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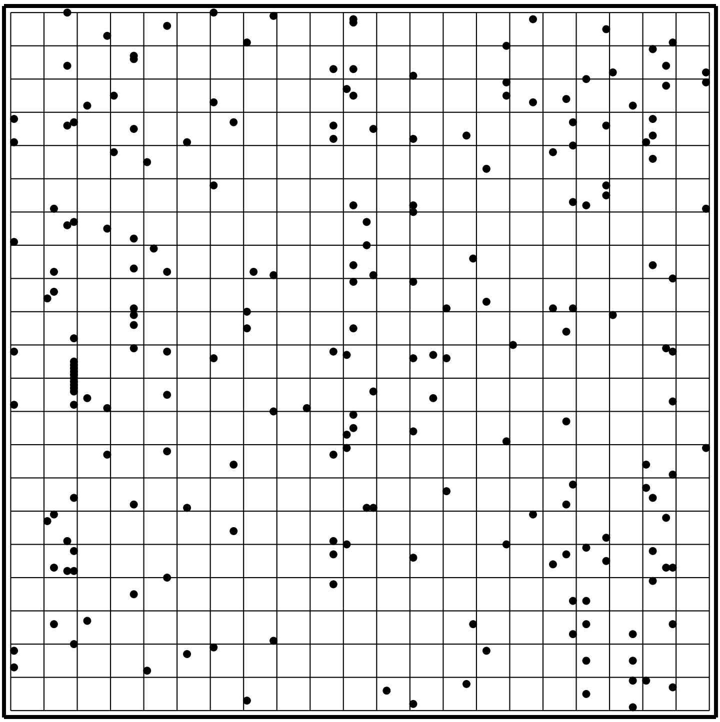
M500 200



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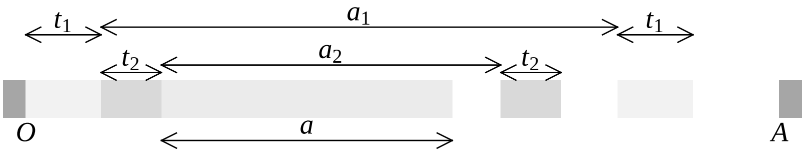
Solution 192.4 – Two boxes

> You have an object of dimensions $a \times b \times c$ and a hole of dimensions $A \times B \times C$. The object fits in the hole. Make two boxes of internal dimensions a_1, b_1, c_1 and a_2, b_2, c_2 out of materials with thickness t_1 and t_2, respectively. Choose $a_1, b_1, c_1, t_1, a_2, b_2, c_2$ and t_2 at random, subject to the constraint that the object fits in each box and each box fits in the hole. What's the probability that one box fits in the other box?

Robin Marks

Since the three-dimensional problem looks difficult, let us simplify things by looking first at the one-dimensional case. We have an object of length a, a hole of length A, a box of internal length a_1 and thickness t_1, and another box of internal length a_2 and thickness t_2.

Figure 1: One–dimensional object, hole and two boxes



Consider pairs of values (a_1, a_2) where $a_1 \in \mathbb{R}$ and $a_2 \in \mathbb{R}$. These pairs fill (span) $\mathbb{R}^2$. The constraints given in the problem are as follows.

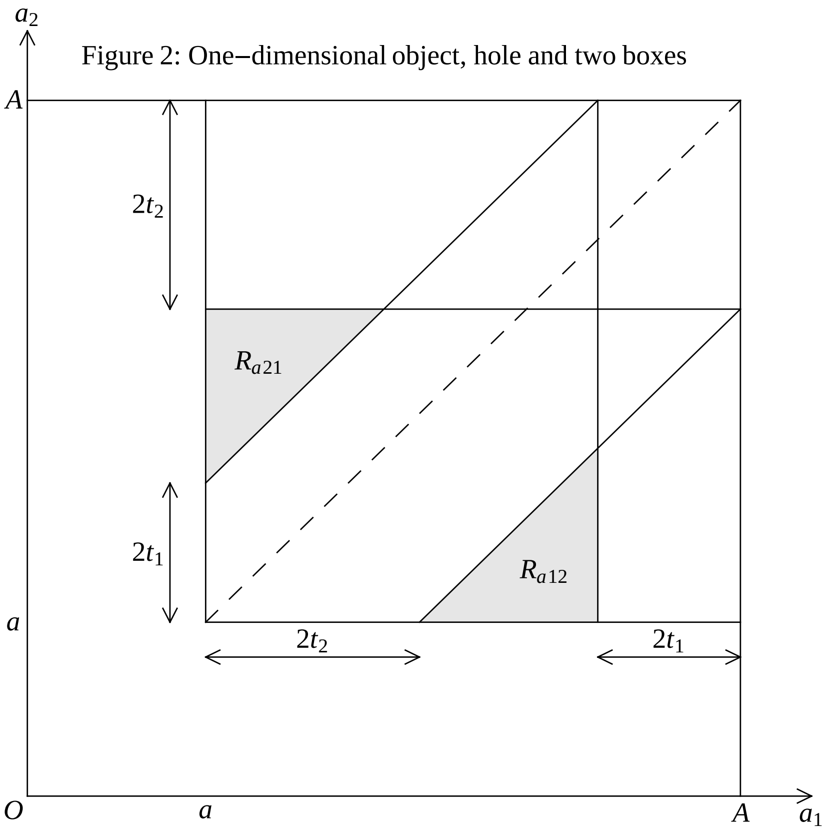
Constraint 1a: $a_1 > a$. Constraint 1b: $a_2 > a$. That is, each box has an internal length greater than a. Points satisfying both constraints 1a and 1b fill the intersection of two half-spaces of $\mathbb{R}^2$: the half-space to the right of the line $a_1 = a$ and the half-space above the line $a_2 = a$.

Constraint 2a: $a_1 < A - 2t_1$. Constraint 2b: $a_2 < A - 2t_2$. That is, each box has an internal length less than the length of the hole minus twice the thickness of the box.

Points satisfying all of the above constraints fill a rectangular region, the intersection of the four half-spaces defined by those constraints. We will call this region R_{a_0}. The area of R_{a_0} is $(A - a - 2t_1)(A - a - 2t_2)$.

Constraint 3a: $a_1 > a_2 + 2t_2$. Constraint 3b: $a_2 > a_1 + 2t_1$. Constraints 3a and 3b cannot both be true; that is, either the internal length of box 1 must be more than the internal length of box 2 by at least twice the thickness of box 2, *or* the internal length of box 2 must be more than the internal length of box 1 by at least twice the thickness of box 1.

Points satisfying constraints 1a, 1b, 2a, 2b and 3a fill a triangular region of $\mathbb{R}^2$ which, because in this region $a_1 > a_2$, we will call R_{a12} (short for $R_{a_1 > a_2}$). Similarly, points satisfying constraints 1a, 1b, 2a, 2b and 3b fill a region which we will call R_{a21}. For one box to fit inside the other within the hole, pairs of values (a_1, a_2) must either lie in region R_{a12} or in region R_{a21}. We can represent these variables on a diagram with axes a_1 and a_2.



The number of ways in which both boxes fit in the hole together without overlapping is the number of (a_1, a_2) pairs in regions R_{a12} and R_{a21}. The number of ways in which the boxes fit in the hole (overlapping or not) is given by the number of (a_1, a_2) pairs in region R_{a0}. Thus the probability that both boxes fit in the hole without overlap is

$$\frac{\text{area of } R_{a12} + \text{area of } R_{a21}}{\text{area of } R_{a0}} \;=\; \frac{(A-a-2t_1-2t_2)^2}{(A-a-2t_1)(A-a-2t_2)}.$$

We can simplify the right-hand side of this expression by dividing top and bottom by $(A-a)^2$ to give

$$\frac{\left(1-\dfrac{2t_1}{A-a}-\dfrac{2t_2}{A-a}\right)^2}{\left(1-\dfrac{2t_1}{A-a}\right)\left(1-\dfrac{2t_2}{A-a}\right)}.$$

We can substitute two new random variables, u_1 for $2t_1/(A-a)$ and u_2 for $2t_2/(A-a)$, giving the probability that one box fits inside the other is

$$h(u_1,u_2) \;:=\; \frac{(1-u_1-u_2)^2}{(1-u_1)(1-u_2)},$$

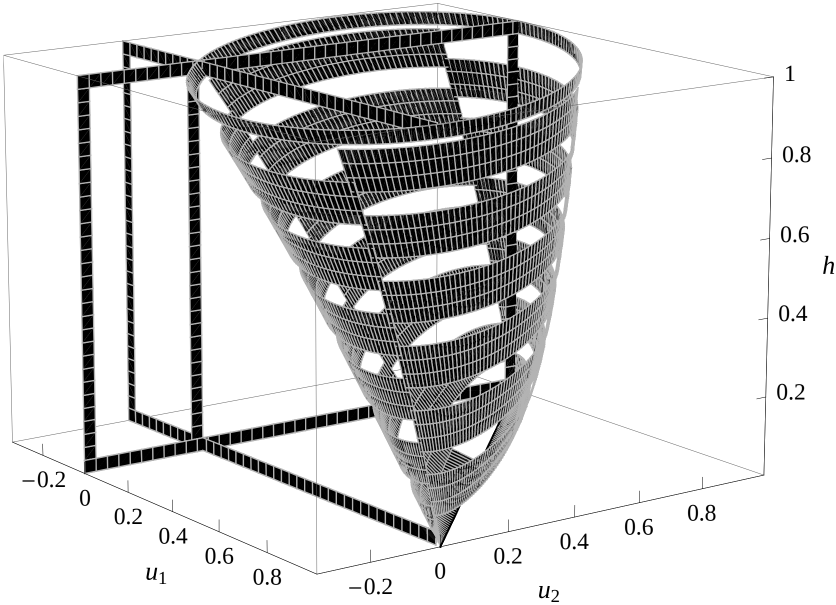
a function of two variables.

If we let both u_1 and u_2 tend to zero, then $h(u_1,u_2)$ tends to 1. In other words, in the one-dimensional case, one box will fit inside the other if the boxes are sufficiently thin and if internal sizes a_1 and a_2 are not exactly equal. Figure 3 is a plot of h with parts cut away for a better view, and with parts of the two planes $u_1=0$ and $u_2=0$ added for visual reference.

We can see that the horizontal cross-section of the function h looks like an ellipse. In fact it is easy to show that the horizontal cross-section is an ellipse when $h(u_1,u_2)=K$, where K is a constant such that $0<K<4$. I leave the proof as an exercise for the interested reader. If $h(u_1,u_2)=0$, then we have $1-u_1-u_2=0$, which is the equation of the straight line passing through $(0,1)$ and $(1,0)$. Parts of this line are visible in Figure 3.

In the two boxes problem h represents a probability, hence $0\le h\le 1$. Also, because the thickness of each box is positive, $u_1>0$ and $u_2>0$, and because each box fits in the hole, $u_1<1$ and $u_2<1$. In addition, if the combined thicknesses are too great to fit in the hole, that is, $u_1+u_2>1$, then the probability is not $h(u_1,u_2)$ but zero. We therefore define a new function $h_2(u_1,u_2)=h(u_1,u_2)$ if $u_1+u_2\le 1$, $h_2(u_1,u_2)=0$ otherwise; see Figure 4.

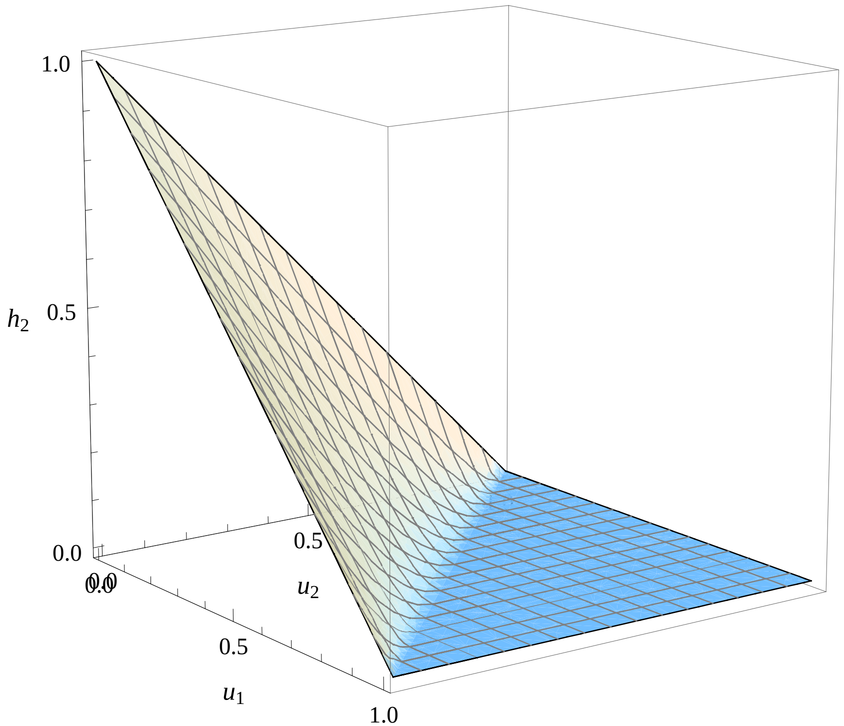
Figure 3: $h(u_1, u_2) = \dfrac{(1 - u_1 - u_2)^2}{(1 - u_1)(1 - u_2)}$



The curved part of the surface in Figure 4 is a section of the surface of Figure 3. To get the average probability we work out the volume under the surface of the function h_2 for relevant values (u_1, u_2), then divide the volume by the area of the base to get an average height.

To do this we integrate h over the triangular region where $u_1 + u_2 \leq 1$, then divide the answer by the total base area, 1×1, giving $\int_0^1 \int_0^{1-u_1} h(u_1, u_2)\, du_2\, du_1 = (\pi^2 - 9)/6$. Hence the average probability of success is $(\pi^2 - 9)/6$. To check this I did a simulation by first picking a pair of random values (t_1, t_2) such that $0 < t_1 < 0.5$ and $0 < t_2 < 0.5$, then picking 10000 pairs of random values (a_1, a_2) such that $0 < a_1 < 1 - 2t_1$ and $0 < a_2 < 1 - 2t_2$ (so that each box fits, separately, in the hole).

Figure 4: $h_2(u_1, u_2) = \dfrac{(1-u_1-u_2)^2}{(1-u_1)(1-u_2)}$ if $u_1 + u_2 \leq 1$, 0 otherwise



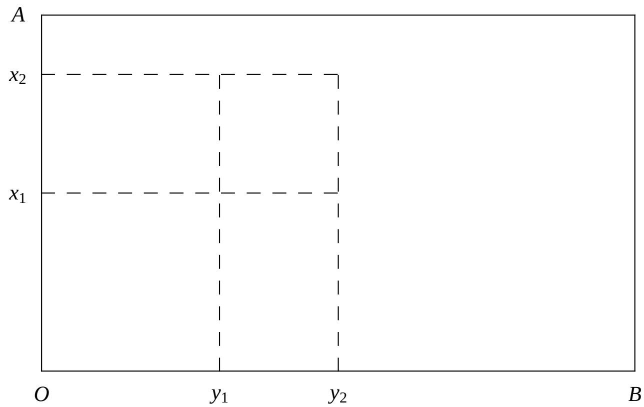
For example, with random values $t_1 = 0.04383$ and $t_2 = 0.2831$, choosing 10000 random (a_1, a_2) pairs gave a success rate for fitting two boxes of 0.3029. The predicted success rate by the function h_2 was 0.3028. The simulation was repeated, using 10000 different (t_1, t_2) pairs, to a total of 10000×10000 sets of random selections. Overall, the average probability of success was 0.144, close to the integrated value $(\pi^2 - 9)/6 = 0.145$. So, amazingly, we can get a reasonable estimate of the value of π by repeatedly trying to fit two one-dimensional boxes into a one dimensional hole!

Furthermore, $(\pi^2 - 9)/6$ is also the probability in a special case in three dimensions; the case when both boxes are cubes. This amounts to the one-dimensional case being simultaneously replicated in the other two dimensions.

Now let us look at the two-dimensional case. We have an object of length a, a hole sized $A \times B$, $B > A$, a box of internal size (a_1, b_1) and thickness t_1, and another box of internal size (a_2, b_2) and thickness t_2. Each box has to fit in the hole in the direction with the smallest gap, that is, the A direction; so the maximum thickness of a box is $(A - a)/2$.

Suppose for now that the boxes have zero thickness. The new condition for one box to fit inside the other in two dimensions is that if $a_1 > a_2$ then $b_1 > b_2$ (Figure 5).

Figure 5: Two thin boxes in a hole sized $A \times B$



The horizontal dashed lines are representative positions for a_1 and a_2; that is, either a_1 is at x_1 and a_2 is at x_2 or vice versa; we will ignore the case where $a_1 = a_2$. Similarly the vertical dashed lines are representative positions for b_1 and b_2. Given that one vertex of a box is at O, the 'opposite' vertex is at one of the four intersections of the dashed lines.

The vertex of one box must be one of the two intersections on the line x_1. Having chosen this vertex, the vertex of the other box can only be in one position on the line x_2; that is, if one vertex is at (x_1, y_1), the other must be at (x_2, y_2), and if one vertex is at (x_1, y_2), the other must be at (x_2, y_1). Thus there are $2 \times 1 = 2!$ configurations (ways of choosing pairs of vertices). In only one of these configurations does one box fit inside the other. So the probability of one box fitting inside the other is $1/2$ when $t_1 = t_2 = 0$.

What if $t_1 > 0$ and $t_2 > 0$? We will now consider the situation in the direction of B. The B dimension is independent of the A dimension; however, the maximum thickness of any box remains $(A - a)/2$. Now we argue as before. Consider pairs of values (b_1, b_2), where $b_1, b_2 \in \mathbb{R}$. The constraints given in the problem in direction B are as follows.

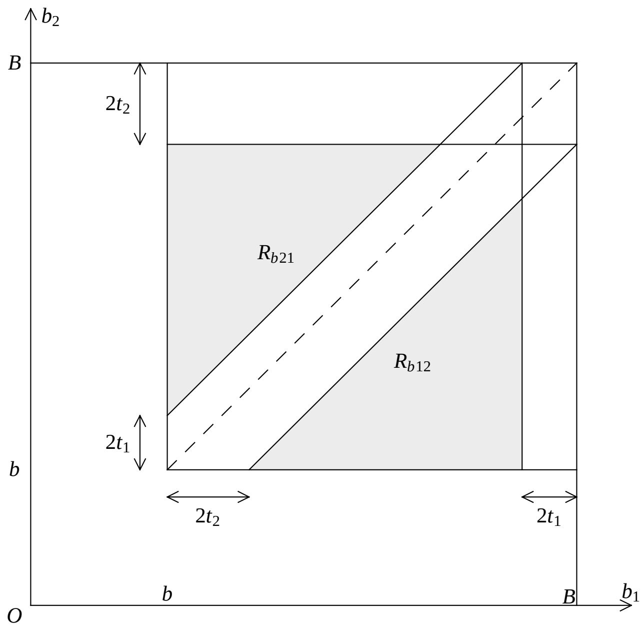
Constraint 1a: $b_1 > b$. Constraint 1b: $b_2 > b$. Constraint 2a: $b_1 < B - 2t_1$. Constraint 2b: $b_2 < B - 2t_2$. Points satisfying all of these constraints fill a rectangular region. We will call this region R_{b_0}. The area of R_{b_0} is $(B - b - 2t_1)(B - b - 2t_2)$.

Constraint 3a: $b_1 > b_2 + 2t_2$. Constraint 3b: $b_2 > b_1 + 2t_1$. Constraints 3a and 3b cannot both be true.

Points satisfying all these constraints fill two triangular regions of $\mathbb{R}^2$, which we will call R_{b12} (short for $R_{b_1 > b_2}$) and R_{b21}.

We can represent these variables on a diagram with axes a_1 and a_2 (Figure 6).

Figure 6: B direction: one-dimensional object, hole, two boxes



The probability that both boxes fit in the hole without overlap is

$$\frac{\text{area of } R_{b12} + \text{area of } R_{b21}}{\text{area of } R_{b0}} = \frac{(B - b - 2t_1 - 2t_2)^2}{(B - b - 2t_1)(B - b - 2t_2)}.$$

We can simplify the right-hand side of this expression by dividing top and bottom by $(B-b)^2$ to give

$$\frac{\left(1-\dfrac{2t_1}{B-b}-\dfrac{2t_2}{B-b}\right)^2}{\left(1-\dfrac{2t_1}{B-b}\right)\left(1-\dfrac{2t_2}{B-b}\right)} = \frac{\left(1-\dfrac{2r_{ab}t_1}{A-a}-\dfrac{2r_{ab}t_2}{A-a}\right)^2}{\left(1-\dfrac{2r_{ab}t_1}{A-a}\right)\left(1-\dfrac{2r_{ab}t_2}{A-a}\right)},$$

where $r_{ab}=(A-a)/(B-b)$. We can write this expression as $h(r_{ab}u_1, r_{ab}u_2)$.

Putting this all together, the probability $\frac{1}{2}$ is reduced in the A direction by a factor of $h(u_1,u_2)$ and is also independently reduced by a factor of $h(r_{ab}u_1, r_{ab}u_2)$ in the B direction. So the probability of one box fitting inside the other is

$$g(u_1,u_2,r_{ab}) = \begin{cases} \frac{1}{2}\,h(u_1,u_2)\,h(r_{ab}u_1,r_{ab}u_2) & \text{if } u_1+u_2\le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Written out in full this becomes

$$g(u_1,u_2,r_{ab}) = \begin{cases} \dfrac{1}{2}\,\dfrac{(1-u_1-u_2)^2}{(1-u_1)(1-u_2)}\,\dfrac{(1-r_{ab}u_1-r_{ab}u_2)^2}{(1-r_{ab}u_1)(1-r_{ab}u_2)} \\ \qquad\qquad \text{if } u_1+u_2\le 1, \\ 0 \qquad\qquad \text{otherwise.} \end{cases}$$

Given a particular value of r_{ab}, integrating g over the triangular region involving u_1 and u_2 gives the probability of both two-dimensional boxes fitting in the hole together. Here is a table of probabilities $\int_0^1\int_0^{1-u_1} g(u_1,u_2,r_{ab})du_2du_1$ for selected values of $r_{ab}=(A-a)/(B-b)$.

r_{ab}	0	1/4	1/2	1
Probability that both boxes fit	0.0725	0.0638	0.0548	0.0363

We obtain an average probability value for a randomly sized hole and object by integrating g over the triangular region involving u_1 and u_2, and from $r_{ab}=0$ to 1, giving

$$\int_0^1\int_0^1\int_0^{1-u_1} g(u_1,u_2,r_{ab})\,du_2\,du_1\,dr_{ab} = 0.0547$$

by numerical integration. A simulation using 10000×10000 sets of random selections gave a success rate of fitting one box inside the other of 0.0538.

Now for the three-dimensional case. Initially suppose $t_1 = t_2 = 0$. Position the two boxes so that each has a vertex at O. As before, consider positions of the two vertices opposite to O.

The vertex of one box must be one of the 2×2 intersections on the plane x_1. Having chosen this vertex, the vertex of the other box can only be in one position on the plane x_2. That is, if one vertex is at (x_1, y_1, z_1), the other is at (x_2, y_2, z_2); if one vertex is at (x_1, y_1, z_2), the other is at (x_2, y_2, z_1); if one vertex is at (x_1, y_2, z_1), the other is at (x_2, y_1, z_2); if one vertex is at (x_1, y_2, z_2), the other is at (x_2, y_1, z_1). Thus there are $(2!)^2 = 4$ configurations (ways of choosing pairs of vertices). In only one of these configurations does one box fit inside the other. So the probability of one box fitting inside the other is $\frac{1}{4}$ when $t_1 = t_2 = 0$.

What if $t_1 > 0$ and $t_2 > 0$? The probability $\frac{1}{4}$ is reduced by a factor of $h(u_1, u_2)$ the A direction and $h(r_{ab}u_1, r_{ab}u_2)$ in the B direction, as in the two-dimensional case above, and also by $h(r_{ac}u_1, r_{ac}u_2)$ in the C direction, where $r_{ac} = (A - a)/(C - c)$. So the probability of one box fitting inside the other is

$$f(u_1, u_2, r_{ab}, r_{ac}) = \begin{cases} \frac{1}{4}\, h(u_1, u_2)\, h(r_{ab}u_1, r_{ab}u_2)\, h(r_{ac}u_1, r_{ac}u_2) \\ \qquad\qquad \text{if } u_1 + u_2 \le 1, \\ 0 \qquad\qquad \text{otherwise} \end{cases}$$

or, when written out in full,

$$f(u_1, u_2, r_{ab}) = \begin{cases} \dfrac{1}{4}\, \dfrac{(1 - u_1 - u_2)^2}{(1 - u_1)(1 - u_2)}\, \dfrac{(1 - r_{ab}u_1 - r_{ab}u_2)^2}{(1 - r_{ab}u_1)(1 - r_{ab}u_2)} \\ \qquad \times \dfrac{(1 - r_{ac}u_1 - r_{ac}u_2)^2}{(1 - r_{ac}u_1)(1 - r_{ac}u_2)} \qquad \text{if } u_1 + u_2 \le 1, \\ 0 \qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Given particular values of r_{ab} and r_{ac}, we can integrate f over the triangular region involving u_1 and u_2 to give the probability of both three-dimensional boxes fitting in the hole together. Here is a table of probabilities $\int_0^1 \int_0^{1-u_1} f(u_1, u_2, r_{ab}, r_{ac})\, du_2\, du_1$ for selected values of $r_{ab} = (A-a)/(B-b)$ and $r_{ac} = (A - a)/(C - c)$.

	$r_{ac} = 0$	$r_{ac} = 1/2$	$r_{ac} = 1$
$r_{ab} = 0$	0.0362	0.0274	0.0181
$r_{ab} = \frac{1}{2}$	0.0274	0.0212	0.0147
$r_{ab} = 1$	0.0181	0.0147	0.0110

As before, we can obtain an average probability value for a randomly sized hole and object by integrating f over the triangular region involving u_1 and u_2, and from $r_{ab} = 0$ to 1 and from $r_{ac} = 0$ to 1. Thus

$$\int_0^1 \int_0^1 \int_0^1 \int_0^{1-u_1} f(u_1, u_2, r_{ab}, r_{ac})\, du_2\, du_1\, dr_{ab}\, dr_{ac} \;=\; 0.0211$$

by numerical integration. A simulation using 10000×10000 sets of random selections gave a success rate of fitting one box inside the other of 0.0208.

Solution 196.5 – Three more friends

> I have three friends, Alan, Bert and Curt. I write an integer greater than zero on the forehead of each of them and I tell them that one of the numbers is the sum of the other two. They take it in turns in alphabetical order to attempt to deduce their own number. The conversation goes as follows.
>
> Alan: "I cannot deduce my number."
> Bert: "I cannot deduce my number."
> Curt: "I cannot deduce my number."
> Alan: "My number is 50."
>
> What are Bert's and Curt's numbers?

David Kerr

Let the numbers be a, b, c with one equal to the sum of the other two and all greater than 0. That A can't deduce a means:

1. $b \neq c$ (else A would be able to deduce that $a = 2b$).

That B can't deduce b means:

2. $a \neq c$ (else B would be able to deduce that $b = 2a$);

3. $a \neq 2c$ (else B would know that $b = c$ or $3c$, but if $b = c$ then A would have been able to deduce that $a = 2b$, hence B would deduce that $b = 3c$).

That C can't deduce c means:

4. $a \neq b$ (else C would deduce that $c = 2a$);

5. $a \neq 2b$ (by similar reasoning to 3, above);

6. $a \neq b/2$ (else C would know that $c = b/2$ or $3b/2$, but if $c = b/2$ then B would have deduced that $b = 2a$, hence C would deduce that $c = 3b/2$;

7. $a \neq 2b/3$ (if so, C would know that $c = b/3$ or $5b/3$, but if $c = b/3$ then B would have seen that $a = 2c$ and been able to deduce that $b = 3c$, hence C would deduce that $c = 5b/3$).

To summarize: we can say that $b \neq c$ and $a \notin \mathcal{S} = \{c, 2c, b, 2b, b/2, 2b/3\}$ otherwise one of A, B or C would have been able to deduce his number.

If A is now able to deduce a, it means that one of $b+c$ or $b-c$ is in the set $\mathcal{S}$. It is easy to see that $b+c \in \mathcal{S}$ implies either $b = c$ or b or c negative. Hence we must have $b - c \in \mathcal{S}$. Consider the various possibilities in turn:

$b - c = c$ implies $b = 2c$ and $a = 3c$, which gives the solution $3x, 2x, x$;

$c - b = c$ implies $b = 0$ and hence no solution;

$b - c = 2c$ implies $b = 3c$ and $a = 4c$, which gives the solution $4x, 3x, x$;

$c - b = 2c$ implies $b = -c$ and hence no solution;

$b - c = b$ implies $c = 0$ and hence no solution;

$c - b = b$ implies $c = 2b$ and $a = 3b$, which gives the solution $3x, x, 2x$;

$b - c = 2b$ implies $c = -b$ and hence no solution;

$c - b = 2b$ implies $c = 3b$ and $a = 4b$, which gives the solution $4x, x, 3x$;

$b-c = b/2$ implies $b = 2c$ and $a = 3c$, which gives the solution $4x, x, 3x$;

$c - b = b/2$ implies $c = 3b/2$ and $a = 5b/2$, which gives the solution $5x, 2x, 3x$;

$b-c = 2b/3$ implies $b = 3c$ and $a = 4c$, which gives the solution $4x, 3x, x$;

$c - b = 2b/3$ implies $c = 5b/3$ and $a = 8b/3$, which gives the solution $8x, 3x, 5x$.

We know that $a = 50$ and the only solution that gives integer answers is $5x, 2x, 3x$, which gives $b = 20$ and $c = 30$.

ADF — If you alter the wording of the problem slightly,

> ‘I have three friends, Alan, Bert and Curt. I write a *different* integer greater than zero on the forehead ...’,

the situation changes significantly. There are now two answers, $50, 10, 40$ (David Porter, M500 **192**, p. 14) and $50, 40, 10$ (Geoff Corris, M500 **194**, p. 10).

Think of two words which have opposite meanings, such that if you add the same letter to the front of each one you make two new words which also have opposite meanings. **Jeremy Humphries**

Solution 195.3 – Doublings

A positive integer N has the property that the number of digits in $2^i N$ is given by the sequence (2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6) for $i = 0, 1, \ldots, 11$. What is N?

Basil Thompson

To satisfy the case for $i = 11$, i.e. 6 digits, the number at this point must lie in the interval $[100000, 999999]$. Using this and working backwards, we repeatedly divide by 2 to obtain the following table.

i		digits		digits
11	100000	6	999999	6
10	50000	5	499999	6
9	25000	5	249999	6
8	12500	5	124999	6
7	6250	4	62499	5
6	3125	4	31249	5
5	1563	4	15624	5
4	782	3	7812	4
3	391	3	3906	4
2	196	3	1953	4
1	98	2	976	3
0	49	2	488	3

Thus $N = 49$ satisfies the original sequence; but is it unique? If $N < 49$, the final term of the sequence is at most 5, and if $N > 49$, the second term is at least 3.

Is it possible to find a method for forecasting the number of digits for any value of N? For small N it is easy to see where these sequences become unique.

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14
i	3	3	5	5	24	24	7	50	50	63	63	23	56	56
digits	1	2	2	3	8	9	3	16	17	20	21	9	18	19

ADF

It is tempting to extend this last table. Let $f(n)$ denote the smallest i such that the sequence $(n, 2n, \ldots, 2^i n)$ differs from $(m, 2m, \ldots, 2^i m)$ for all $m \neq n$.

n	$f(n)$	n	$f(n)$	n	$f(n)$	n	$f(n)$	n	$f(n)$	n	$f(n)$	n	$f(n)$
1	3	31	15	61	117	91	133	121	209	151	212	181	431
2	3	32	88	62	14	92	226	122	209	152	212	182	328
3	5	33	88	63	97	93	226	123	106	153	305	183	328
4	5	34	68	64	97	94	123	124	106	154	305	184	421
5	24	35	58	65	180	95	123	125	292	155	202	185	421
6	24	36	48	66	180	96	113	126	292	156	202	186	225
7	7	37	38	67	77	97	113	127	189	157	99	187	318
8	50	38	28	68	160	98	103	128	282	158	388	188	318
9	50	39	18	69	160	99	103	129	282	159	388	189	215
10	63	40	91	70	150	100	93	130	179	160	285	190	215
11	63	41	91	71	150	101	186	131	272	161	285	191	308
12	23	42	81	72	47	102	279	132	272	162	378	192	308
13	56	43	71	73	130	103	279	133	169	163	378	193	401
14	56	44	61	74	130	104	176	134	262	164	275	194	401
15	26	45	51	75	120	105	176	135	262	165	275	195	205
16	69	46	41	76	120	106	166	136	159	166	368	196	298
17	69	47	31	77	110	107	259	137	448	167	368	197	298
18	49	48	21	78	110	108	259	138	448	168	265	198	391
19	29	49	11	79	193	109	156	139	345	169	265	199	391
20	82	50	94	80	193	110	249	140	345	170	358	200	484
21	82	51	94	81	183	111	249	141	242	171	358	201	484
22	62	52	84	82	183	112	146	142	242	172	255	202	577
23	42	53	167	83	173	113	146	143	139	173	255	203	577
24	22	54	167	84	173	114	136	144	232	174	348	204	670
25	85	55	64	85	163	115	229	145	232	175	348	205	670
26	85	56	54	86	163	116	229	146	129	176	245	206	278
27	75	57	137	87	153	117	126	147	418	177	245	207	371
28	55	58	137	88	153	118	219	148	418	178	338	208	371
29	45	59	34	89	143	119	219	149	222	179	338	209	464
30	25	60	117	90	143	120	116	150	119	180	431	210	464

An observation that might be worth investigating: a local maximum of $f(n)$ occurs for a pair of consecutive values of n.

Fibonacci and all that

Ron Potkin

Here are some 'what comes next?' sequences:

(a) 1, 1, 2, 3, 5, 8, 13, 21, ?

(b) 1, 1, 1, 3, 5, 9, 17, 31, ?

(c) 1, 1, 3, 7, 17, 41, ?

(d) 1, 1, 1, 6, 21, 76, 276, ?

Answers:

(a) 34. No points for guessing this; it is the Fibonacci sequence. It begins with the numbers 1, 1 and subsequent numbers are the sum of the preceding two. In other words, $S_n = S_{n-2} + S_{n-1}$.

(b) 57. This is usually called the *tribonacci* sequence. It is the sum of the preceding three numbers; $S_n = S_{n-3} + S_{n-2} + S_{n-1}$.

(c) 99. This starts with 1,1 but now we add twice the last number plus the number before that; $S_n = S_{n-2} + 2S_{n-1}$.

(d) 1001. Starts with 1,1,1 and sums three times the last number plus twice the prior number plus the number before that; $S_n = S_{n-3} + 2S_{n-2} + 3S_{n-1}$.

The Fibonacci sequence is well known. It has the property that the value of S_n divided by S_{n-1} approaches the irrational number $\phi = 1.618033\ldots$. The numbers ϕ and $1-\phi$ are the roots of the quadratic equation $x^2-x-1 = 0$, which can be rearranged in a slightly different form as $1 + x = x^2$. The sequence is usually given by $S_{n-2} + S_{n-1} = S_n$ but it could equally be expressed as $aS_{n-2} + bS_{n-1} = S_n$, where $a = 1$ and $b = 1$.

By introducing variables a and b, we can generate many more Fibonacci-like sequences. For example, if $a = 1$ and $b = 2$ then we have $S_{n-2}+2S_{n-1} = S_n$, giving the sequence 1, 1, 3, 7, 17, 41, …, and S_n divided by S_{n-1} approaches $2.41421\ldots$. This is one root of the quadratic $1 + 2x = x^2$.

And, if $a = 2$ and $b = 2$ then $2S_{n-2}+2S_{n-1} = S_n$, giving 1, 1, 4, 10, 28, 76, …, and the ratio approaches $2.732\ldots$, which is a root of $x^2-2x-2 = 0$.

So it appears that there may be a relationship between the sequence and the associated quadratic which can be expressed in terms of a and b; i.e. $a+bx = x^2$. The rule is not 'add' but 'multiply and add.' Can this rule be extended to all such equations; i.e. $a + bx + cx^2 + dx^3 + \cdots = x^n$? (The coefficient of x^n must be 1.)

I checked out a few reference books and discovered that some start the Fibonacci sequence with 0, 1 while others start 1, 1 and relate it to the rabbit population. Personally, I find the former more satisfying knowing that the sequences were created from a single seed, although I don't think the rabbits would agree. The Fibonacci sequence will be the same whichever is used, but other sequences will give different results. For example, the tribonacci sequence will vary depending on whether it starts with 0, 0, 1 or 1, 1, 1.

The 0, 1 sequence will be used in the following text. We also define a sequence by $S_{m,n}$, where m is the order of the sequence (Fibonacci is of order 2, tribonacci is of order 3) and n is the index of an item in the sequence.

The quadratic equation $x^2 - (p+q)x + pq = 0$ has the real roots p and q which, rearranged, is $-pq + (p+q)x = x^2$. The equivalent sequence is $-pqS_{2,n-2} + (p+q)S_{2,n-1} = S_{2,n}$, as follows:

$$\begin{aligned}
S_{2,0} &= 0,\\
S_{2,1} &= 1,\\
S_{2,2} &= -p \cdot q \cdot 0 + (p+q) \cdot 1 \ = \ p+q,\\
S_{2,3} &= p^2 + pq + q^2,\\
S_{2,4} &= p^3 + p^2q + pq^2 + q^3,\\
&\dots,\\
S_{2,n} &= p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + p^2q^{n-3} + pq^{n-2} + q^{n-1},\\
S_{2,n+1} &= p^n + p^{n-1}q + p^{n-2}q^2 + \cdots + p^2q^{n-2} + pq^{n-1} + q^n.
\end{aligned}$$

Remarkably, no term contains a coefficient, otherwise it is similar to a binomial expression and its symmetry enables us to sum $S_{2,n}$ in two ways: $S_{2,n+1} - pS_{2,n} = q^n$ and $S_{2,n+1} - qS_{2,n} = p^n$, so that, provided $|p| \neq |q|$, we have $S_{2,n} = (p^n - q^n)(p - q)$. It follows that

$$\frac{S_{2,n+1}}{S_{2,n}} = \frac{p^{n+1} - q^{n+1}}{p^n - q^n}$$

and so, as n approaches infinity, the expression approaches the greater of $|p|$ and $|q|$.

The higher orders are not so easy. The equation $pqr - (pq + pr + qr)x + (p+q+r)x^2 = x^3$ has the real roots p, q and r. The equivalent sequence is

$$pqrS_{3,n-3} - (pq + pr + qr)S_{3,n-2} + (p+q+r)S_{3,n-1} = S_{3,n}$$

of which the first few lines are

$$\begin{aligned}
S_{3,0} &= 0,\\
S_{3,1} &= 0,\\
S_{3,2} &= 1,
\end{aligned}$$

$$\begin{aligned}
S_{3,3} &= p+q+r,\\
S_{3,4} &= p^2+pq+q^2+qr+r^2+pr,\\
S_{3,5} &= p^3+p^2q+pq^2+q^3+q^2r+qr^2+r^3+p^2r+pr^2+pqr.
\end{aligned}$$

This time, when we try to sum $S_{3,4}$, we obtain $S_{3,5}-pS_{3,4} = q^3+q^2r+qr^2+r^3$. But this is equal to $S_{2,4}$! Is there a relationship between the tribonacci and the Fibonacci sequences?

In the general case, $S_{m,n} = \sum x_1^{n_1}x_2^{n_2}\dots x_m^{n_m}$ (taken over all non-negative integers $n_1, n_2, \dots, n_m$ such that $n_1+n_2+\cdots+n_m = n-m+1$).

There is a strong relationship between $S_{m,n} - x_iS_{m,n-1} = S_{m-1,n-1}$ and the identity ${}^{p+1}C_{k+1} = {}^{p}C_{k+1} + {}^{p}C_k$ which is the basis for Pascal's triangle. The number of terms in $S_{m,n}$ is equal to ${}^{n}C_{m-1}$ and the number of terms containing each variable in $S_{m,n}$ is equal to the number of terms in $S_{m,n-1}$.

If we multiply $S_{m,n-1}$ by x_i and deduct this from $S_{m,n}$, we remove all terms containing x_i; thus

$$S_{m,n} - x_iS_{m,n-1} = \sum x_1^{n_1}\dots x_{m-1}^{n_{m-1}} = S_{m-1,n-1}$$

(the order of the sequence is reduced by one), and so

$$\frac{S_{m,n}}{S_{m,n-1}} = x_i + \frac{S_{m-1,n-1}}{S_{m,n-1}}.$$

If $|x_i|$ is the greatest root then, since it has been removed from $S_{m-1,n-1}$, the expression will approach x_i as n approaches infinity.

Solving equations using this method, however, is not recommended. First, it will only return one root. Secondly, it may take several iterations to reach an accurate answer; the closer the roots, the longer it will take. Finally, and more importantly, it cannot solve for complex roots.

Imagine a string containing an infinite number of cells each containing zero with a single 1 in the middle of the string at position x. You require two real variables, a and b. A read/write head is placed at position $x-1$. It multiplies the number at this position in the string by a, then moves forward one cell and multiplies the new number by b, adding the two products together. Now you may continue in one of two ways:

Method F writes the sum on the string at $x+1$ and repeats the operation, this time adding the number at x times a to the number at $x+1$ times b and writing the sum at $x+2$, and so on.

Method P writes the sum at x on a second string of zeros placed under the first one and starts again. This time it adds the number at x times a to the number at $x+1$ times b and writes the sum at $x+1$ in the second string and so on. It will repeat the loop until the sum of the products is zero, when it jumps down to the second string at position $x-1$ and continues the operation.

If $a = 1$ and $b = 1$, what are the results of method F and method P? They are as follows.

$$
\begin{array}{ll}
\text{Method } F\text{:} & 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots \\
\text{Method } P\text{:} & 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0 \\
 & 0,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0 \\
 & 0,\ 1,\ 2,\ 1,\ 0,\ 0,\ 0 \\
 & 0,\ 1,\ 3,\ 3,\ 1,\ 0,\ 0 \\
 & 0,\ 1,\ 4,\ 6,\ 4,\ 1,\ 0
\end{array}
$$

Method F will give us the Fibonacci sequence. Method P gives the Pascal triangle although in this case it will be a right-angled triangle rather than the more familiar isosceles.

Of course, we are not limited to two variables; we could start with three variables and place the read/write head at $x-2$, or n variables and start at $x-n+1$.

With such a strong similarity in the two methods, you will not be surprised to learn that there is a strong relationship between them; best described by a slight rearrangement of the results above.

$$
\begin{array}{lll}
 & n & \quad 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots \\
\text{Method } F\text{:} & & 0,\ 0,\ 1,\ 1,\ 2,\ 3,\ \mathbf{5},\ 8,\ 13,\ 21,\ \ldots \\
\text{Method } P\text{:} & 0 & 0,\ 1,\ 0,\ 0,\ 0,\ \mathbf{0},\ 0 \\
 & 1. & 0,\ 1,\ 1,\ 0,\ \mathbf{0},\ 0,\ 0 \\
 & 2. & 0,\ 1,\ 2,\ \mathbf{1},\ 0,\ 0,\ 0 \\
 & 3. & 0,\ 1,\ \mathbf{3},\ 3,\ 1,\ 0,\ 0 \\
 & 4. & 0,\ \mathbf{1},\ 4,\ 6,\ 4,\ 1,\ 0 \\
 & 5. & 0,\ 1,\ \ldots
\end{array}
$$

The Fibonacci numbers are derived by summing the diagonals. For example, $5 = 1+3+1$ and $8 = 3+4+1$, or

$$F_n = \sum_{i \le n/2} \binom{n-i}{i} a^i b^{n-2i}, \qquad a > 0, b > 0.$$

Solution 196.4 – Snub cube

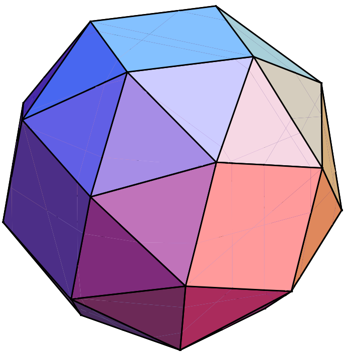
Compute the following parameters for the snub cube:

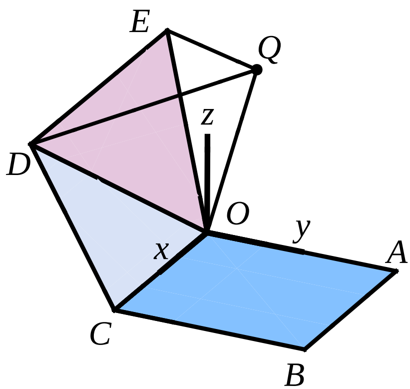
δ_1: the square–triangle dihedral angle,
δ_2: the triangle–triangle dihedral angle,
d_s: the distance between opposite squares,

and α, the angle through which the square faces have been turned from their cube orientation.

Dick Boardman

Imagine a snub cube placed on a plane with a square face downward. Label the bottom left corner of the square O, the origin, and label the vertices clockwise $OABC$ so that OC is the x-axis and OA the y-axis. The z-axis is vertically upward. The edge OC is the intersection of the square and an equilateral triangle. Call the third vertex of the triangle D. There is a second equilateral triangle touching OD. Call its third vertex E. Choose the lengths of all edges to be 1. Consider the point D. Its coordinates are $(\frac{1}{2}, -h\cos\theta_1, h\sin\theta_1)$, where $h = \sin 60° = \sqrt{3}/2$. The angle θ_1 is $180° - \delta_1$.





Consider the point E. By symmetry, its x and y coordinates will be equal. Let OE make an angle θ_2 with the plane. Then the coordinates of E will be $(-w\cos\theta_2, -w\cos\theta_2, \sin\theta_2)$, where $w = 1/\sqrt{2}$. The angle between OD and OE is 60 degrees, so that

$$OD \cdot OE \;=\; \cos 60° \;=\; \tfrac{1}{2}. \tag{1}$$

The bounding sphere of the snub cube will go through all of O, A, B,

C, D and E. The coordinates of its centre will be $Q = (\frac{1}{2}, \frac{1}{2}, r)$, where r is to be determined. (Note that r is not the radius.) Thus

$$(OE - OQ) \cdot (OE - OQ) \;=\; OQ \cdot OQ, \tag{2}$$

$$(OD - OQ) \cdot (OD - OQ) \;=\; OQ \cdot OQ. \tag{3}$$

Equations (2) and (3) make all the radii the same.

Equations (1)–(3) may be solved numerically to yield $\theta_1 \approx 37.0166°$, $\theta_2 \approx 53.5967°$ and $r \approx 1.14261$. For an exact solution, it seems best to start by solving (2) and (3) for θ_1 and θ_2 in terms of r. Thus

$$\theta_1 \;=\; \arccos\left(\frac{2r\sqrt{12r^2+2}-1}{\sqrt{3}\,(4r^2+1)}\right), \tag{4}$$

$$\theta_2 \;=\; \arccos\left(\frac{2r\sqrt{4r^2+1}-\sqrt{2}}{4r^2+2}\right).$$

Then, substituting θ_1 and θ_2 in (1), we obtain

$$\begin{aligned} &\left(1-\sqrt{2}\,r\sqrt{1+4\,r^2}\right)\left(1+2\,r^2-r\sqrt{2+12\,r^2}\right) \\ &\qquad + \;\sqrt{1+6\,r^2+2\,r\sqrt{2+8\,r^2}}\;\sqrt{1+8\,r^2+2\,r\sqrt{2+12\,r^2}} \\ &= \;\left(1+2\,r^2\right)\left(1+4\,r^2\right). \end{aligned}$$

This is a complicated equation involving square roots of square roots. However, we can transform it to a polynomial by repeatedly rearranging so that the equation takes the form $\mathcal{A} - \sqrt{\mathcal{B}} = 0$ and squaring to get $\mathcal{A}^2 - \mathcal{B} = 0$. The final result is that r satisfies

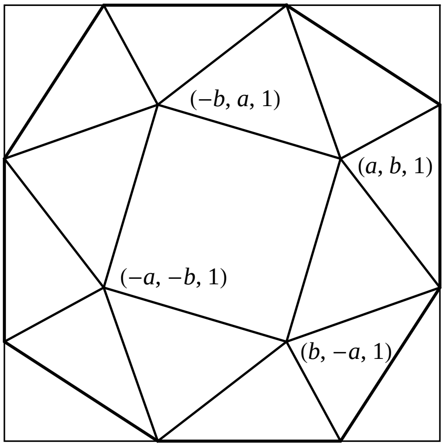
$$\begin{aligned} &16384\,r^{16} + 73728\,r^{14} - 109568\,r^{12} - 25600\,r^{10} \\ &\qquad + 7680\,r^8 + 1216\,r^6 - 160\,r^4 - 16\,r^2 + 1 \;=\; 0. \end{aligned}$$

Amongst the 16 solutions is the only one that is relevant to our problem:

$$\begin{aligned} r &= \sqrt{\frac{\sqrt[3]{199-3\sqrt{33}}}{12} + \frac{\sqrt[3]{199+3\sqrt{33}}}{12} + \frac{1}{3}} \\ &= 1.1426135089259620934794 84\ldots. \end{aligned}$$

Substituting r in (4) gives a complicated expression for θ_1 and hence for $\delta_1 = 180° - \theta_1$. And, of course, $d_s = 2r$. I leave δ_2 and α for someone else.

John Smith



By encasing a snub cube in a cube of side 2, with vertices at $(\pm 1, \pm 1, \pm 1)$ we can write down the vertices of the snub cube in the form $(a, b, 1)$, $(-b, a, 1)$, $(-a, -b, 1)$, $(b, -a, 1)$ and similar expressions on the other five faces. If we have $a > b$, then some algebra shows that b satisfies the cubic

$$b^3 + b^2 + 3b - 1 \;=\; 0$$

and $a = \sqrt{b}$. Formulae exist for solving cubics, so a dash to the M433 course book (Ian Stewart, *Galois Theory*) provides the result:

$$b \;=\; \frac{\sqrt[3]{6\sqrt{33} + 26} - \sqrt[3]{6\sqrt{33} - 26} - 1}{3}$$

and $a = \sqrt{b}$ or, numerically,

$$a \;=\; 0.54368901269207, \quad b \;=\; 0.29559774252208.$$

In principle it is then straightforward to generate expressions for the normals to the faces and the dihedral angles. In practice, my manipulations have little hope of simplifying the expressions to something reasonable. On the other hand, it is easily seen that the ratio of snub cube edge length to cube edge length is

$$\frac{1}{d_s} \;=\; \frac{\sqrt{(a+b)^2 + (a-b)^2}}{2} \;=\; \frac{\sqrt{b^2 + b}}{\sqrt{2}}.$$

Furthermore, $\tan(\alpha)$ is given by $(a - b)/(a + b)$ or, by using $a = \sqrt{b}$,

$$\alpha \;=\; \arctan \frac{1 - a}{1 + a} \;=\; \arctan b \;\approx\; 16.46756040038636^\circ.$$

Perhaps there is a simpler worthwhile question: if we label the triangular faces as either octahedral faces or snub faces, then why is the dihedral angle between two snub faces the same as the dihedral angle between a snub face and an octahedral face? There may be a simple symmetry argument, but I cannot see it. However, it is sort of responsible for the $a = \sqrt{b}$ result above.

Solution 197.5 – Toilet paper

Find a formula that relates the radius of a toilet roll (R), the total length (L), the paper thickness (t) and the radius of the cylindrical cardboard thing at the centre (r).

Simon Geard

If we consider the toilet roll to be an Archimedean spiral then in polar coordinates (s, θ)

$$\Theta \;=\; \frac{2\pi}{t}\,(s-r)\,.$$

The length of the paper can then be expressed by the integral

$$L \;=\; \int_r^R \sqrt{1+s^2\left(\frac{2\pi}{t}\right)^2}\,ds \;=\; \left[\frac{s}{2}\sqrt{1+\left(\frac{s}{a}\right)^2}+\frac{a}{2}\sinh^{-1}\frac{s}{a}\right]_r^R,$$

where $a = t/2\pi$.

For toilet paper the sheet thickness is very small compared with the roll thickness; so the messiness of the above can be simplified:

$$L \;=\; \frac{\pi}{t}\left(R^2-r^2\right)+\frac{t}{4\pi}\log\frac{R}{r}.$$

According to the packet there are ~ 240 sheets per roll, and each sheet is 125 mm long; so $L \approx 30000$ mm. I measured $r = 20$ mm and $R = 60$ mm; so

$$30000 \;=\; \frac{3200\pi}{t}+\frac{\log 3}{4\pi}t,$$

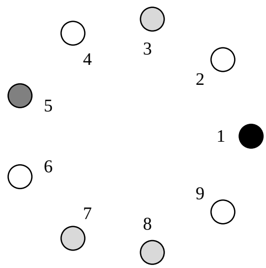
giving $t \approx 0.335$ mm, which seems to be about the right value.

Years ago I did a related calculation which had a practical application that readers might like to try. I wanted to know how much recording time I had left on a tape, given the value displayed by the tape counter. The counter is connected to the left hand tape spool and is set to zero at the beginning of the tape. Since the tape is transported at a constant velocity v the counter's speed increases as the tape is used up. The tape manufacturers supplied the length of the tape and the above calculation was used to estimate the thickness—the radii can be measured sufficiently accurately with a ruler. Also the value of the counter when the tape has been used up, N, is easily determined. So the question is: given a value n on the tape counter, what is the time remaining τ as a function of L, N, R, r, t and v?

Stuart Cresswell and **Ralph Hancock** found the minus-oneth order approximation to L by dividing the cross-section area, $\pi(R^2 - r^2)$, by t. Curiously, there is no zeroth order term. — **ADF**

Problem 200.1 – Well spaced

There are n slots, numbered 1 to n, arranged in a circle. They are to be occupied by n objects, one by one, such that at all times the objects are as well spaced as possible. The first object can go anywhere. Thereafter, when an object is added to the system it must be placed such that the minimum distance to its two neighbours is as large as possible. In how many ways can this be achieved?

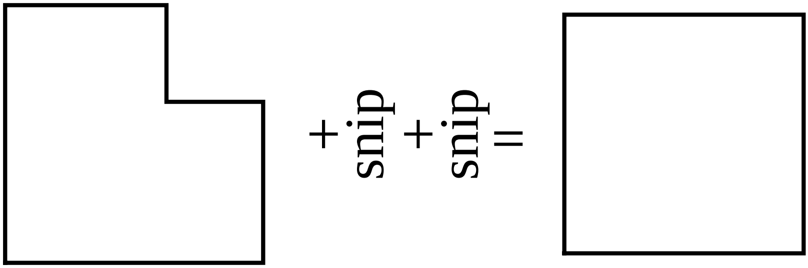


By rotational symmetry we can always choose slot 1 first and then multiply the answer by n. With this observation in mind, let us look in detail at a small but non-trivial case, $n = 9$. We start with 1. The next choice is between 5 and 6, each of which is at least 4 away from 1. After 15, we can choose between 3, 7 and 8, as illustrated on the right. We now have 153, 157 and 158. Then 153 extends to 1537 or 1538, 157 to 1573, and 158 to 1583. Similarly we also have 1638, 1648, 1683 and 1684. Thereafter, all permutations of the remaining five numbers are valid. Thus, after including the rotations, the answer is $8 \cdot 5! \cdot 9 = 8640$.

Can you find a general formula?

Problem 200.2 – Square with corner missing

Take two integers, $0 < m < n$. Take an $n \times n$ square of suitable sheet material. Cut out an $m \times m$ square from a corner. Then make two straight-line cuts and rearrange the pieces to make a perfect square. For what values of m and n is this possible?



Conversion factors

ADF

As a follow-up to the list of conversion factors in M500 **198**, I would like to offer some novel and interesting ideas for converting time to temperature.

You may be aware of John Cage's *4′33″*, four minutes and thirty-three seconds of musical silence. Whilst I am happy to leave it for the experts to judge the work, I was amused to learn that *4′33″* was rather neatly discredited by the composer himself! 'Why *4′33″*?' Cage was asked. Apparently his answer was that 4:33 = 273 seconds, and minus 273 degrees Celsius is the temperature of absolute zero. So any deep significance there might have been in the title amounts to nothing more than a meaningless numerical coincidence.

Thermodynamically speaking, the number 273 is significant only because water happens to freeze at about 273°K. *It has nothing to do with absolute zero.* Nevertheless, if you really want to construct a scientifically respectable time interval corresponding to the freezing point of water, surely a good place to start is Boltzmann's constant k, which has units energy/temperature. This is not quite what we want, but if we throw in c, the velocity of light and G, the gravitational constant, we can obtain a suitable factor for converting temperature to time, namely

$$\Gamma \;=\; G\,k/c^5.$$

Using the values $k = 1.38062 \cdot 10^{-23}$ J/°K, $c = 299792459$ m/s and $G = 6.673 \cdot 10^{-11}$ N m^2/kg^2, we have

$$\Gamma \;=\; 3.804 \cdot 10^{-76}\ \text{s/°K}.$$

Therefore 273°K becomes $1.038 \cdot 10^{-73}$ seconds. Of course, adopting this suggestion would have produced a considerably shorter composition.

Similar crackpot numerology appears to have been the inspiration for the title of Michael Moore's film, *Fahrenheit 9/11: The temperature where freedom burns.* The idea, I think, is that the date of the terrorist attack against the USA is transformed as follows.

> September 11th 2001 → 11/9/2001 → 9-11-2001 → 9-11 → nine eleven → nine hundred and eleven → 911 → 911°F.

Clearly this extends to a general method for converting dates to temperatures. On the same scale, New Year's Day works out at a chilly $-11\frac{2}{3}$°C.

According to Eric Blake of the National Hurricane Centre in Miami, sea temperatures are now 5°C (41°F) higher than recent averages and these have been directly responsible for spawning Hurricanes Ivan, Frances and Charley this summer. — *Observer*, 12 September 2004. [Spotted by **John Bull**]

What prime is even?

Tony Forbes

Jeremy Humphries informs me that the following was once a £32,000 question on the television quiz show *Who Wants to be a Millionaire?*.

What is the only even prime number? A: 2, B: 4, C: 10, D: 12.

The contestant didn't have a clue, so he asked the audience, who voted:

A: 62%, B: 10%, C: 18%, D: 10%.

Of course, this proves nothing—except that in a particular group of people 38 per cent (plus perhaps as many as 10 per cent who accidentally guessed A) are unconcerned with that obscure branch of pure mathematics which deals with the classification of numbers into various types. I imagine a similar frequency distribution would occur if B, C and D were odd.

Anyway, in case you happen to be a contestant and a similar question comes up again, here are some definitions of $\mathbb{P}$, the set of primes. Choose whichever one you feel comfortable with.

(i) $\mathbb{P}$ is the set of integers $p > 1$ for which p is not divisible by any positive integer other than 1 and p.

(ii) $\mathbb{P}$ is the set of positive integers with precisely two positive integer divisors.

(iii) $\mathbb{P}$ is a multiplicative basis for the positive integers. Furthermore, $\mathbb{P}$ is the only set with this property. Thus every positive integer has a unique representation as a product of elements of $\mathbb{P}$. This includes 1, which is the empty product:

$$1 = \prod_{p \in \{\}} p.$$

(iv) $\mathbb{P}$ is the unique set of positive integers for which

$$\prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1} = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for all complex values of s with $\Re s > 1$.

To see how it works, we use this last definition to prove the question setter's assertion. Put $s = 2$. Then we have

$$\prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^2}\right)^{-1} = \zeta(2) = \frac{\pi^2}{6}. \tag{1}$$

Clearly 1 cannot be in $\mathbb{P}$, for otherwise the left-hand side of (1) would have a zero in the denominator. Suppose $2 \notin \mathbb{P}$. Then

$$\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p^2}\right)^{-1} \le \prod_{n=3}^{\infty}\left(1-\frac{1}{n^2}\right)^{-1} = \frac{3}{2} < \frac{\pi^2}{6}.$$

So without including the factor for $p = 2$ the product in (1) can never achieve its true value, $\pi^2/6 \approx 1.644934$. Hence 2 must be included in $\mathbb{P}$, and therefore 2 is prime.

Equation (1) also provides a very simple proof that there are infinitely many primes. Since π^2 is irrational, the product on the left of (1) has more than a finite number of factors.

Problem 200.3 – An arithmetic geometric mean

Traditionally the *arithmetic–geometric mean* of a pair of numbers $\{a, b\}$ is the common limit of the process $\{a, b\} \to \{(a + b)/2, \sqrt{ab}\}$. Here we adopt a slightly skewed definition.

Let $a_1 = (a + b)/2$, $b_1 = \sqrt{a_1 b}$, and for $n > 1$ let $a_n = (a_{n-1} + b_{n-1})/2$, $b_n = \sqrt{a_n b_{n-1}}$. Show that

$$a_\infty = b_\infty = \frac{\sqrt{b^2 - a^2}}{\arccos a/b}.$$

Problem 200.4 – Circle in a box

What is the locus of the centre of a unit-radius circle placed such that the circumference is in contact with the positive (x, y)-plane, the positive (x, z)-plane and the positive (y, z)-plane? (I'm sorry, this is too difficult to draw convincingly. To get a better understanding of the problem, drop a 2p piece into the corner of a box and move it about whilst ensuring that it remains touching the three sides of the box which meet there.)

Problem 200.5 – Bouncing ball

There are two fixed spheres of radius 1; sphere A is at position $(-10, 0, 0)$ and B is at $(10, 0, 0)$. A third ball, C, also of radius 1, bounces back and forth with perfect elasticity along the x-axis between A and B. Then, just as it is bouncing off A, the trajectory of C changes by 10^{-100} radians. How many further bounces does C experience before it leaves the system?

Question. Where do no badgers live? *Answer.* The empty set.

Doctor Dave

Eddie Kent

Earlier this year David Bradley retired from IBM after 28 years toiling in the bowels of the PC. Always known as 'Dr Dave' on the IBM campus in North Carolina, he was one of the original 12 creators of the personal computer.

Now 55, he was in the team that delivered a prototype of the IBM PC to a small company in December 1980. That company, with fewer than 40 employees, was called Microsoft.

The rudimentary PC was a crude bundle of cables wired by hand in a rack board. It was so secret that Bradley had to smuggle it into a back room.

Why am I bothering you with this? Well. For one thing Bradley was once a clue in the television quiz show *Jeopardy*. He said, 'If I can be a clue in the *New York Times* crossword puzzle I will have met all my life's goals.'

One other thing. He was the man who once spent five minutes writing a code to restart a recalcitrant computer. This was intended for engineers, to save them having to wait through a reboot. But somehow the news leaked out, and Ctrl-Alt-Del is now universally recognized as the command of last resource. He was looking for keys that are far apart and originally contemplated Ctrl-Alt-+, but thought Del made more sense.

On the 20th anniversary of the IBM PC he appeared on a panel with Bill Gates and said, referring to the well-known bug-infested nature of the PC, 'I may have invented it but Bill made it famous.' On another occasion he said, 'I didn't know it was going to be a cultural icon. I did a lot of other things than Ctrl-Alt-Del but I'm famous for that.'

He intends to lecture at North Carolina State University. For more on Bradley, visit http://www.fastcompany.com/articles/archive/pc_bday.html.

A golden rule for computer experts: The customer is always a block-head.

Two computer experts will never agree on anything except that the other computer expert is an arsehole.

How does a computer expert deal with a flat tyre? He gets the spare wheel out and tries replacing each wheel in turn with it until the car drives normally.

How to trash all your computer data: Lend your computer to a computer expert.

Solution 198.1 – Two knights

What is the probability that two knights attack each other on an $n \times n$ board?

Ian Bruce Adamson

Let $m(r)$ be number of cells attacked from the rth cell and let $f(q)$ be the number of cells attacking q squares. We have at once

$$\sum_{r=1}^{n^2} m(r) \;=\; \sum_{q=2}^{8} q f(q),$$

where $\sum f(q) = n^2$.

An arbitrarily placed knight attacks $m(r)$ cells out of $n^2 - 1$ and its probability of being placed there is $1/n^2$. Thus the required probability is $\sum q f(q)/(n^2(n^2-1))$, where $f(q)$ and $qf(q)$ are computed according to the following table.

q	2	3	4	6	8
$f(q)$	4	8	$4(n-3)$	$4(n-4)$	$(n-4)^2$
$qf(q)$	8	24	$16(n-3)$	$24(n-4)$	$8(n-4)^2$

(Note that $\sum f(q) = n^2$, as we would hope.) Thus

$$\sum q f(q) \;=\; 16 - 24n + 8n^2 \;=\; 8(n-1)(n-2).$$

So the required probability is

$$\frac{8(n-1)(n-2)}{n^2(n^2-1)} \;=\; \frac{8(n-2)}{n^2(n+1)},$$

the limit of which as n tends to infinity being $8/n^2$.

We have considered only a board where $n > 4$. If $n = 1$ or 2, the probability is zero; if $n = 3$ or 4, it can be easily calculated.

Anne Robinson – What ‘A’ is the term for the arithmetic mean of a set of numbers?

Contestant – Algebra. [*The Weakest Link*, BBC 2]

Letters to the Editor

Paradoxical dice

Consider David Singmaster's paradoxical dice problem in M500 **199**, where we throw a k-sided die ($k \geq 2$) until a particular pair comes up in order on consecutive throws and we ask ourselves why it is counterintuitive that not all pairs have the same expected time to appear.

There is a simple example which shows what is going on. Imagine a 3-sided die (!), i.e. let $k = 3$ and all throws give 1, 2 or 3. The probability of getting $(1, 2)$ in two throws is 1/9, and similarly for $(2, 2)$. However the probability of the bet ending in exactly 3 throws is different.

Case 1: Probability of getting $(x, 1, 2)$. There are 27 possible results for three throws. Three of the results $(1, 2, x)$ end after two throws, so delete them. There are three results $(x, 1, 2)$, so the probability of getting $(x, 1, 2)$ is 3/24.

Case 2: Probability of getting $(x, 2, 2)$. Again, there are 27 possible results. Three of the results, including $(2, 2, 2)$ end after 2 throws, so delete them. Now there are only two ways of getting $(x, 2, 2)$, since $(2, 2, 2)$ has been deleted, so the probability of getting $(x, 2, 2)$ is only 2/24.

This discrepancy carries on with the probability of $(x, x, \ldots, 2, 2)$ always being less than that of $(x, x, \ldots, 1, 2)$.

Dick Boardman

Re: M500 198

I was puzzled by the What's missing? problem in the same way as Keith Drever, after I had earnestly examined the numbers 339–343 for primeness. But I would never have suspected Tony of having put in such a trivial trick. Now all is revealed, but it is a rather sad glimpse into the tatty undergrowth of number theory, like seeing the Gap in the Matrix in that silly film.

Tony's conversion factors had a macabre interest, and henceforward I shall certainly convert miles into feet with $e^{\sqrt{67}\pi/3}$, rather than boringly multiplying by 5280. This reminds me of the furlong-firkin-fortnight system of measurements (instead of metre-kilogram-second), an attractive but flawed arrangement as a firkin is a measure of capacity rather than mass. Probably a firkin of ale is intended, but this can be between 8 and 9 imperial gallons, weighing 82 lb to 102 lb 8 oz multiplied by the specific gravity of the ale. I suppose one excludes the barrel. Well, worse things are done in cosmology.

Ralph Hancock

M500 Special Issue

Eddie Kent

There will be no Special Issue this year, or next. The number of contributions is too small to justify a separate mailing, so from now on all comments on courses will be on the Net. They will be anonymous unless the authors give permission for their names to be used.

This publication has been running for some time now. It was an idea dreamed up by the OU Mathematics Faculty; the name Bob Margolis springs to mind. Early labourers included Marion Stubbs and Michael Gregory, and of course I have always been hovering around.

Now it is over. It has been slain before and dragged itself back into consciousness. But there are now too many competing sources of information for the Special Issue to be justified.

In the future your views are still wanted. Contributions will be edited in exactly the way they have always been, but instead of being printed and sent out by our gallant Post Office, they will be lodged at http://www.m500.org.uk/.

M500 Winter Weekend 2005

The twenty-fourth M500 Society Winter Weekend will be held on **Friday 7th to Sunday 9th January 2005** at

Trevelyan College, **Durham University**.

This is an annual residential weekend to dispel the withdrawal symptoms due to courses finishing in October and not starting again until February. It's an excellent opportunity to get together with friends, new and old, and do some interesting mathematics in a leisurely and congenial atmosphere.

Ian Harrison is running the event and this year's theme will be announced on the M500 web site nearer the time. Cost: £170.00 for M500 members. This includes accommodation and all meals from dinner on Friday to lunch on Sunday. Add £30.00 for *en suite* accommodation. For full details and a booking form, send a stamped, addressed envelope to

Norma Rosier.

Enquiries by email to norma@m500.org.uk. A booking form is also available at www.m500.org.uk.

'These are our permanent prices until further notice.' — Takeaway menu.
[EK]

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