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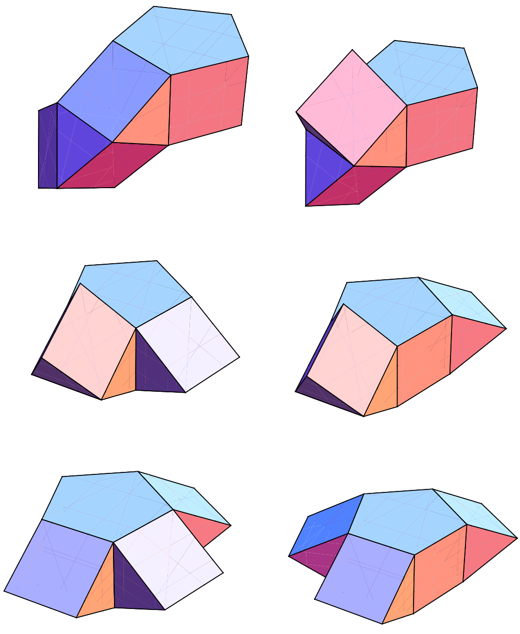
M500 195



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Solution 191.8 – Infinite exponentiation

> I want to know the value of $y = x^{x^{x^{x^{\cdots}}}}$ when $x = 1.1$. Writing it as $y = x^y$ I get $\log y = y \log x$; so $\log x = (\log y)/y$. Putting $x = 1.1$, appears to give two solutions: $y = 1.111782011...$ and $y = 38.22873285....$ They can't both be right. Explain.

Basil Thompson

Euler proved that the function $y = x^{x^{x^{x^{\cdots}}}}$, where the height of the tower of exponents tends to ∞, has a limit if

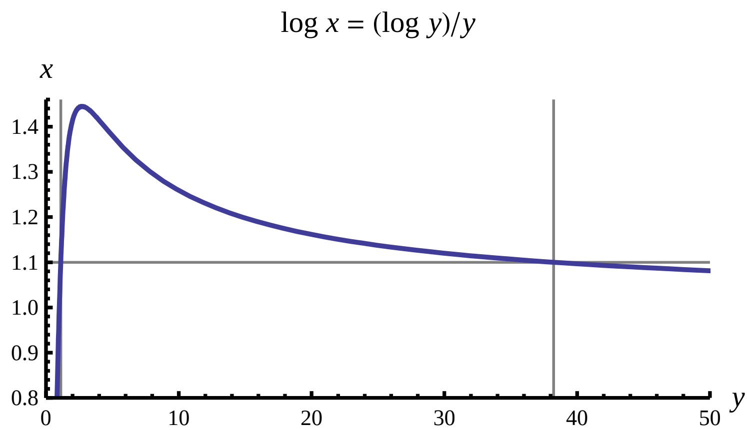
$$e^{-e} < x < e^{1/e}, \qquad 0.065988\ldots < x < 1.444667\ldots.$$

See the entry for $1.444667861\ldots$ in David Wells, *The Penguin Dictionary of Curious and Interesting Numbers*.

So it is not surprising that when $x = 1.1$, y will converge (to $1.111782011\ldots$). But $y = x^{x^{x^{x^{\cdots}}}}$ can only be written as $y = x^y$ if x is within the Euler limits, with the implied limits for y, $1/e < y < e$, i.e. $0.367879\ldots < y < 2.718281\ldots.$

By taking logs we get $\log x = (\log y)/y$, from which a curve for x against y can be plotted. By differentiating, it can be shown that the maximum of x is at $y = e$ and $x = e^{1/e}$. As $y \to \infty$, $x \to 1$.

The line $x = 1.1$ cuts the graph in two points, $y = 1.111782011...$ within the Euler limits, and $y = 38.22873285...$ outside the limits. There is no mystery, but is it possible to find the two solutions other than by numerical methods?



On the cellular automaton of Ulam and Warburton

David Singmaster

Introduction

Mike Warburton's 'One-edge connections' [3] is an example of a cellular automaton or a cell growth pattern. It seems to have been first considered by Stanislaw Ulam in one of the original papers in the field [2].

We consider the square lattice as an infinite chessboard of cells, with each cell having as neighbours the four cells which share an edge with it. In generation 0, the cell at $(0,0)$ is transformed. In each succeeding generation, the cells which share one edge with already transformed cells are transformed. One can think of the situation as an infinite array of cells and the transformation being that they are infected, and perhaps die. Or we can think of the plane as a nutrient surface and the cells are becoming alive and propagating. The latter interpretation is more common and agrees with the idea of generation as used in Conway's *Life*, etc.

We first simplify by noting that the pattern has the symmetry of the square and so we need only look at one quadrant. It is convenient to take the fourth quadrant. Figure 1 shows part of the fourth quadrant with cells labelled with the generation in which they are born. Up through the sixth generation, the pattern coincides with the pattern of odd binomial coefficients, and would continue to do so if we required that life had to spread outward. After seeing Warburton's note, I did some analysis of the latter pattern and sent it to Tony Forbes since it is a well-known pattern, related to the Tower of Hanoi, the fractal known as Sierpiński's Gasket and pathological curves—see [1]. However, my friend Chris Base recently asked about a pattern which had arisen in a school investigation, and this was the Ulam-Warburton pattern. She pointed out that in the 7th generation there is an inward growth, and this gets more important in higher generations. So here I do the analysis for the Ulam-Warburton pattern and determine the number of cells born in each generation.

The number in each generation

Let $A(i)$ be the number of cells born in generation i. Table 1 (page 7) lists the values of $A(i)$, up through $i = 63$.

Looking at Figure 1, we see that the growth from 2^n to $2^{n+1}-1$ can be viewed as three triangles, as shown in Figure 2. Triangles A and C are identical to triangle A advanced by 2^n. To see what is going on in triangle D, we need to subdivide as in Figure 3. Because the growth from point $P = (0, 2^n)$ is into virgin territory up until generation $2^{n+1}-1$, the growth is symmetric with respect to the horizontal through P. So triangle D1 is the reflection of triangle C2. However, these triangles share the common horizontal through P. So the number of cells of a given generation in the interior of D1 is one less than the number counted in C2. Also by symmetry, C2 is the reflection of C1, with a slight counting problem at the point P. Putting together all the triangles, we see that

$$A(2^n + i) = 2A(i) + 2[A(i)/2 - 1] = 3A(i) - 2, \quad i = 1, ..., 2^n - 1. \tag{1}$$

When $i = 0$, we have $A(2^n) = 2$, except $A(0) = 1$.

Since $A(i)$ is always even (except at $i = 0$), let us set $B(i) = A(i)/2$, $B(0) = 1$. Then we have

$$B(2^n + i) \;=\; 3B(i) - 1, \quad \text{except } B(2^n) \;=\; 1. \tag{2}$$

Looking at

$$\begin{aligned}
B(11) &= 3B(3) - 1 = 3(3B(1) - 1) - 1 = 9 - 3 - 1, \\
B(12) &= 3B(4) - 1 = 3 - 1, \\
B(13) &= 3B(5) - 1 = 3(3B(1) - 1) - 1 = 9 - 3 - 1, \\
B(14) &= 3B(6) - 1 = 3(3B(2) - 1) - 1 = 9 - 3 - 1, \\
B(15) &= 3B(7) - 1 = 3(3B(3) - 1) - 1 \\
&= 3(3(3B(1) - 1) - 1) - 1 = 27 - 9 - 3 - 1,
\end{aligned}$$

we see that the expressions depend just on the number, d, of ones in the binary representation of i, being $3^{d-1} - 3^{d-2} - \cdots - 1$. The tail of this is just $(3^{d-1} - 1)/2$, so we have a total of $B(i) = (3^{d-1} + 1)/2$. Doubling this gives us

$$A(i) \;=\; 3^{d-1} + 1, \tag{3}$$

in agreement with the values in Table 1. This holds even when $d = 1$, i.e. $i = 2^n$, but $A(0) = 1$ is still exceptional.

I cannot yet see any simple way to describe the cells born in the ith generation, nor how to determine for a given cell whether it is ever born nor in which generation. I have a rather complicated method for the latter questions, but I will postpone this until the end of this note. Such descriptions depend on the binary representation in some way.

The number in the first 2^n generations

Now let $C_n = \sum_{i=0}^{2^n-1} A(i)$ be the total number born in generations $0, 1, \ldots, 2^n - 1$. We have $C_0 = 1$, $C_1 = 3$, $C_2 = 9$, $C_3 = 29$, $C_4 = 101$, $C_5 = 373$, $C_6 = 1429$, ... Either by counting as done to find equation (1), or by adding up equation (1) from $2^n + 1$ to $2^{n+1} - 1$, and using $A(0) = 1$, $A(2^n) = 2$, we find

$$C_n = 4C_{n-1} - (2^n - 1), \quad \text{for } n > 0. \tag{4}$$

This is not a common type of recurrence because of the non-homogeneous terms $2^n - 1$. After some fiddling based on the idea that the solution should include terms like the non-homogeneous part, I realized I could eliminate this part by considering $C_n = D_n + a2^n + b$. This yields $D_{n+1} + a2^{n+1} + b = 4D_n + 4a2^n + 4b - 2^{n+1} + 1$. Setting $b = -1/3$ makes the constant part cancel out, and setting $a = 1$ makes the 2^{n+1} part cancel out. We are then left with $D_{n+1} = 4D_n$, whose solution is obviously $D_n = \alpha 4^n$. Hence $C_n = \alpha 4^n + 2^n - 1/3$. Using $C_1 = 3$, we find $\alpha = 1/3$, so

$$C_n = (4^n - 1)/3 + 2^n. \tag{5}$$

The total number of cells in levels 0 through $2^n - 1$ is $1 + 2 + 3 + \cdots + 2^n = 2^n(2^n + 1)/2 = t(2^n)$, the 2^nth triangular number. Hence the density of live cells up through $2^n - 1$ is $C_n/t(2^n) = 2/3 - 1/(3 \cdot 2^{n-1}) + 2/(2^n + 1)$, which is asymptotic to 2/3.

To relate to Mike Warburton's expression, we let E_0 be the number in his pattern for levels 0 through $2^n - 1$. We have $E_0 = 1$, $E_1 = 5$, $E_2 = 21$, $E_3 = 85$, $E_4 = 341$, ... Note that Warburton includes the level 2^n, which adds four to E_n. Since the whole figure is four quadrants, with some overlap along the axes, we find $E_n = 4C_n - 3 - 4(2^n - 1)$, and using equation (5), we have

$$E_n = (4^{n+1} - 1)/3. \tag{6}$$

Adding 4 and slightly rearranging gives $E_n + 4 = 4 \cdot 4^n/3 + 11/3$, which is Warburton's expression.

Surprisingly, neither $A(i)$ nor $B(i)$ appears in Sloane's *On-Line Encyclopedia of Integer Sequences* (www.research.att.com/cgi-binlaccess.cgi/as/njas/sequences/eismum.cgi) and E_n appears as in equation (6) but only in quite different contexts.

Concluding remarks

Let me now describe my somewhat complicated process of determining whether and when a given cell becomes alive. Basically we use the recursive observations about Figures 2 and 3 to reduce the coordinates. Let a cell have coordinates (i, j), where j is taken positive in the downward direction. Let $G(i, j)$ be the generation number in which cell (i, j) is born. Since

Figure 1 is symmetric with respect to its diagonal, i.e. $G(i,j) = G(j,i)$, we can assume $i < j$ and we need only look at the lower part of Figure 1.

The growth from generation 2^n to generation $2^{n+1} - 1$ all lies in the triangles B, C, D of Figure 2, i.e. where $2^n \le i + j \le 2^{n+1} - 1$.

If $j \ge 2^n$, we are in triangle C and we have

$$G(i,j) \;=\; 2^n + G(i, j - 2^n). \tag{7}$$

But if $i \le j < 2^n$, then we are interior to triangle D1 and reflection in the horizontal through P gives $G(i,j) = G(i, 2^{n+1} - j)$. Use of (7) then gives us

$$G(i,j) \;=\; 2^n + G(i, 2^n - j). \tag{8}$$

Rotating triangle D1 into triangle C1 gives us $G(i,j) = G(2^n - j, 2^n + i)$ and use of (7) gives

$$G(i,j) \;=\; 2^n + G(2^n - j, i), \tag{9}$$

which is a symmetric form of (8). However, when $i = j = 2^{n-1}$, both transformations bring us back to the same point we started with. If $i = j = 0$, we are at the end of our process, but if $i = j > 0$, our point is never born. Other general rules such as $G(0,j) = j$ and $G(i, 2^n - 1 - i) = 2^n - 1$ help shorten any calculation. One can describe these rules in terms of the binary expansions of i and j, but the dichotomy of the rules and the use of 2s complement in (8) or (9) make it quite unclear what the overall result of the process will be.

As a final remark, observe that the pattern of unborn cells is the same fractal as the pattern of born cells, but rotated by 45° and shrunk by a factor of $\sqrt{2}$.

References

[1] Ian Stewart, Mathematical Recreations: Sierpiński's ubiquitous gasket, *Sci. Amer.* **281:2** (Aug 1999) 74–75.

[2] Stanislaw M. Ulam, On some mathematical problems connected with patterns of growth of figures, *Proc. Symp. Appl. Math.* **14**: Mathematical Problems in the Biological Sciences, R. E. Bellman, ed. Amer. Math. Soc. (1962) 215–224. Reprinted in: Arthur W. Burks, ed., *Essays on Cellular Automata*, Univ. of Illinois Press, 1970, 219–231.

[3] Mike Warburton, One-edge connections, M500 **188** (Oct 2002) 11.

Figure 1: $G(i,j)$, $0 \le i \le j \le 63$, part (i): $j \le 31$

00	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
01		03		05		07		09		11		13		15		17		19		21		23		25		27		29		31	
02	03		07	06	07		11	10	11		15	14	15		19	18	19		23	22	23		27	26	27		31	30	31		35
03		07		07		15		11		15		15		23		19		23		23		31		27		31		31		39	
04	05	06	07		15	14	13	12	13	14	15		23	22	21	20	21	22	23		31	30	29	28	29	30	31		39	38	37
05		07		15		15		13		15		31		23		21		23		31		31		29		31		47		39	
06	07		15	14	15		15	14	15		31	30	31		23	22	23		31	30	31		31	30	31		47	46	47		39
07		11		13		15		15		31		29		27		23		27		29		31		31		47		45		43	
08	09	10	11	12	13	14	15		31	30	29	28	27	26	25	24	25	26	27	28	29	30	31		47	46	45	44	43	42	41
09		11		13		15		31		31		29		27		25		27		29		31		63		47		45		43	
10	11		15	14	15		31	30	31		31	30	31		27	26	27		31	30	31		63	62	63		47	46	47		43
11		15		15		31		29		31		31		31		27		31		31		63		61		63		47		47	
12	13	14	15		31	30	29	28	29	30	31		31	30	29	28	29	30	31		63	62	61	60	61	62	63		47	46	45
13		15		23		31		27		31		31		31		29		31		63		63		59		63		55		47	
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17		19		21		23		25		27		29		31		63		63		61		59		57		55		53		51	
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41		43		45		47		63		63		61		59		57		59		61		63									
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43		47		47		63		61		63		63		63		59		63		63											
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45		47		55		63		59		63		63		63		61		63													
46	47		55	54	55		59	58	59		63	62	63		63	62	63														
47		51		53		55		57		59		61		63		63															
48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63																
49		51		53		55		57		59		61		63																	
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51		55		55		63		59		63		63																			
52	53	54	55		63	62	61	60	61	62	63																				
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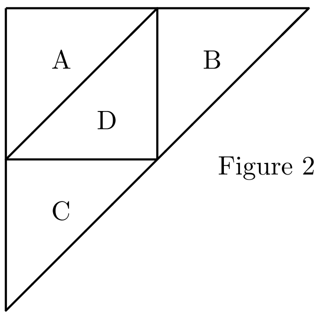

Figure 2

Figure 1: $G(i,j)$, $0 \le i \le j \le 63$, part (ii): $j \ge 32$

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32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63
33    35    37    39    41    43    45    47    49    51    53    55    57    59    61    63
34 35    39 38 39    43 42 43    47 46 47    51 50 51    55 54 55    59 58 59    63 62 63
35    39    39    47    43    47    47    55    51    55    55    63    59    63    63
36 37 38 39    47 46 45 44 45 46 47    55 54 53 52 53 54 55    63 62 61 60 61 62 63
37    39    47    47    45    47    63    55    53    55    63    63    61    63
38 39    47 46 47    47 46 47    63 62 63    55 54 55    63 62 63    63 62 63
39    43    45    47    47    63    61    59    55    59    61    63    63
40 41 42 43 44 45 46 47    63 62 61 60 59 58 57 56 57 58 59 60 61 62 63
41    43    45    47    63    63    61    59    57    59    61    63
42 43    47 46 47    63 62 63    63 62 63    59 58 59    63 62 63
43    47    47    63    61    63    63    63    59    63    63
44 45 46 47    63 62 61 60 61 62 63    63 62 61 60 61 62 63
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63
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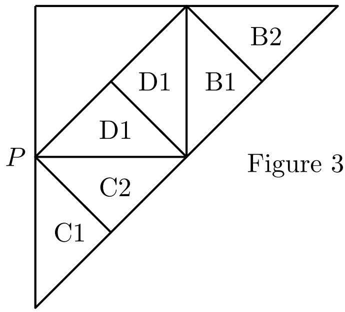


Figure 3

Table 1

i	$A(i)$	i	$A(i)$	i	$A(i)$	i	$A(i)$
0	1	16	2	32	2	48	4
1	2	17	4	33	4	49	10
2	2	18	4	34	4	50	10
3	4	19	10	35	10	51	28
4	2	20	4	36	4	52	10
5	4	21	10	37	10	53	28
6	4	22	10	38	10	54	28
7	10	23	28	39	28	55	82
8	2	24	4	40	4	56	10
9	4	25	10	41	10	57	28
10	4	26	10	42	10	58	28
11	10	27	28	43	28	59	82
12	4	28	10	44	10	60	28
13	10	29	28	45	28	61	82
14	10	30	28	46	28	62	82
15	28	31	82	47	82	63	244

LCM

Dick Boardman

I learned about the LCM when I learned about adding fractions but we were never told an algorithm to calculate it nor its relation to other functions. This fills some of the gap.

If $\{a, b, c\}$ have no common factor, the LCM is simply the product abc.

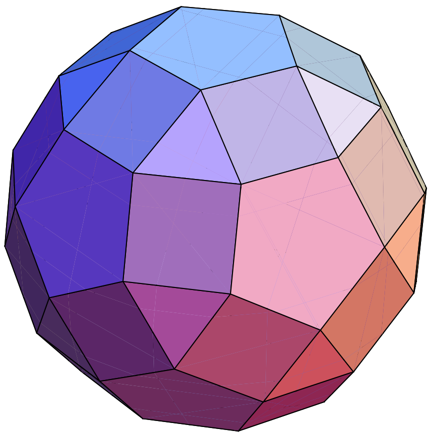
If $\{a, b, c\}$ have a highest common factor f so that $a = fA$, $b = fB$ and $c = fC$, then the LCM is $fABC$. In general, for N numbers, let $f = \mathrm{HCF}(a_1, a_2, \ldots, a_N)$ then the LCM is $a_1 a_2 \ldots a_N / f^{N-1}$

Crucially the LCM of N numbers is at most the product of N numbers. Hence your statement in M500 **191** (page 10) that $\mathrm{LCM}(a, b, c) \le ((a + b + c)/3)^3$ is always true (not just for sufficiently large n) and is a special case of the well-known theorem which states that the geometric mean of N numbers is not greater than the arithmetic mean.

Solution 192.5 – 16 Polygons

> Make a 16-faced polyhedron out of two regular pentagons, eight squares and six equilateral triangles.

Chris Pile



Well, isn't that a picture of the polyhedron on the front cover of **192**? I made one very easily in a few minutes—just had to bend it a little! The arrangement of the polygons is similar to part of the Archimedean solid, the (small) rhombicosidodecahedron, right. Unfortunately, the dihedral angle between pentagon and square is $148^\circ 17'$, not 150° as would be required by effectively gluing together triangular wedges on to a pentagonal prism.

If the decagonal polar caps of the rhombicosidodecahedron are glued together, the pentagonal faces are 1.054 edge units apart.

Robin Marks

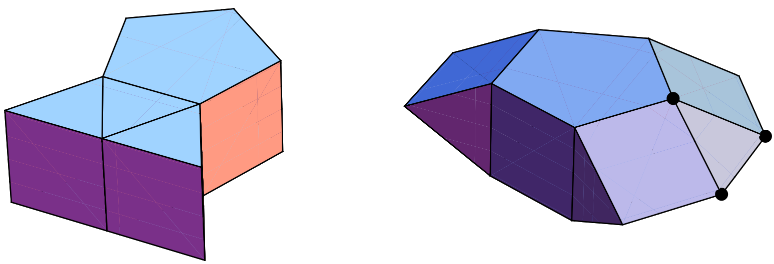
The answer to the question is ‘yes’. Start with a pentagonal prism and three triangular prisms. Together these polyhedra have two pentagonal faces, 14 square faces and six triangular faces.

1. Take any two polyhedra. Choose a square face on each. Align the polyhedra so that one square lies flat against the other and the square centres are lined up. Rotate until all the the edges are aligned (four possible ways). Join the two objects where they touch.

2. Repeat step 1 until there is only one polyhedron.

This procedure leaves one polyhedron with two pentagonal, eight square and six triangular faces. I can find 64 examples. However, most of these polyhedra, like the one shown on the left, below, have two of the component polygons joined together at an angle of 180 degrees. Consequently the resulting shape cannot be regarded as a 16-hedron. If we disallow such objects, we are left with just nine different 16-sided polyhedra, as illustrated on the front cover of this magazine. On the other hand if we allow such objects there are 64 polyhedra altogether. In such an object we must consider each set of polygons-joined-together-at-an-angle-of-180-degrees to be combined into a single (irregular) polygon. Such an object is thus a polyhedron with less than 16 sides.

And, yes, I did notice the diagram on the front cover of M500 **192** (below, right). The triangle marked by the three black dots is not equilateral: by my calculations it has side lengths 1, 1 and $\sqrt{3}\sin\pi/5 \approx 1.01807$.



ADF — This was in fact the shape that inspired the problem. I accidentally made it whilst playing with one of my toys, a construction set comprising finite numbers of plastic regular 3-, 4- and 5-gons which clip together in an ingenious manner. There was sufficient tolerance to allow the two odd triangles to be replaced by equilateral ones.

Solution 190.1 – 50 pence

Starting on his sixth birthday, a child is given 50 pence every day but always in a different combination of coins. The money stops when this is no longer achievable. How old is the child when that happens?

Ralph Hancock

I cheated foully and wrote a BASIC program, right, which seems to be watertight. It told me that there are 451 ways of combining the coins. So if the child was 6 at the start, his/her age at the end will be 7 years and 86 days (assuming that the first year is not a leap year).

```
r = 0
FOR f = 0 TO 1
FOR e = 0 TO 2
FOR d = 0 TO 5
FOR c = 0 TO 10
FOR b = 0 TO 25
FOR a = 0 TO 50
IF a+2*b+5*c+10*d+20*e+50*f = 50 THEN
   r = r + 1
   PRINT r; a; b; c; d; e; f
END IF
NEXT: NEXT: NEXT: NEXT: NEXT: NEXT
```

ADF writes — An alternative to such foul play but involving even more computation is to evaluate the Taylor series expansion:

$$
\begin{aligned}
&\frac{1}{(1-x)(1-x^2)(1-x^5)(1-x^{10})(1-x^{20})(1-x^{50})} \\
&\quad = 1 + x + 2\,x^2 + 2\,x^3 + 3\,x^4 + 4\,x^5 + 5\,x^6 + 6\,x^7 + 7\,x^8 + 8\,x^9 \\
&\qquad + 11\,x^{10} + 12\,x^{11} + 15\,x^{12} + 16\,x^{13} + 19\,x^{14} + 22\,x^{15} + 25\,x^{16} \\
&\qquad + 28\,x^{17} + 31\,x^{18} + 34\,x^{19} + 41\,x^{20} + 44\,x^{21} + 51\,x^{22} + 54\,x^{23} \\
&\qquad + 61\,x^{24} + 68\,x^{25} + 75\,x^{26} + 82\,x^{27} + 89\,x^{28} + 96\,x^{29} + 109\,x^{30} \\
&\qquad + 116\,x^{31} + 129\,x^{32} + 136\,x^{33} + 149\,x^{34} + 162\,x^{35} + 175\,x^{36} \\
&\qquad + 188\,x^{37} + 201\,x^{38} + 214\,x^{39} + 236\,x^{40} + 249\,x^{41} + 271\,x^{42} \\
&\qquad + 284\,x^{43} + 306\,x^{44} + 328\,x^{45} + 350\,x^{46} + 372\,x^{47} \\
&\qquad + 394\,x^{48} + 416\,x^{49} + 451\,x^{50} + \dots .
\end{aligned}
$$

The coefficient of x^n then gives the number of ways of expressing n as a sum of integers taken from the set $\{1, 2, 5, 10, 20, 50\}$. So the answer we want is the coefficient of x^{50}. Lo and behold! It is the same, 451.

Solution 191.5 – Another magic square

Paul Terry

Here is the solution to Claudia Gioia's magic square.

47	58	69	80	1	12	23	34	45
57	68	79	9	11	22	33	44	46
67	78	8	10	21	32	43	54	56
77	7	18	20	31	42	53	55	66
6	17	19	30	41	52	63	65	76
16	27	29	40	51	62	64	75	5
26	28	39	50	61	72	74	4	15
36	38	49	60	71	73	3	14	25
37	48	59	70	81	2	13	24	35

Using the usual formula, $n(n^2+1)/2$, with $n = 9$, the rows, columns and diagonals all sum to 369. Notice the relationship of the entries in the middle column and in the diagonal from bottom left to top right.

Solution 190.2 – Six celebrities

> How many celebrities would I need to know for there to be a greater than 50 per cent chance of knowing six celebrities with my birthday?

Sheldon Attridge

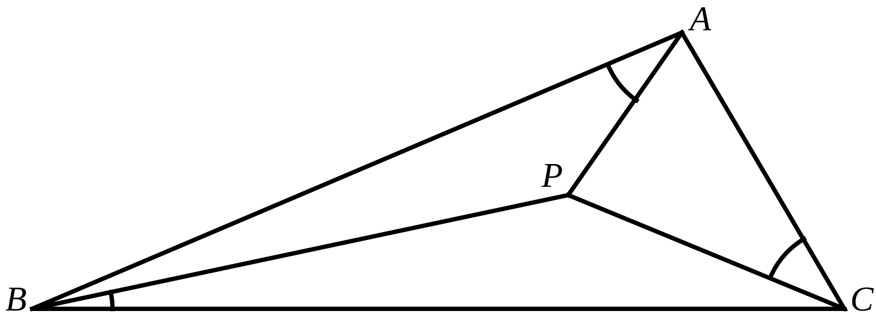
We use the binomial theorem. Let $p = 1/365$ and

$$P(n) \;=\; \sum_{k=0}^{5} \binom{n}{k} p^k (1-p)^{n-k}.$$

Here, the k term is the probability that exactly k people out of n share my birthday. Hence the probability of 6 or more of them sharing my birthday is $1 - P(n)$. So we want to determine the smallest n for which $1 - P(n) > 1/2$. We now have to use a method of trial and error to obtain $n = 2070$.

Maximum Brocard angle

In the original problem we asked for a proof that not all of the angles PAB, PBC and PCA can exceed 30°. When the angles are equal, P is known as a *Brocard point*, and the angle is called the *Brocard angle*. So the Brocard angle can never exceed 30°.



Rob Evans

With respect to Solution 189.8, printed in M500 **191**, I would like to make the following comments.

1. One should not assume without justification that where $D_{12} = \{(x,y) \in (0,\pi)^2 : x + y < \pi\}$ the function $w_{12} : D_{12} \to \mathbb{R}^+$ defined by $w_{12}(x,y) = \operatorname{arccot}(\cot x + \cot y - \cot(x+y))$ has a global maximum.

2. The most that one can infer about the graph of w_{12} from the *symmetry between x and y* in the above expression for $w_{12}(x,y)$ is that it has *mirror symmetry with respect to the plane $x = y$.*

3. The most that one can infer about the graph of w_{12} from the fact that the function $w_1 : (0,\pi/2) \to \mathbb{R}^+$ defined by $w_1(x) = \operatorname{arccot}(2\cot x - \cot 2x)$ has a *zero derivative* (only) when $x = \pi/3$ is that in the plane $x = y$ it is *stationary in the direction parallel to the plane $x = y$* (only) when $(x,y) = (\pi/3, \pi/3)$.

As an approach to showing that a maximum Brocard angle exists, and finding its value, I propose the following.

Start with the function that gives the Brocard angle of a triangle in terms of its three interior angles, i.e. where

$$D = \{(x,y,z) \in (0,\pi)^3 : x + y + z = \pi\}.$$

Let the function $w : D \to \mathbb{R}^+$ be defined by

$$w(x, y, z) = \operatorname{arccot}(\cot x + \cot y + \cot z).$$

Since each of the two Brocard points of a triangle lies inside the open triangular region defined by that triangle, for each $(x, y, z) \in D$ we have that

$$w(x, y, z) \;<\; x, y, z. \tag{1}$$

From the definition of w, we have that

$$w\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) \;=\; \frac{\pi}{6}. \tag{2}$$

From simple considerations of arithmetic, for each $(x, y, z) \in D$ such that each of x, y, z is greater than $\pi/6$, we have that

$$x, y, z \;<\; \frac{2\pi}{3}. \tag{3}$$

Consequently, from (1), (2) and (3) together with simple considerations of logic, we have that w attains a global maximum at $(x, y, z) = (x_0, y_0, z_0)$ if and only if w^* attains a global maximum at $(x, y, z) = (x_0, y_0, z_0)$, where the function $w^* : D^* \to \mathbb{R}^+$ is defined by $w^*(x, y, z) = w(x, y, z)$ and $D^* = \{(x, y, z) \in D : (x, y, z) \in (0, \pi/3)^3\}$.

From elementary differential calculus we have that

$$\cot' x \;=\; -\frac{1}{\sin^2 x}, \qquad 0 < x < \pi. \tag{4}$$

From the graph of the sine function, we have that

$$\begin{aligned} 0 \;<\; \sin x \;<\; \sin\frac{\pi}{3}, &\qquad 0 < x < \frac{\pi}{3}, \\ \sin x \;>\; \sin\frac{\pi}{3} \;>\; 0, &\qquad \frac{\pi}{3} < x < \frac{2\pi}{3}. \end{aligned} \tag{5}$$

Consequently, from (4) and (5) together with simple considerations of arithmetic, we have that

$$\begin{aligned} \cot' x \;<\; \cot'\frac{\pi}{3}, &\qquad 0 < x < \frac{\pi}{3}, \\ \cot' x \;>\; \cot'\frac{\pi}{3}, &\qquad \frac{\pi}{3} < x < \frac{2\pi}{3}. \end{aligned} \tag{6}$$

Hence from (6) together with the Fundamental Theorem of Calculus, for each $(\pi/3+\Delta_1, \pi/3+\Delta_2, \pi/3+\Delta_3) \in D^*$ we have that

$$
\begin{aligned}
&\sum_{i\in N_3} \cot\left(\frac{\pi}{3}+\Delta_i\right) \\
&= 3\cot\frac{\pi}{3} + \left(\sum_{i\in N_3^+} \int_{\frac{\pi}{3}}^{\frac{\pi}{3}+\Delta_i} (\cot' x)dx\right) + \left(\sum_{i\in N_3^-} \int_{\frac{\pi}{3}+\Delta_i}^{\frac{\pi}{3}} (\cot' x)dx\right) \\
&\le 3\cot\frac{\pi}{3} + \left(\sum_{i\in N_3^+} \int_{\frac{\pi}{3}}^{\frac{\pi}{3}+\Delta_i} \left(\cot'\frac{\pi}{3}\right)dx\right) + \left(\sum_{i\in N_3^-} \int_{\frac{\pi}{3}+\Delta_i}^{\frac{\pi}{3}} \left(\cot'\frac{\pi}{3}\right)dx\right) \\
&\quad = 3\cot\frac{\pi}{3} + \left(\cot'\frac{\pi}{3}\right)\sum_{i\in N_3}\Delta_i \\
&\quad = 3\cot\frac{\pi}{3},
\end{aligned}
$$

where $N_3 = \{1,2,3\}$ and, in turn, $N_3^{\pm} = \{i \in N_3 : \pm\Delta_i > 0\}$. Note that equality is attained if and only if $N_3^+ \cup N_3^- = \emptyset$, i.e. $(\Delta_1,\Delta_2,\Delta_3) = (0,0,0)$.

However, arccot is a decreasing function on $\mathbb{R}$. Consequently, for each $(\pi/3+\Delta_1, \pi/3+\Delta_2, \pi/3+\Delta_3) \in D^*$ we have that

$$\operatorname{arccot}\left(\sum_{i\in N_3}\cot\left(\frac{\pi}{3}+\Delta_i\right)\right) \;\le\; \operatorname{arccot}\left(3\cot\frac{\pi}{3}\right),$$

which implies that

$$w^*\left(\frac{\pi}{3}+\Delta_1, \frac{\pi}{3}+\Delta_2, \frac{\pi}{3}+\Delta_3\right) \;\le\; w^*\left(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3}\right) \;=\; \frac{\pi}{6}.$$

Hence, from the argument put forward above, for each $(\pi/3+\Delta_1, \pi/3+\Delta_2, \pi/3+\Delta_3) \in D$ we have that

$$w\left(\frac{\pi}{3}+\Delta_1, \frac{\pi}{3}+\Delta_2, \frac{\pi}{3}+\Delta_3\right) \;\le\; w\left(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3}\right) \;=\; \frac{\pi}{6}.$$

Equality is attained if and only if $(\Delta_1,\Delta_2,\Delta_3) = (0,0,0)$.

Solution 192.6 – 500 factors

What is the smallest number that has exactly 500 factors?

David Turtle

Any natural number x has a unique prime factorization

$$x = \prod_{a=1}^{N} p_a^{n_a},$$

where the p_a are distinct primes and each $n_a \in \mathbb{N}$. A natural number y divides x if and only if

$$y = \prod_{a=1}^{N} p_a^{m_a},$$

where each $0 \leq m_a \leq n_a$, the cofactor being

$$x/y = \prod_{a=1}^{N} p_a^{n_a - m_a}.$$

As a consequence of the uniqueness of prime factorization each distinct sequence $\{m_a\}$ determines a different factor of x; so x has a total of

$$\prod_{a=1}^{N} (n_a + 1)$$

factors. This means that the least x having exactly K factors will be

$$x = \prod_{a=1}^{N} p_a^{q_a - 1},$$

where p_a is the ath prime and $K = \prod_{a=1}^{N} q_a$ is some factorization (not necessarily prime) of K. Clearly the q_a should be in descending order of magnitude so that the smallest primes are raised to the highest powers and vice versa.

For many K the prime factorization of K gives the recipe for the smallest x, but not all. For example, when $K = 32 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ we get $x = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$ but using $K = 4 \cdot 4 \cdot 2$ we get $x = 2^3 \cdot 3^3 \cdot 5 = 1080$. If we consider a pair of the factors of K, say q_a, q_b, then we may be able to reduce the value of x by replacing the two terms $p_a^{(q_a-1)}p_b^{(q_b-1)}$ with the single term $p_a^{(q_a q_b-1)}$. The value of x will decrease if and only if

$$p_a^{(q_a q_b-1)}p_b < p_a^{(q_a-1)}p_b^{(q_b-1)}p_m,$$

where p_m is the largest prime in the factorization of x. Dividing both sides by $p_a^{(q_a-1)}$ we get

$$(p_a^{q_a})^{(q_b-1)} < p_b^{(q_b-1)}(p_m/p_b)$$

and taking (q_b-1)th roots we get

$$p_a^{q_a} < p_b(p_m/p_b)^{1/(q_b-1)}.$$

Is it possible that combining more than two factors of K would reduce x when none of the pairs of the factors involved meet this criterion? If this were so then, if there are f factors, $p_b, ..., p_z$, we would have

$$p_a^{(q_a q_b\,\cdots\,q_z-1)}p_b \ldots p_z < p_a^{(q_a-1)}p_b^{(q_b-1)} \ldots p_z^{(q_z-1)}p_m \ldots p_{m-f+1}.$$

So

$$(p_a^{q_a})^{(q_b\,\cdots\,q_z-1)}p_b \ldots p_z < p_b^{(q_b-1)} \ldots p_z^{(q_z-1)}p_m \ldots p_{m-f+1};$$

but also

$$p_a^{q_a} \geq p_n(p_m/p_n)^{1/(q_n-1)}$$

for all $n \in \{b, ..., z\}$. So

$$(p_a^{q_a})^{(q_b\,\cdots\,q_z-1)}p_b...p_z < (p_a^{q_a})^{(q_b-1)}(p_b/p_m)$$
$$\ldots (p_a^{q_a})^{(q_z-1)}(p_z/p_m)p_m \ldots p_{m-f+1};$$

i.e.

$$(p_a^{q_a})^{(q_b\,\cdots\,q_z-1)} < (p_a^{q_a})^{(q_b-1)} \ldots (p_a^{q_a})^{(q_z-1)}p_m \ldots p_{m-f+1}/p_m^f,$$

and taking the log to base $p_a^{q_a}$ of both sides we get

$$(q_b \ldots q_z - 1) < (q_b - 1) + \ \ldots\ + (q_z - 1) + \log_{p_a^{q_a}}(p_m \ldots p_{m-f+1}/p_m^f);$$

that is,

$$(q_b \ldots q_z) - (q_b + \ \ldots\ + q_z) < 1 - f + \log_{p_a^{q_a}}(p_m \ldots p_{m-f+1}/p_m^f),$$

which is impossible since all the q_n are integers greater than 1 and $\log_{p_a^{q_a}}(p_m...p_{m-f+1}/p_m^f) < 1$. This means that testing pairs of factors for

$$p_a^{q_a} < p_b(p_m/p_b)^{1/(q_b-1)}$$

is sufficient to detect whether any reduction in the value of x is possible. In the case $K = 500$ we have $K = 2^2 \cdot 5^3$ giving $x = 2^4 \cdot 3^4 \cdot 5^4 \cdot 7 \cdot 11$ which clearly cannot be reduced by using a non-prime factorization of K (since, for example, $2^5 > 11$ and all other pairs are even worse). So the smallest natural number with exactly 500 factors is 62370000.

The checking of individual pairs of factors also provides an algorithm for finding x for any K, but unfortunately it is quite inefficient in cases when K has many small prime factors and few large ones, since the order of combining factors does affect the final result. For example in the case of $K = 32$ above if we went from $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ to $2^3 \cdot 3 \cdot 5 \cdot 7$ we would find no further reductions, but going first to $2 \cdot 3^3 \cdot 5 \cdot 7$ we can then go on to $2^3 \cdot 3^3 \cdot 5$. In fact the single reduction to $2^3 \cdot 3 \cdot 5 \cdot 7$ yields the smallest $x = 840$. This means that the algorithm needs to search the whole tree of possible factor-pair combinations to be sure of finding the smallest x. Of course it is still more efficient than a brute force search through all possible factorizations of K, which is in turn more efficient than searching through possible values of x.

One heuristic that often seems to work for choosing the best values of a, b to combine first is to find the smallest $p_a^{q_a}$ and combine it with the largest p_b. Is there a number of factors (a value of K) for which this does not work? if so, what is the smallest such K?

Tony Forbes writes — We also had responses from **Barbara Lee**, **Claudia Gioia** and **Adrian Cox**, all of whom used elementary reasoning to obtain the same answer.

Interestingly, and somewhat perversely, if you increase the number of factors from 500 to 504, you get a considerably *smaller* answer, namely $14414400 = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, which, as you can see, has precisely $7 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = 504$ divisors. Furthermore, 14414400 has the special property that it has more factors than any smaller number. Srinivasa Ramanujan describes a number with this property as *highly composite* and he states that he does not know of any method for determining consecutive highly composite numbers except by trial.

In a lengthy paper ('Highly Composite Numbers', *Proc. London Math. Soc.* Series 2 **XIV** (1915), 347–409), Ramanujan provides a table of all such numbers up to $6746328388800 = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$ (10080 factors). We would be very interested to see how much further modern computer power can extend his results.

Four Colours Suffice by Robin Wilson

Eddie Kent

Four colours suffice to fill in any map so that neighbouring countries are always coloured differently. Neighbouring countries are those that share a section of border; for countries that meet at a point there is no number of colours that would suffice.

Although the problem as stated was thought of by someone colouring a map, in fact cartographers have no interest in solving it. The number of colours used is one of the less interesting things about a map. However, from the beginning, or nearly so, it was seen to be an intriguing mathematical problem. Of course it should not hold anyone long. Since it is obvious that if four colours are actually needed to colour a map then one of the countries involved must be completely enclosed by three others, it clearly requires just a little tidying up to prove the theorem.

Robin Wilson has long been fascinated by this problem; I remember his lectures on the subject even before it finally yielded to Appel and Haken in 1977. He is thus eminently suited to write this book. He easily demolishes the above argument in the early pages, and then goes on to describe, clearly and in great detail, every other argument that was made along the way, giving credit where due to anyone whose work helped the theorem, or indeed mathematics, forward. Kempe, for instance very nearly succeeded, and it was only later discovered that all he had proved was that five colours suffice.

But you probably know all this. What you don't know, unless you've read it, is how good this book is. Clearly it is complete. How could it not be after Robin's years of devotion to the subject? But it is also readable, memorable, and funny. One tiny example concerns Percy 'Pussy' Heywood, the man who wrecked Kempe's argument. A colleague mentioned to him that his watch was two hours fast. Pussy denied this, claiming that it was in fact ten hours slow.

This book came out more than a year ago (in fact it is now in paperback) and I have only just got around to writing about it because, apart from sheer laziness of course, I have given it to a few people to read and tell me what they thought. One of them surprised me by saying "It hooked me from the beginning. It is technical but still gives you an idea, if not an understanding, of how and why the proof worked, and even of why earlier ones didn't. I had to go on and finish it" And that from a man who watches birds for a living and is not entirely certain what a mathematic is.

You of course have a very clear idea about mathematics, so you might like to know that there is more than one way to read this book. As Henry said, it is technical, in that every piece of mathematics is expounded with full rigour. Thus you can read it as a mathematician, in which case you will need a pencil and paper. Otherwise, if you just want the general drift, and a reasonable idea of which way things are going, you can read it in bed (my preferred method). This is perfectly reasonable since no statement is left unjustified, even if sometimes you have to wait a little for the punch line. And the pictures help—they are excellent.

The final chapter gets a bit philosophical. What is a proof? How do we know if it is? There is a scattering of quotations from people on both sides of the debate about whether a computer can produce a proof, especially if, as here, there is no way of checking the working. And was Hardy right when he said there is no permanent place in the world for ugly mathematics? On one thing just about everybody who has seen this proof is agreed: it is not elegant. However, people have been hacking away at it since its first publication, and have succeeded in at least reducing its size. It might yet one day grow up to be a swan.

The book is completed by a set of notes and references including a remarkable bibliography, a glossary, a list of significant dates and an efficient index. Riches indeed.

Freddy's third function

Colin Davies

In 1950, we had a very pedantic maths teacher called Freddy. At that stage we had learned how to integrate straightforward polynomials and trig functions, but not the techniques of integration by parts or by substitution. So Freddy started his lesson by saying:

> There are three types of function. Some can be integrated and some cannot. An example of the first type is $x^6 - 5x^4 + 6$. That can be integrated, and you can integrate it. An example of the second type is $e^x \cos x$. That can be integrated, but you cannot integrate it because you are not sufficiently accomplished mathematicians.

I can't remember just what Freddy wrote on the board as an example of the third type of function; I think it involved $\tan x^2$, but whatever it was, Freddy said about it:

> That cannot be integrated. Nobody could integrate that. (He

meant by analytical methods to get a primitive function; Freddy never told us about numerical methods of integration.)

We then went on to learn about integrating the second type of function; the third type was never mentioned again. However, I have always wondered just what Freddy meant when he said that nobody could integrate it. Which of these did he mean?

1. Nobody can integrate Freddy's third function because nobody is a sufficiently accomplished mathematician.

2. Nobody has ever found a function which, when differentiated gives Freddy's third function.

3. There is a non-existence theorem about the integral of that function.

Or is there another possibility that I have not thought of?

At the time I wondered: If the reason was number 1, how did Freddy know how accomplished some other mathematician might be? And anyway, new mathematics was being discovered every day all over the world. How did Freddy know that somebody in Montevideo had not worked out how to integrate his third function the previous day?

I have since decided that the answer is most likely to be number 2, but that is unsatisfactory because it does not prove that the function cannot be integrated.

Is there a non-existence theorem for some integrals? Can anybody comment?

Freddy's classroom was upstairs, and when his piece of chalk got too short to use, Freddy would first remind us that the path taken by an object in free fall above the earth would be an ellipse, rather than a parabola, because the earth was spherical. Then he would ceremoniously cross to the open window and hurl the piece of chalk out. His object of was to land it on the ledge over a doorway on the next building, but it usually landed on the gravel between the buildings. Freddy would then tell us that it had improved the drainage. I myself never saw Freddy actually land a piece of chalk on the ledge, so I don't know what he told the class when that happened. There were a dozen or so pieces lying on the ledge, so presumably he succeeded occasionally. However, I was privileged to see him hit a telephone wire outside. On that occasion, Freddy announced that he had hit the catenary. He immediately abandoned his prepared lesson and introduced us to hyperbolic functions instead. Meanwhile, the piece of chalk had presumably gone on to improve the drainage.

And now for a little light relief ...

Sheldon Attridge

Most of us would like a bit of recognition for our hard efforts from time to time. After all, who doesn't like to be appreciated? Well, I have come up with a foolproof method for dealing with the problem of ambition based on the 'Completeness Axiom' from which the 'greatest lower bound principle' in M203 can be found:

Consider the scenario: You have been considered to be the greatest mathematician of all time.

And why?

We start with the hypothesis that 'you are considered to be the greatest mathematician of all time', if n people consider this to be true, where n is an integer greater than zero.

Now, by the greatest lower bound property of such a subset of integers, this set has a least member, namely 1. It doesn't matter that it is you yourself that has considered yourself the 'greatest of all time', because 1 is a member of the set. I mean, how large does n have to be anyway? Splendid.

And now we come on to the cunning bit. As we know, Descartes, of Cartesian geometry fame, said once, 'I think I am a turnip, therefore I am a turnip.'

Of course, the modern form of this can be found in most off-the-shelf pop psychology books: 'You are not what you think you are; but, what you think, you are.' (*How profound.*)

So the next stage is to stand in front of the mirror and say to oneself, 'I am considered the greatest mathematician of all time.' 'I am considered the greatest mathematician of all time.' ... And so on, by a process of induction.

Based on the correct reasoning of the first part of our proof, the truth of this statement won't be able to resist slipping into your unconscious; and, before you can say, 'Gauss stole all his results from ancient cuneiform texts written by aliens', your head will no longer get through standard doorways, and crowds of adulating Royal Society profs and dons will be waiting outside your home, begging for an autograph.

... And all because of the Completeness Axiom used for the integers.

Question. What's yellow and is equivalent to the Axiom of Choice?

Answer. Zorn's lemon.

Crossnumber

Tony Forbes

Across

3. $\left(\sqrt{(12\text{ across})} - 2\right)^2$

5. $4\left(\sqrt{(8\text{ down})} - 1\right)^2$

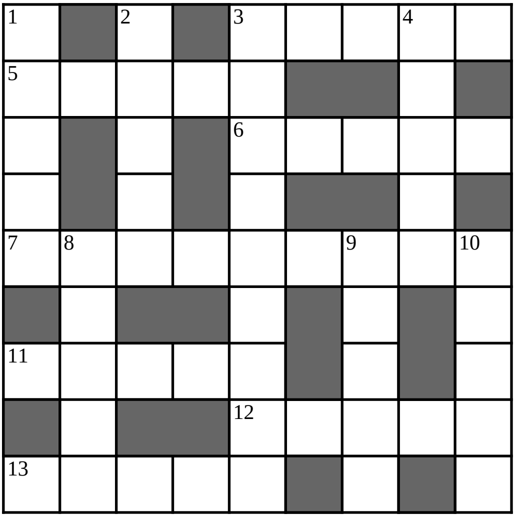
6. $\left(\sqrt{(11\text{ across})} - 5\right)^2$

7. $(9\text{ down})^2/3$

11. $\left(\sqrt{(6\text{ across})} + 5\right)^2$

12. $\left(\sqrt{(3\text{ across})} + 2\right)^2$

13. $\sqrt{(3\text{ down})}$



Down

1. $\left(\sqrt{(2\text{ down})} - 2\right)^2$

2. $3\,(9\text{ down})$

3. $(13\text{ across})^2$

4. $\sqrt{(10\text{ down})}\sqrt{(2\text{ down})}$

8. $\left(\sqrt{(5\text{ across})}/2 + 1\right)^2$

9. $\sqrt{3\,(7\text{ across})}$

10. $(4\text{ down})^2/(2\text{ down})$

Problem 195.1 – Two queens

Two queens are placed on different squares of an $n \times n$ chessboard. What is the probability they 'attack' each other?

Problem 195.2 – Six tans

If $\theta = \pi/13$, prove that

$$\tan\theta\ \tan 2\theta\ \tan 3\theta\ \tan 4\theta\ \tan 5\theta\ \tan 6\theta\ =\ \sqrt{13}.$$

What's next?

ADF

What's next in the following sequences?

(i) twins, triplets, quadruplets, quintuplets, sextuplets, septuplets, octuplets, nonuplets, decuplets, ?, dodecuplets, ?, ?, ?, ?, ?, ?, ...

(ii) *Sir, I seek a rhyme excelling*
In mystic force and magic spelling.
Celestial spheres elucidate,
But my own feelings can't relate
...

I am really serious about the second one. If you can determine the rest of that useful poem, please send it in! In case you haven't spotted it, the lengths of the words are 3 1 4 1 5 9 2 6 5 3 5 8 9 7 9 3 2 3 8 4 6, and therefore in the continuation they must be 2 6 4 3 3 8 3 2 7 9 5 0 2 8 8 4 1 9 7 1 6 9 3 9 9 3 7 5 1 0 5 8 2 0 9 7 .... No, I don't know of any zero-letter words.

Problem 195.3 – Doublings

Tony Forbes

A positive integer N has the property that the number of digits in $2^i N$ is given by the sequence

$$(2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 5,\ 5,\ 5,\ 6)$$

for $i = 0, 1, \ldots, 11$. What is N?

A general problem: Investigate the possibility of identifying a number by its number of digits and the number of digits in its doublings. Given n, let $D_f(n)$ denote the sequence $(d_0, d_1, \ldots, d_f)$, where d_i is the number of digits in $2^i n$. How big must f be such that n is uniquely identified by $D_f(n)$?

From a modicum of experimentation it seems that often f must be quite large. For instance, I found that for $n = 967$ you need to go as far as $f = 2538$. Thus $D_{2538}(n)$ differs from $D_{2538}(967)$ for all other n, but there exists an $m \neq 967$ such that $D_{2537}(m) = D_{2537}(967)$. However, I claim that the short sequence given above, which is $D_{11}(N)$, contains sufficient information for you to determine N unambiguously. For a start, N can't be 10 because $D_{11}(10) = (2, 2, 2, \ldots)$.

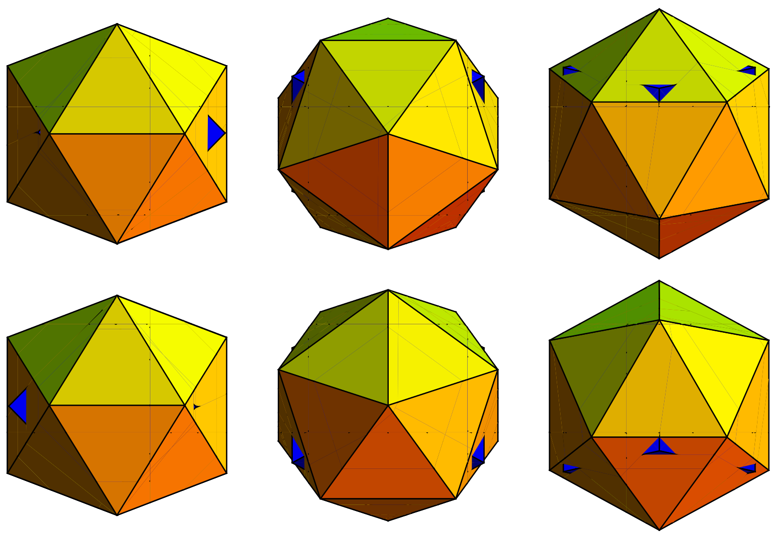
Letters to the Editor

Platonic solids

Re: Problem 192.3 [Can the regular polyhedra fit one inside the next when arranged in order of increasing volume?]. I think that each solid can be fitted inside the next (if the solids are hollow!) but they are not 'stackable' in the same way as the Russian dolls. A tetrahedron can be covered by an octahedron with one triangular face removed—they have the same altitude. The octahedron can be dropped into a cube with one face removed, as the diameter of the octahedron is the same as a face diagonal of the cube. The cube will obviously not pass through one face of the icosahedron—I think one pentagonal cap must be removed. Similarly, the icosahedron will not pass through one face of the dodecahedron although there is ample room inside.

Chris Pile

ADF writes — Will somebody put me out of my misery regarding the cube and the icosahedron! All I can say is that after spending a long time doodling with MATHEMATICA I cannot see how to avoid bits of the cube protruding through the surface. And if it can't be done, it does not seem obvious how to orient the cube to minimize the volume of the protrusions.



M500 191

The problems in **191** all require real mathematics beyond my grasp. But Tony's remark that setting something in 11 point instead of 10 point produces a 21 per cent increase in paper costs isn't quite right. There is also a fitting problem of words per line (even when hyphenation is used—breaks can only be in certain places) and lines per page which means that the figure will always be a bit over 21 per cent. And that's when the same font is used. The new version has a different font, with considerably wider letter spacing and line spacing than the Times New Roman used in earlier versions. It is worth noting that Stanley Morison designed Times to be as compact as possible, so that as many words as possible could be got on to a newspaper page. I don't think any other fonts have been designed with this criterion, at least not as a principal one—except Matthew Carter's Bell Centennial, a narrow sans serif font designed specially for telephone directories.

Colin Davies's 'eleven plus two = twelve plus one' question is very boringly answerable by citing 'seventy-one plus seven = seventy-seven plus one' and many similar. But I can't think of an example involving real respelling of single words in any language. Jeremy Humphries's 'add one w to stall' question gives 'stonewall'—anyone who does crosswords would get that right away. But these are not problems.

Best wishes,

Ralph Hancock

Fruit cakes

ADF

While doing your Christmas shopping imagine that you have to obey every single notice, price label, advertisement, inducement, special offer, whatever, that you come across during your wanderings around the supermarket. For example, you see a sign which reads 'Fruit cakes. Now only £3. Save 30p'. Interpret the 'Save 30p' part as an order to be obeyed. So you must spend £3.00 on a fruit cake. But the notice is still there, so you buy another fruit cake, and another, ..., until you clean the shelf out. Then you move on to the next offer: 'Toilet paper £4.50. Buy 2 get 1 free'. And so on.

Now you have got the idea, we ask: Approximately how much would you spend?

I did try it once at a largish Sainsbury's but I ran out of time and had to stop after spending about £20,000.00.

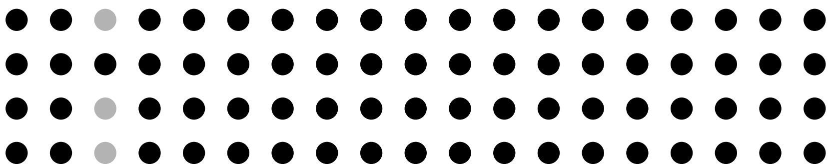
Parades and resolvable Steiner systems

Tony Forbes

If you went to this year's M500 Revision Weekend, you may have witnessed Professor Donald Preece, dressed in the 18th century uniform of an officer in the Ruritanian Army, delivering a lecture on combinatorial designs whilst playing the piano and on several occasions breaking into song. I think that's an accurate description of the Saturday guest lecture; if you weren't there, you will just have to use your imagination.

Here I would like to follow up on some of the material of Donald's talk.

You remember that song about *76 Trombones in the Big Parade*. (Yes, I'm afraid we were forced to sing it during the lecture.) One imagines that every year for a period of 925 years the 76 trombonists march in the annual 'big parade' as a neat rectangular array of 19 rows of four. Donald wishes to answer the question: Is it possible to arrange the parades such that each triple of trombonists appears together in the same row exactly once? Let's be clear about this. Suppose Edwina, Wendy and Felicity are three typical trombonists. Then the rule implies that there is precisely one parade during the 925 years when Edwina, Wendy and Felicity will march in the same row.

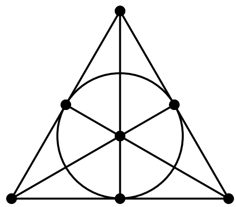


Before we go on we need a couple of definitions.

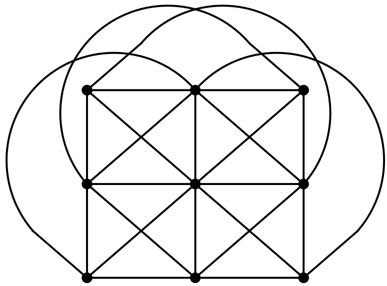
First definition. Suppose we have a set, V, of v things and suppose t and k are integers with $1 \leq t \leq k \leq v$. A *Steiner system*, $S(t, k, v)$, is a set of k-tuples of elements of V, called *blocks*, such that each t-tuple of elements of V is contained in precisely one block.

This is a lot to hold in the mind all at once, so let's look at a few examples. When $t = 1$ the situation is trivial and uninteresting. Thus $\{\{0, 1, 2\}, \{3, 4, 5\}\}$ is an $S(1, 3, 6)$; it is a set of triples and each single element occurs in exactly one triple. It is clear that in general an $S(1, k, v)$ exists if and only if v/k is an integer.

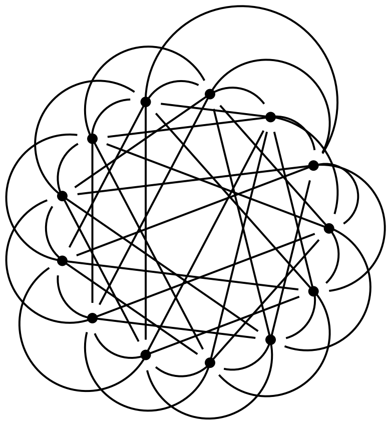
When $t = 2$, things are much more exciting. An $S(2, 3, v)$, is usually called a *Steiner triple system*, or an STS(v). The system is a set of *triples*, and each *pair* of elements occurs in exactly one triple. In 1853 Jakob Steiner (the person whose name is associated with these systems) conjectured that there exists an STS(v) for every $v \equiv 1$ or 3 (mod 6). The condition on v turns out to be necessary, as a simple calculation shows, and with a little patience you can verify that {012} is an STS(3), {012, 034, 135, 236, 146, 245, 056} is an STS(7) and {012, 345, 678, 036, 147, 258, 057, 138, 246, 048, 156, 237} is an STS(9). Furthermore, if you look at the front cover of this year's Special Issue of M500 you will see a picture of an STS(13) (reproduced on the right), where an element is represented by a blob and a triple by an unbroken line consisting of a circular arc and a straight segment.



STS(7)



STS(9)



STS(13)

The systems with $v = 3, 7$ and 9, are unique (up to isomorphism) but there are two distinct STS(13)s. If you want to construct the other one, take the Special Issue picture, remove the four triples that look as if they don't belong and add four new triples to give the thing a nice 13-fold symmetry.

Thus we have proved Steiner's conjecture for all valid $v < 15$. However, the conjecture had already been completely proved in 1847 by The Rev. Thomas P. Kirkman. Presumably at that time he was unaware of Steiner's 1853 deliberations.

In general, an $S(t,k,v)$ can exist only if

$$\binom{v-i}{t-i}\bigg/\binom{k-i}{t-i} \text{ is an integer for } i = 0, 1, \ldots, t. \tag{1}$$

For if you take an $S(t,k,v)$ and remove an element, v_0, say, as well as all the blocks which contain v_0, you will end up with a set of $(k-1)$-tuples which form an $S(t-1, k-1, v-1)$.

As t gets larger, the condition (1) becomes more and more restrictive. When $t = 5$ only a handful of systems are known, and it is currently a major unsolved problem of combinatorial design theory to find an $S(t,k,v)$ system with $t \geq 6$ (and $v > k$).

Second definition. A Steiner system, $S(t,k,v)$, is said to be *resolvable* if the blocks can be partitioned into subsets of v/k blocks such that in every subset each of the v elements occurs once. The additional condition that must be satisfied is

$$v \equiv 0 \pmod{k}. \tag{2}$$

Of the examples given above, the STS(3) and the STS(9) are resolvable but the STS(7) and STS(13)s are not. The STS(9) has the partitioning

$$\{\{012, 345, 678\}, \{036, 147, 258\}, \{057, 138, 246\}, \{048, 156, 237\}\}.$$

It appears that the mathematical community has tried to make amends for the rather unfair naming of Steiner triple systems after someone other than Kirkman, their true discoverer. A resolvable $S(2,3,v)$, is known as a *Kirkman triple system*, or a KTS(v), and the classic example occurs in a problem set by Kirkman in the 1850 edition of *The Lady's and Gentleman's Diary*:

> Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily such that no two shall walk twice abreast.

From a careful reading of the puzzle, it becomes evident that the solution calls for a KTS(15); in other words a resolvable STS(15). Notice that 15 works because both (1) and (2) are satisfied for $t = 2$, $k = 3$ and $v = 15$. I could give you an actual system but I don't have a KTS(15) to hand and anyway I'm sure you will have more fun discovering one for yourself.

Now at last we can get back to the 76 trombonists marching in a 19×4 array once a year for 925 years. The numbers do add up: there are $925 \cdot 19 = 17575$ rows altogether and each row contains four triples. Hence there are

70300 triples in all, and that is equal to $\binom{76}{3}$. Therefore it is possible for each triple to occur exactly once.

To put it in combinatorial language, we are asking if there exists a resolvable $S(3,4,76)$. In this case the answer is 'yes', and indeed there is nothing special about the number 76. It is known there exists a resolvable $S(3,4,v)$ system whenever $v \equiv 4$ or 8 (mod 12) except possibly for $v \in \{220,$ 236, 292, 364, 460, 596, 676, 724, 1076, 1100, 1252, 1316, 1820, 2236, 2308, 2324, 2380, 2540, 2740, 2812, 3620, 3820, 6356$\}$ (C. Colbourn & J. Dinitz, *The CRC Handbook of Combinatorial Designs*).

Observe that resolvability is an important consideration for our 76 trombones, and it is fortunate that both (1) and (2) hold for $t = 3$, $k = 4$, $v = 76$. If we tried to arrange the marches using an unresolvable $S(3,4,76)$, some of the trombonists would have to occupy two places at once!

That deals with the trombones. The problem is solved. But, as Donald pointed out, the song goes on to say that the big parade also features 110 cornets. Presumably they march in 22 rows of 5, and, as before, we want to arrange their parades—this time over the next 981 years—such that no three cornettists shall march twice abreast.

Again, the numbers work: $\binom{110}{3} = 215820 = 981 \cdot 22 \cdot \binom{5}{3}$ and the job can be done with a resolvable $S(3,5,110)$. Furthermore, you can check that both conditions (1) and (2) hold for $t = 3$, $k = 5$ and $v = 110$. Hence there is no obvious reason why an $S(3,5,110)$ should not exist. However, as far as we (Donald and I) are aware, no such system is known.

According to the Colbourn & Dinitz book, $S(3,5,v)$s are rare. The only known non-trivial ones having $v \le 200$ are for $v \in \{17, 26, 65, 101\}$ but apart from 65 these systems are of no use for scheduling marching bands because the resolvability condition (2) fails. And until I go and look it up I don't know whether the $S(3,5,65)$ is resolvable.

So here are a few things for you to do over the Christmas / New Year holiday period:

(i) Arrange the walks for the fifteen young ladies.

(ii) Knock a v or two off the list of possible exceptions for $S(3,4,v)$.

(iii) Find an $S(3,5,110)$. Any $S(3,5,110)$ will do for a start but if you can construct a resolvable one, so much the better.

(iv) Given t and k, are there infinitely many v for which (1) holds?

(v) And if you really want to make a name for yourself: Find a non-trivial $S(6,k,v)$.

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