the outside world, it still rules supreme in a lot of pure mathematicians heads. Why should a species very different from ours in physiology and social habits develop a comparable mathematics? If mathematics is eternal and there is only one sort of mathematics, all species everywhere will partake of it. But there is no evidence to support such a view, only the prejudices of pure mathematicians. I would expect a different species to have a different mathematics with, however, certain overlaps.

Problem 184.6 – Limit

Show that $\dfrac{a\sin b - b\sin a}{a\cos b - b\cos a} \to \tan(a - \arctan a)$ as $b \to a$.

Problem 184.7 – Deux nombres

Nicolas Guerrero

Ce sont deux mathématiciens russes qui connaîssent l'un le produit et l'autre la somme de deux nombres compris entre 2 et 100. Ils se rencontrent et le premier dit à l'autre: "Je connais le produit de deux nombres mais pas les deux nombres."

L'autre répond: "Ah! Je connais la somme mais pas les deux nombres."

Le premier dit alors: "Ca y est j'ai trouvé!"

Et l'autre répond: "Eh bien, moi aussi!"

Question: Quels sont ces deux nombres?

Solution 182.2 – Nine tarts

> There are nine jam tarts. Some jam has been removed from one and put back into another so that seven weigh the same, one weighs a bit more and one a bit less. Find the light and heavy tarts in the minimum number of weighings.

Tony Forbes

There are 72 ways this can happen. Three weighings provide only 27 sets of observations and therefore in some cases at least a fourth will be required to identify the two non-standard tarts. I show not only that four weighings suffice but that *the weighings can be fixed in advance.* I am pleased with my solution because it avoids a tedious case-by-case analysis where the description of the $(n+1)$th test depends upon the outcome of the nth.

Label the tarts 0, 1, 2, 3, 4, 5, 6, 7, 8. Then

W1. weigh $\{0, 1, 2\}$ against $\{3, 4, 5\}$,

W2. weigh $\{0, 3, 6\}$ against $\{1, 4, 7\}$,

W3. weigh $\{0, 4\}$ against $\{1, 3\}$,

W4. weigh $\{2, 5\}$ against $\{6, 7\}$.

From the results of W1–W4 it is easy to identify the heavy and light tarts.

To make it even easier I have provided the table on the next page from which you can read off the answer directly by looking up the results of W1–W4 in column 2. Letter 'L' is used to indicate that the set of tarts on the left pan of the scales is light, 'H' means the set of tarts on the left is heavy, and 'B' means a perfect balance. The third column identifies the heavy tart and the light tart.

For example, the first row reads '0 : L L L L : 7 0'. This means that if the set of tarts on the left is light in W1, W2, W3 and W4, then 7 is heavy and 0 is light.

We now prove that the table is correct.

First observe that W1 and W2 cannot both balance; hence the nine entries 36 to 44 that begin 'B B' have column 3 empty. That leaves 72 cases to deal with. However the work is much easier than it looks. Since every 'hl' pair appears at least once in column 3 it is sufficient to show that in each entry of the table the 'hl' labelling is consistent with the four results. Uniqueness follows automatically by a simple counting argument. The 72 rows of the table contain the 72 different heavy-light combinations. This is impossible unless they are distributed precisely one per set of results. Furthermore, by exploiting the table's obvious symmetry we only have to check entries 0 to 35.

Consider entry number 0. Tarts 0 and 7 do not occur together on the same side of the scales; 0 is on the left in W1, W2 and W3, and 7 is on the right in W4. Thus '7 heavy, 0 light' is consistent with 'L L L L'.

I leave the remaining 35 cases to the reader.

	results	h l
0	L L L L	7 0
1	L L L B	8 0
2	L L L H	5 0
3	L L B L	7 2
4	L L B B	4 0
5	L L B H	5 6
6	L L H L	4 2
7	L L H B	4 8
8	L L H H	4 6
9	L B L L	6 0
10	L B L B	3 0
11	L B L H	3 6
12	L B B L	8 2
13	L B B B	5 2
14	L B B H	5 8
15	L B H L	7 1
16	L B H B	4 1
17	L B H H	4 7
18	L H L L	3 2
19	L H L B	3 8
20	L H L H	3 7
21	L H B L	6 2
22	L H B B	3 1
23	L H B H	5 7
24	L H H L	6 1
25	L H H B	8 1
26	L H H H	5 1

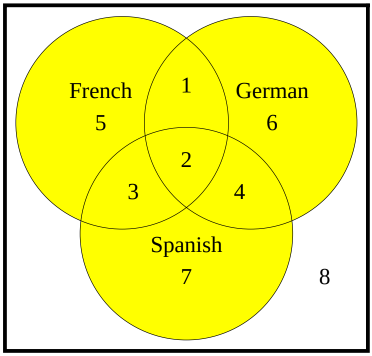
	results	h l
27	B L L L	1 2
28	B L L B	1 0
29	B L L H	2 0
30	B L B L	7 8
31	B L B B	7 6
32	B L B H	8 6
33	B L H L	4 5
34	B L H B	4 3
35	B L H H	5 3
36	B B L L	
37	B B L B	
38	B B L H	
39	B B B L	
40	B B B B	
41	B B B H	
42	B B H L	
43	B B H B	
44	B B H H	
45	B H L L	3 5
46	B H L B	3 4
47	B H L H	5 4
48	B H B L	6 8
49	B H B B	6 7
50	B H B H	8 7
51	B H H L	0 2
52	B H H B	0 1
53	B H H H	2 1

	results	h l
54	H L L L	1 5
55	H L L B	1 8
56	H L L H	1 6
57	H L B L	7 5
58	H L B B	1 3
59	H L B H	2 6
60	H L H L	7 3
61	H L H B	8 3
62	H L H H	2 3
63	H B L L	7 4
64	H B L B	1 4
65	H B L H	1 7
66	H B B L	8 5
67	H B B B	2 5
68	H B B H	2 8
69	H B H L	6 3
70	H B H B	0 3
71	H B H H	0 6
72	H H L L	6 4
73	H H L B	8 4
74	H H L H	2 4
75	H H B L	6 5
76	H H B B	0 4
77	H H B H	2 7
78	H H H L	0 5
79	H H H B	0 8
80	H H H H	0 7

Problem 184.8 – Four regions

Mike Adams

A *Venn diagram* provides a representation of, say, how many pupils in a class take which combinations of languages. For example, in the diagram, opposite, five students are learning French only, six German only, seven Spanish only, one French and German but not Spanish, three French and Spanish but not German, four German and Spanish but not French, and two all three; eight students are doing other things.



(i) Construct a diagram that represents four languages, where each region is an ellipse or its topological equivalent.

(ii) Is there an upper limit to how many subjects can be represented in this manner?

Problem 184.9 – States

Zoe Forbes

What is the probability of winning a game of *Hangman* where the words are restricted to the names of USA states. Assume only one life. Assume also that you and your opponent always play sensibly.

(The game involves two players. At the start you choose a USA state, S, say, and you tell your opponent how many letters there are in each word of S (in the correct order if more than one). This is conveniently done by drawing an appropriate sequence of dashes.

If S is identifiable, the game ends and you lose. If not, your opponent chooses a letter, α, say. If α does not occur in S, the game ends and you win. Otherwise you reveal the position(s) of α in S and the game continues.

For example, if faced with '_ _ _   _ _ _ _ _ _', one can safely choose 'E' to identify NEW JERSEY or NEW MEXICO. On the other hand, '_ _ _ _' is not so easy to resolve—IOWA, OHIO or UTAH?

If you are unhappy with the phrase 'probability of winning', consider instead the proportion of games that you can expect to win as the number of games tends to infinity.)

Threes are best

Dilwyn Edwards

The problem I was originally given was to choose a set of positive integers adding up to 100, which would give the maximum result when multiplied together. After some rather disorganized searching I came up with, I believe, the correct answer, 32 threes and 1 four. On generalizing to an arbitrary natural number N, I realized that threes are always the 'most wanted' integers for this problem. This is because $3 > 2 \cdot 1$ and $4 = 2^2 > 3 \cdot 1$, while $5 < 2 \cdot 3$ so we will never want to use integers greater than 4. We might want one four but not more because $4^2 < 3^2 \cdot 2$. Based on this reasoning, my solution to the problem

> Given a natural number N find a set of positive integers with sum N and with maximal product P

is as follows.

If $N = 3x$ then $P = 3^x$. (1)
If $N = 3x + 1$ then $P = 3^{x-1} \cdot 4$. (2)
If $N = 3x + 2$ then $P = 3^x \cdot 2$. (3)

In going from (1) to (2) the extra 1 allows us to replace a 3 with a 4. In going from (2) to (3) the extra 1 allows us to replace the 5 with a $3 \cdot 2$.

I think this is a complete solution. Can anyone prove me right (or wrong)?

Twenty-five years ago

From M500 39

Sue Davies—Has anyone noticed that M500 is being taken over by machines? ...

The extent to which dependence upon machines undermines human intelligence is clearly shown in the 196th root problem. This was an interesting problem which, as was pointed out in M500 **37**, is instantly solvable by inspection given that the answer is an integer. Those not wishing to use this information could indulge in a little elementary factorization and the whole thing cancels out nicely in a few minutes. It should be obvious to anyone that to feed the calculation of this size into an electronic calculator will produce rounding errors and only an approximate answer can be obtained. In which case, since the approximate answer is immediately obtainable by inspection, why in the sacred name of Babbage should anyone want to put it into a calculator? Nevertheless, button-pushers by the dozen sent in 'solutions' to nine meaningless decimal places.

Then it was suggested that we should run the CoRA algorithm on a machine to find out how in many iterations 196 produces a palindrome. Who cares? The mathematical interest of the CoRA process is in whether it produces a palindromic for all numbers and if so, why? ...

Contents — M500 184 – February 2002