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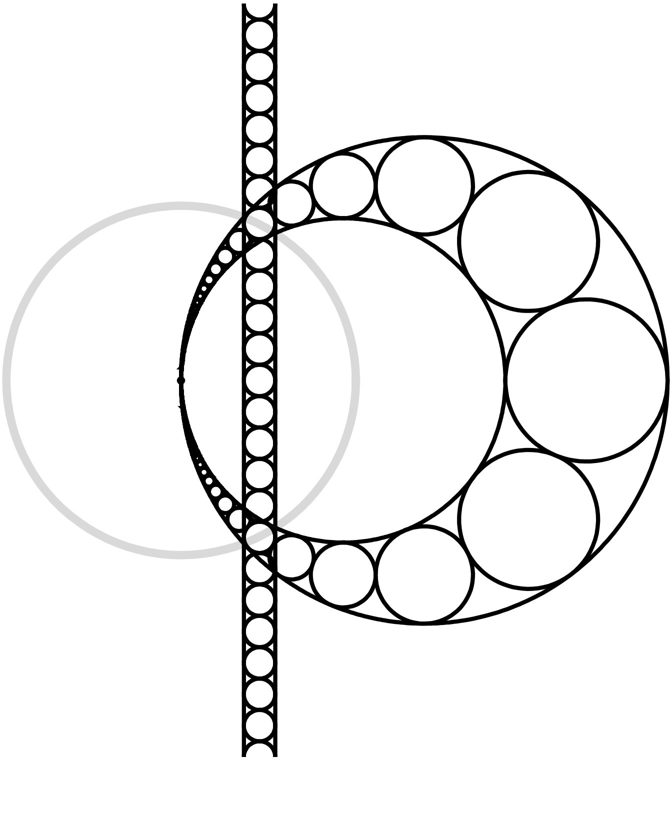

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Report upon Fibonacci numbers and quaternions

Dennis Morris

The real numbers, the complex numbers, the quaternions and the octonions (Cayley numbers), $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{K}$, are the only normed algebras over the real field; $\mathbb{R}$ has order, commutativity and associativity; $\mathbb{C}$ has commutativity and associativity but not order; $\mathbb{H}$ has associativity but neither order nor commutativity; $\mathbb{K}$ does not have order or commutativity or associativity. A complex number is two real numbers, one of which is associated with the square root of minus one. A quaternion number is four real numbers, three of which are associated with three different square roots of minus one. An octonion number is eight real numbers, seven of which are associated with seven different square roots of minus one. One can view a quaternion as being two complex numbers and an octonion as being two quaternion numbers. Quaternions are used for calculating rotations of space ships and appear in quantum mechanics. As far as I know, octonians are used for nothing.

Fibonacci numbers are numbers generated by the iteration $F_{n+1} = F_n + F_{n-1}$. Traditionally, the first two Fibonacci numbers are taken to be $F_1 = F_2 = 1$. The sequence of numbers generated by this iteration with these two starting numbers is thus 1, 1, 2, 3, 5, 8, 13, 21, 34, ... We will generalize the sequence by allowing that F_1 and F_2 can be any real numbers other than zero. Remarkably, for any values of $F_1, F_2 \in \mathbb{R} \setminus \{0\}$, the ratio F_n/F_{n-1} tends to the same limit as $n \to \infty$. That limit is the golden ratio: $\phi = 1.6180339887\ldots$. This much is well known among mathematicians, and it is this aspect of Fibonacci numbers that we are going to consider in the complex, quaternion, and octonion number systems. Quite surprisingly, if $F_1, F_2 \in \mathbb{C} \setminus \{0\}$, then the real part of F_n/F_{n-1} tends to the golden ratio and the imaginary part tends to zero as $n \to \infty$. Taking $F_n = a + ib$ and $F_{n-1} = c + id$, we find

$$\frac{F_n}{F_{n-1}} = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}.$$

With reflection, we see that $\lim_{n\to\infty} \left(\frac{a}{c} + \frac{b}{d}\right) = 2\phi$, and so $\lim_{n\to\infty} \frac{ad + bc}{dc + dc} = \phi$ and, in an operational sense, $c \equiv d$. Thus we might expect that the real part tends to the golden ratio. Similarly, in an operational sense, $a \equiv b$. Thus we might expect that the imaginary part tends to zero.

Quaternions are of the form $a + \hat{\imath}b + \hat{\jmath}c + \hat{k}d$ for $a, b, c, d \in \mathbb{R}$. Similarly to complex numbers, we have the multiplicative identities $\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = -1$. However, quaternions are not commutative. We have the further identities $\hat{\imath}\hat{\jmath} = \hat{k}$, $\hat{\jmath}\hat{k} = \hat{\imath}$, $\hat{k}\hat{\imath} = \hat{\jmath}$, $\hat{\jmath}\hat{\imath} = -\hat{k}$, $\hat{k}\hat{\jmath} = -\hat{\imath}$ and $\hat{\imath}\hat{k} = -\hat{\jmath}$. Addition and multiplication are as one would expect:

$$\begin{aligned}
&a + \hat{\imath}b + \hat{\jmath}c + \hat{k}d + e + \hat{\imath}f + \hat{\jmath}g + \hat{k}h \\
&\qquad = (a+e) + \hat{\imath}(b+f) + \hat{\jmath}(c+g) + \hat{k}(d+h), \\
&(a + \hat{\imath}b + \hat{\jmath}c + \hat{k}d)(e + \hat{\imath}f + \hat{\jmath}g + \hat{k}h) \\
&\qquad = (ae - bf - cg - dh) + \hat{\imath}(af + be + ch - dg) \\
&\qquad\quad + \hat{\jmath}(ag - bh + ce + df) + \hat{k}(ah + bg - cf + de).
\end{aligned}$$

Division is done, as with complex numbers, by use of the conjugate of a quaternion. The conjugate of $a + \hat{\imath}b + \hat{\jmath}c + \hat{k}d$ is $a - \hat{\imath}b - \hat{\jmath}c - \hat{k}d$. The product of a quaternion and its conjugate is $a^2 + b^2 + c^2 + d^2$ regardless of the order of multiplication;

$$\frac{a + \hat{\imath}b + \hat{\jmath}c + \hat{k}d}{e + \hat{\imath}f + \hat{\jmath}g + \hat{k}h} = \begin{cases} \dfrac{(a + \hat{\imath}b + \hat{\jmath}c + \hat{k}d)(e - \hat{\imath}f - \hat{\jmath}g - \hat{k}h)}{e^2 + f^2 + g^2 + h^2} \\ \text{or} \\ \dfrac{(e - \hat{\imath}f - \hat{\jmath}g - \hat{k}h)(a + \hat{\imath}b + \hat{\jmath}c + \hat{k}d)}{e^2 + f^2 + g^2 + h^2}. \end{cases}$$

Taking $F_n = a + \hat{\imath}b + \hat{\jmath}c + \hat{k}d$ and $F_{n-1} = e + \hat{\imath}f + \hat{\jmath}g + \hat{k}h$, we find two expressions for F_n/F_{n-1} from post-multiplying and from pre-multiplying:

$$\begin{aligned}
(1) \qquad \frac{F_n}{F_{n-1}} &= \frac{ae + fb + gc + hd}{e^2 + f^2 + g^2 + h^2} + \hat{\imath}\,\frac{eb - fa + gd - hc}{e^2 + f^2 + g^2 + h^2} \\
&\quad + \hat{\jmath}\,\frac{hb - ga + ec - fd}{e^2 + f^2 + g^2 + h^2} + \hat{k}\,\frac{fc + ed - gb - ha}{e^2 + f^2 + g^2 + h^2}, \\
(2) \qquad \frac{F_n}{F_{n-1}} &= \frac{ae + fb + gc + hd}{e^2 + f^2 + g^2 + h^2} + \hat{\imath}\,\frac{eb - fa - gd + hc}{e^2 + f^2 + g^2 + h^2} \\
&\quad + \hat{\jmath}\,\frac{-hb - ga + ec + fd}{e^2 + f^2 + g^2 + h^2} + \hat{k}\,\frac{-fc + ed + gb - ha}{e^2 + f^2 + g^2 + h^2}.
\end{aligned}$$

The reader might like to reflect upon the symmetry of these answers, in particular the correspondence between $\{a, e\}$, $\{b, f\}$, $\{c, g\}$ and $\{d, h\}$. Considering only the vector part of the quaternion (the i, j and k part), we count the occurrences of positive and negative signs.

	a	e	b	f	c	g	d	h
+ve	0	3	2	1	2	1	2	1
−ve	3	0	1	2	1	2	1	2

Bearing in mind that $\{a, b, c, d\} > \{e, f, g, h\}$ because of the way the sequence is constructed and that there is a correspondence between $\{a, e\}$, $\{b, f\}$, $\{c, g\}$ and $\{d, h\}$, we see that the $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ parts of this quaternion tend to zero as n tends to infinity (at different rates). The scalar part, $\dfrac{ae + fb + gc + hd}{e^2 + f^2 + g^2 + h^2}$, tends to the golden ratio. As with complex numbers, for quaternions, we have $\lim_{n\to\infty} F_n/F_{n-1} = \phi$, and this in spite of the non-commutativity of quaternions.

Octonions are of the form

$$a + b\hat{\imath}_0 + c\hat{\imath}_1 + d\hat{\imath}_2 + e\hat{\imath}_3 + f\hat{\imath}_4 + g\hat{\imath}_5 + h\hat{\imath}_6.$$

As with quaternions, there are multiplicative relations: $\hat{\imath}_m^2 = -1$, $\hat{\imath}_m\hat{\imath}_n = -\hat{\imath}_n\hat{\imath}_m$ and $\{\hat{\imath}_m\hat{\imath}_{m+1} = \hat{\imath}_{m+3}$, $\hat{\imath}_{m+3}\hat{\imath}_m = \hat{\imath}_{m+1}$, $\hat{\imath}_{m+1}\hat{\imath}_{m+3} = \hat{\imath}_m$, where m is modulo 7$\}$. In the bracketed part of these relations, the $\hat{\imath}_m$s form into triads, $\{\hat{\imath}_m, \hat{\imath}_{m+1}, \hat{\imath}_{m+3}\}$, that copy the triad $\hat{\imath}, \hat{\jmath}, \hat{k}$ in quaternions. Addition is as you would expect and, with regard to the above relations and to non-commutativity, so is multiplication. The conjugate of $a + b\hat{\imath}_0 + c\hat{\imath}_1 + d\hat{\imath}_2 + e\hat{\imath}_3 + f\hat{\imath}_4 + g\hat{\imath}_5 + h\hat{\imath}_6$ is $a - b\hat{\imath}_0 - c\hat{\imath}_1 - d\hat{\imath}_2 - e\hat{\imath}_3 - f\hat{\imath}_4 - g\hat{\imath}_5 - h\hat{\imath}_6$, and this is used to do division as with complex numbers and quaternions. Taking

$$F_n = a + b\hat{\imath}_0 + c\hat{\imath}_1 + d\hat{\imath}_2 + e\hat{\imath}_3 + f\hat{\imath}_4 + g\hat{\imath}_5 + h\hat{\imath}_6$$

and

$$F_{n-1} = z + y\hat{\imath}_0 + x\hat{\imath}_1 + w\hat{\imath}_2 + v\hat{\imath}_3 + u\hat{\imath}_4 + t\hat{\imath}_5 + s\hat{\imath}_6,$$

we seek expressions for F_n/F_{n-1}. The product of an octonion and its conjugate is

$$z^2 + y^2 + x^2 + w^2 + v^2 + u^2 + t^2 + s^2$$

regardless of the order of multiplication. So

$$\begin{aligned}\frac{F_n}{F_{n-1}} &= \frac{(a + b\hat{\imath}_0 + c\hat{\imath}_1 + d\hat{\imath}_2 + e\hat{\imath}_3 + f\hat{\imath}_4 + g\hat{\imath}_5 + h\hat{\imath}_6)}{z^2 + y^2 + x^2 + w^2 + v^2 + u^2 + t^2 + s^2} \\ &\quad \times (z - y\hat{\imath}_0 - x\hat{\imath}_1 - w\hat{\imath}_2 - v\hat{\imath}_3 - u\hat{\imath}_4 - t\hat{\imath}_5 - s\hat{\imath}_6)\end{aligned}$$

$$
\begin{aligned}
&= \frac{az+by+cx+dw+ev+fu+gt+hs}{z^2+y^2+x^2+w^2+v^2+u^2+t^2+s^2} \\
&\quad + \hat{i}_0 \frac{-ay+bz-cv-ds+ex-ft+gu+hw}{z^2+y^2+x^2+w^2+v^2+u^2+t^2+s^2} \\
&\quad + \hat{i}_1 \frac{-ax+bv+cz-du-ey+fw-gs+ht}{z^2+y^2+x^2+w^2+v^2+u^2+t^2+s^2} \\
&\quad + \hat{i}_2 \frac{-aw+bs+cu+dz-et-fx+gv-hy}{z^2+y^2+x^2+w^2+v^2+u^2+t^2+s^2} \\
&\quad + \hat{i}_3 \frac{-av-bx+cy+dt+ez-fs-gw+hu}{z^2+y^2+x^2+w^2+v^2+u^2+t^2+s^2} \\
&\quad + \hat{i}_4 \frac{-au+bt-cw+dx+es+fz-gy-hv}{z^2+y^2+x^2+w^2+v^2+u^2+t^2+s^2} \\
&\quad + \hat{i}_5 \frac{-at-bu+cs-dv+ew+fy+gz-hx}{z^2+y^2+x^2+w^2+v^2+u^2+t^2+s^2} \\
&\quad + \hat{i}_6 \frac{-as-bw-ct+dy-eu+fv+gx+hz}{z^2+y^2+x^2+w^2+v^2+u^2+t^2+s^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{F_n}{F_{n-1}} &= \frac{(z-y\hat{i}_0-x\hat{i}_1-w\hat{i}_2-v\hat{i}_3-u\hat{i}_4-t\hat{i}_5-s\hat{i}_6)}{z^2+y^2+x^2+w^2+v^2+u^2+t^2+s^2} \\
&\quad \times (a+b\hat{i}_0+c\hat{i}_1+d\hat{i}_2+e\hat{i}_3+f\hat{i}_4+g\hat{i}_5+h\hat{i}_6) \\
&= \frac{az+by+cx+dw+ev+fu+gt+hs}{z^2+y^2+x^2+w^2+v^2+u^2+t^2+s^2} \\
&\quad + \hat{i}_0 \frac{-ay+bz+cv+ds-ex+ft-gu-hw}{z^2+y^2+x^2+w^2+v^2+u^2+t^2+s^2} \\
&\quad + \hat{i}_1 \frac{-ax-bv+cz+du+ey-fw+gs-ht}{z^2+y^2+x^2+w^2+v^2+u^2+t^2+s^2} \\
&\quad + \hat{i}_2 \frac{-aw-bs-cu+dz+et+fx-gv+hy}{z^2+y^2+x^2+w^2+v^2+u^2+t^2+s^2} \\
&\quad + \hat{i}_3 \frac{-av+bx-cy-dt+ez+fs+gw-hu}{z^2+y^2+x^2+w^2+v^2+u^2+t^2+s^2} \\
&\quad + \hat{i}_4 \frac{-au-bt+cw-dx-es+fz+gy+hv}{z^2+y^2+x^2+w^2+v^2+u^2+t^2+s^2} \\
&\quad + \hat{i}_5 \frac{-at+bu-cs+dv-ew-fy+gz+hx}{z^2+y^2+x^2+w^2+v^2+u^2+t^2+s^2} \\
&\quad + \hat{i}_6 \frac{-as+bw+ct-dy+eu-fv-gx+hz}{z^2+y^2+x^2+w^2+v^2+u^2+t^2+s^2}.
\end{aligned}
$$

The $\hat{i}_n$ parts of this octonion tend to zero as n tends to infinity (at different rates). The scalar part,

$$\frac{az + by + cx + dw + ev + fu + gt + hs}{z^2 + y^2 + x^2 + w^2 + v^2 + u^2 + t^2 + s^2},$$

tends to the golden ratio. As with complex numbers and quaternions, for octonions we have $\lim_{n\to\infty} F_n/F_{n-1} = \phi$.

For the convenience of the reader, we reproduce the scalar parts of F_n/F_{n-1} for complex numbers, quaternions and octonions. All of these expressions tend to ϕ:

$$\frac{ac + bd}{c^2 + d^2},$$
$$\frac{ae + fb + gc + hd}{e^2 + f^2 + g^2 + h^2},$$
$$\frac{az + by + cx + dw + ev + fu + gt + hs}{z^2 + y^2 + x^2 + w^2 + v^2 + u^2 + t^2 + s^2}.$$

They rather look like normalized inner products, don't they? So what have inner products got to do with ϕ?

The above is but a small part of quaternion/octonion arithmetic. As with the real numbers, you can do number theory, geometry and group theory with quaternions and octonions. People wanting to know more are referred to the following books.

J. P. Ward, *Quaternions and Cayley Numbers*, Kluwer, ISBN: 0-7923-4513-4.

John H. Conway and Derek A. Smith, *On Quaternions and Octonions*, A. K. Peters, ISBN:1-56881-134-9.

What's next?

Chris Jones

In one of his intriguing books Douglas Hofstadter gives the beginning of a sequence:

$$0,\ 1,\ 2,\ \ldots.$$

And asks – What comes next? M500 readers will quickly see that the answer is (approximately) 2.601×10^{1746}. Which leaves two questions—how is the sequence derived, and can readers give any better examples of surprising sequences produced from simple rules?

Generalized Fibonacci numbers

Tony Forbes

Let us start with a set of d different numbers

$$R = \{r_1, r_2, \ldots, r_d\}$$

and form the polynomial

$$(x - r_1)(x - r_2) \ldots (x - r_d). \tag{1}$$

Write c_k for the coefficient of x^k in (1), so that c_k is $(-1)^{d-k}$ times the sum of all possible products of the r_i taken $d - k$ at a time. Define the $d \times d$ matrix A by

$$A = \left[\begin{array}{ccccccc} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ & & & & \ldots & & \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ -c_0 & -c_1 & -c_2 & -c_3 & \ldots & -c_{d-2} & -c_{d-1} \end{array} \right].$$

Given R, we can now define a sort of d-dimensional *generalized Fibonacci sequence* whose nth term is given by

$$F(n) = F(n, R) = [A^n]_{1,d},$$

i.e. the top right-hand element of A^n. The corresponding linear recurrence formula is encoded in the last row of A,

$$F(n + d) = [A]_d \cdot (F(n), F(n + 1), \ldots, F(n + d - 1)).$$

For example, if we put $R = \{\phi, \tilde{\phi}\}$, where $\phi = (1 + \sqrt{5})/2$ and $\tilde{\phi} = (1 - \sqrt{5})/2$, we have

$$(x - \phi)(x - \tilde{\phi}) = x^2 - x - 1.$$

The coefficients are $c_0 = -1$ and $c_1 = -1$; so

$$A = \left[\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right],$$

the familiar matrix that generates the Fibonacci numbers, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, ...

If R consists of a single number, $R = \{\alpha\}$ say, then $A = [\alpha]$ and obviously $F(n, \{\alpha\}) = \alpha^n$. With a little more effort we can generalize the two-dimensional case,

$$F(n, \{\alpha, \beta\}) \;=\; \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

and immediately this gives a closed formula for the nth Fibonaccci number:

$$F(n, \{\phi, \tilde{\phi}\}) \;=\; \frac{\phi^n - \tilde{\phi}^n}{\phi - \tilde{\phi}} \;=\; \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{\sqrt{5}\; 2^n}.$$

Similarly,

$$F(n, \{\alpha, \beta, \gamma\}) \;=\; \frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)},$$

$$\begin{aligned} F(n, \{\alpha, \beta, \gamma, \delta\}) \;=\; & \frac{\alpha^n}{(\alpha-\beta)\,(\alpha-\gamma)\,(\alpha-\delta)} + \frac{\beta^n}{(\beta-\alpha)\,(\beta-\gamma)\,(\beta-\delta)} \\ & + \frac{\gamma^n}{(\gamma-\alpha)\,(\gamma-\beta)\,(\gamma-\delta)} + \frac{\delta^n}{(\delta-\alpha)\,(\delta-\beta)\,(\delta-\gamma)} \end{aligned}$$

and so on. The pattern is clear.

It is possible to obtain these formulae because by its construction the matrix A has eigenvalues R, and therefore A can be diagonalized by a suitable linear transformation. Indeed,

$$D \;=\; \begin{bmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & r_d \end{bmatrix} \;=\; Q^{-1} \cdot A \cdot Q,$$

where

$$Q \;=\; \begin{bmatrix} r_1^{-(d-1)} & r_2^{-(d-1)} & \dots & r_d^{-(d-1)} \\ r_1^{-(d-2)} & r_2^{-(d-2)} & \dots & r_d^{-(d-2)} \\ & & \dots & \\ r_1^{-1} & r_2^{-1} & \dots & r_d^{-1} \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

Since D^n is just a diagonal matrix with elements $[D^n]_{i,i} = r_i^n$, we can obtain an explicit formula for $A^n = Q \cdot D^n \cdot Q^{-1}$ and hence for $[A^n]_{1,d}$,

$$F(n, \{r_1, r_2, \dots, r_d\}) \;=\; \sum_{i=1}^{d} \frac{r_i^n}{(r_i - r_1)\dots(r_i - r_{i-1})(r_i - r_{i+1})\dots(r_i - r_d)}.$$

Recall that in the theory of the Fibonacci sequence the ratio of successive terms tends to a certain limit. In the general case we can use the formula to compute this limit. Suppose for the time being that there is a unique element α of R with maximum absolute value. Then the formula for $F(n,R)$ is dominated by a term of the form $\alpha^n/((\alpha-\beta)(\alpha-\gamma)\ \ldots)$. Hence

$$\lim_{n\to\infty}\frac{F(n,R)}{F(n-1,R)} \;=\; \alpha.$$

For the ordinary Fibonacci numbers, $\alpha=\phi$, the golden ratio.

Things get more complicated if there are two elements of R with the same largest absolute value, α and $-\alpha$, say. We assume that R has other elements besides α and $-\alpha$ (if not, $F(n,R)$ is identically zero). Call them β_1, β_2, …, β_{d-2}, so that $R=\{\alpha,-\alpha,\beta_1,\beta_2,\ldots,\beta_{d-2}\}$, and let

$$\chi \;=\; (-1)^d\,\frac{(\alpha-\beta_1)(\alpha-\beta_2)\ \ldots\ (\alpha-\beta_{d-2})}{(\alpha+\beta_1)(\alpha+\beta_2)\ \ldots\ (\alpha+\beta_{d-2})}.$$

Since now $F(n,R)$ is dominated by the terms involving α^n and $(-\alpha)^n$, we have

$$\begin{aligned}F(n,R) \;=\; &\frac{\alpha^n}{(\alpha+\alpha)(\alpha-\beta_1)(\alpha-\beta_2)\ \ldots\ (\alpha-\beta_{d-2})}\\ &+\frac{(-\alpha)^n}{(-\alpha-\alpha)(-\alpha-\beta_1)(-\alpha-\beta_2)\ \ldots\ (\alpha-\beta_{d-2})}+\cdots,\end{aligned}$$

and therefore

$$\frac{F(n,R)}{F(n-1,R)} \;\to\; \alpha\,\frac{1-(-1)^n\chi}{1+(-1)^n\chi};$$

that is, the ratio doesn't tend to anything. As you can see, it tends to oscillate between the values $\alpha(1+\chi)/(1-\chi)$ and $\alpha(1-\chi)/(1+\chi)$. However, if we consider instead two successive-plus-one terms, we do get a definite limit,

$$\frac{F(n,R)}{F(n-2,R)} \;\to\; \alpha^2.$$

For example, put $S=\{0,1,-2,2\}$. Then $d=4$, $\alpha=2$, $\chi=1/3$, the two limits are 1 and 4, the general term of the sequence is

$$F(n,S) \;=\; \frac{3\cdot 2^n-(-2)^n-8+6\cdot 0^n}{24}$$

and the recurrence is

$$F(n+4,S) \;=\; F(n+3,S)+4F(n+2,S)-4F(n+1,S),$$

which is obtained by expanding the polynomial

$$x\,(x-1)(x+2)(x-2) \;=\; x^4 - x^3 - 4x^2 + 4x$$

and then replacing x^k by $F(n+k, S)$ in the equation

$$x^4 - x^3 - 4x^2 + 4x \;=\; 0.$$

The first few terms are 0, 0, 0, 1, 1, 5, 5, 21, 21, 85, 85, 341, 341, 1365, 1365, 5461, 5461, 21845, 21845, 87381, 87381, ... from which it is evident that the two limits really are 1 and 4 and that

$$\lim_{n\to\infty} \frac{F(n,S)}{F(n-2,S)} \;=\; \alpha^2 \;=\; 4.$$

For another familiar sequence, let $T = \{t_1, t_2, t_3\}$,

$$t_1 = \frac{1+u+v}{3}, \quad t_2 = \frac{1+\rho\, u + \bar{\rho}\, v}{3}, \quad t_3 = \frac{1+\bar{\rho}\, u + \rho\, v}{3},$$

where

$$u = \sqrt[3]{19+3\sqrt{33}}, \quad v = \sqrt[3]{19-3\sqrt{33}},$$

and

$$\rho = \frac{-1+\sqrt{3}\, i}{2} \;\text{ and }\; \bar{\rho} = \frac{-1-\sqrt{3}\, i}{2}$$

are the two non-real cube roots of 1. After some elementary algebra during which it helps to note that $1+\rho+\bar{\rho} = 0$, $\rho\bar{\rho} = 1$, $\rho^2 = \bar{\rho}$, $\bar{\rho}^2 = \rho$, $uv = 4$ and $u^3 + v^3 = 38$, we obtain

$$(x-t_1)(x-t_2)(x-t_3) \;=\; x^3 - x^2 - x - 1.$$

The recurrence relation is therefore

$$F(n+3, T) = F(n+2, T) + F(n+1, T) + F(n, T),$$

which is just like Fibonacci except that each term is the sum of the preceding three. The sequence begins 0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, 5768, 10609, 19513, 35890, ..., and the ratio of two consecutive terms tends to $t_1 \approx 1.83929$, the element of T with the largest absolute value.

How the other half thinks – Part I

Dennis Morris

When we view the real number line we see a left–right symmetry about the number 0. The number 0 is the additive identity, and it occupies the position at the symmetrical centre of the real number line. While ever we do nothing more than add real numbers, the symmetry is preserved, but as soon as we introduce multiplication we break that symmetry because the multiplicative identity is $+1$. By symmetrical considerations, -1 ought to be a multiplicative identity also. It does not take long to realize that any multiplicative identities must be the same number, and so we cannot have both $+1$ and -1 as multiplicative identities in the same algebra. But still, -1 ought to be a multiplicative identity.

With $+1$ as the multiplicative identity, we have:

$(+1) \times (+1) = +1$ because any number times the multiplicative identity equals itself;

$(+1) \times (-1) = -1$ because any number times the multiplicative identity equals itself;

$(-1) \times (+1) = -1$ because any number times the multiplicative identity equals itself;

$(-1) \times (-1) = +1$ because $(-1) \times (-1) = -1 \Rightarrow (-1) = (+1)$.

All the above means that positive numbers have two square roots within the real number line but that the square roots of negative numbers are not within the real number line. Because of this, we call the square roots of negative numbers imaginary numbers. Anyone familiar with the complex number plane will be aware that multiplication by $\sqrt{-1}$ corresponds to an anti-clockwise rotation.

Now let us try it with -1 as the multiplicative identity; we have:

$(-1) \times (-1) = -1$ because any number times the multiplicative identity equals itself;

$(+1) \times (-1) = +1$ because any number times the multiplicative identity equals itself;

$(-1) \times (+1) = +1$ because any number times the multiplicative identity equals itself;

$(+1) \times (+1) = -1$ because $(+1) \times (+1) = +1 \Rightarrow (-1) = (+1)$.

All the above means that negative numbers have two square roots within the real number line but that the square roots of positive numbers are not within the real number line. Because of this, we call the square roots of positive numbers imaginary numbers. Anyone familiar with the complex number plane will be aware that multiplication by $\sqrt{+1}$ corresponds to a clockwise rotation.

With a little consideration, we see that the whole structure of mathematics will work just as well with -1 as the multiplicative identity as it does with $+1$ as the multiplicative identity. Thus, the symmetry that once was broken is now reforged.

How the other half thinks – Part II

Part I is a bit of a swizz in that the same can be achieved by running the number-line from right to left rather than from left to right and using minus to mean plus. In essence, $+1$ is a rotation by $360°$ and -1 is a rotation by $180°$ and these are fundamentally different.

How the other half thinks – Part III

Scratch Part II for hyperbolic rotations.

Solution 199.2 – 30 matches

> Use thirty matches to make a polygon of area 8 square matches such that the vertices of the polygon have integer match coordinates.

In the only reply we have, the offered solution (consisting of six 1×1 squares and a 2×1 rectangle) was stretching the meaning of the word 'polygon' much too far. Assuming it is not without interest, we give an answer involving a legitimate (but non-convex) polygon—and to make it a little more of a challenge we do not provide a picture of any kind.

Place thirteen matches pointing north $\left(\frac{180}{\pi} \arctan \frac{5}{12}\right)^\circ$ west in a straight line from $(5, 0)$ to $(0, 12)$, twelve matches pointing north at $(4, 0)$, $(4, 1)$, $(3, 2)$, $(3, 3)$, $(2, 4)$, $(2, 5)$, $(2, 6)$, $(1, 7)$, $(1, 8)$, $(0, 9)$, $(0, 10)$ and $(0, 11)$, and five matches pointing west at $(1, 9)$, $(2, 7)$, $(3, 4)$, $(4, 2)$ and $(5, 0)$. The result is a $5 \times 12 \times 13$ right-angled triangle with some bits missing. The area is

$$\tfrac{1}{2} \cdot 5 \cdot 12 - 9 - 7 - 4 - 2 \;=\; 8.$$

Always keep matches away from children.

Solution 200.4 – Bouncing ball

There are two fixed spheres of radius 1; sphere A is at position $(-10, 0, 0)$ and B is at $(10, 0, 0)$. A third ball, C, also of radius 1, bounces back and forth with perfect elasticity along the x-axis between A and B. Then, just as it is bouncing off A, the trajectory of C changes by 10^{-100} radians. How many further bounces does C experience before it leaves the system?

Steve Moon

The basic set-up is shown in the upper diagram, opposite. The centre of sphere C travels ≈ 16 before the collision with B. On colliding with B, the vertical displacement is $h_1 \approx 16\theta_1$, where $\theta_1 = 10^{-100}$.

Now consider the n^{th} collision, as illustrated by the lower diagram for C colliding with B. In this case Σh_n is the combined vertical displacement, $|BC| = 2$,

$$\psi_n = \arcsin \frac{\Sigma h_n}{2} \approx \frac{\Sigma h_n}{2},$$

and for angle of incidence θ_n, the rebound angle is $2\psi_n + \theta_n$. We compute the first few values of θ_n and Σh_n.

Horizontal deviation θ_n	Vertical displacement h_n	Aggregate $\sum_{k=1}^{n} h_k$	Ratio $\sum_{k=1}^{n} h_k \,/\, \sum_{k=1}^{n-1} h_k$
θ_1	$16\theta_1$	$16\theta_1$	–
$\theta_1 + 16\theta_1 = 17\theta_1$	$16 \cdot 17\theta_1 = 272\theta_1$	$272\theta_1 + 16\theta_1 = 288\theta_1$	18
$17\theta_1 + 288\theta_1 = 305\theta_1$	$16 \cdot 305\theta_1 = 4880\theta_1$	$5168\theta_1$	$17.94444\ldots$
$5473\theta_1$	$87568\theta_1$	$92736\theta_1$	$17.94427\ldots$
$98209\theta_1$	$1571344\theta_1$	$1664080\theta_1$	$17.94427\ldots$

Thus, after the first three bounces we have $h_1 + h_2 + h_3 \approx 5168\theta_1$ and then the ratios seem to settle at about 17.94427. Ball C leaves the system when $\Sigma h_k \approx 2$. Therefore we need to solve the equation

$$5168\theta_1 + 5168\theta_1(17.94427)^{n-3} = 2 \tag{1}$$

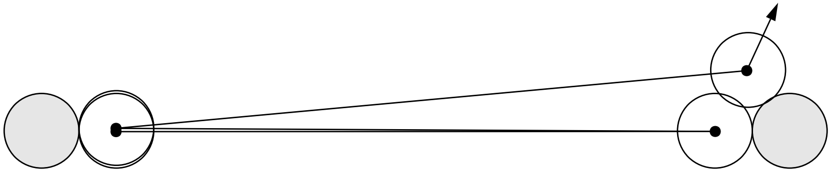
for n. Ignoring $5168\theta_1$, which is small compared with 2, (1) simplifies to

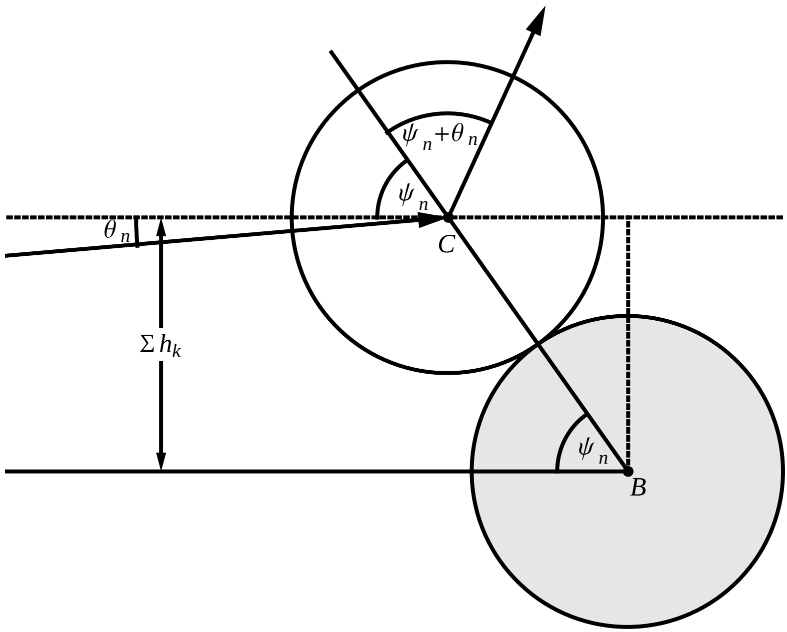
$$(17.94427)^{n-3} = \frac{2 \cdot 10^{100}}{5168} \approx 3.86997 \cdot 10^{96}$$

or, on taking logs and rearranging,

$$n - 3 \approx \frac{\log(3.86997 \cdot 10^{96})}{\log 17.94427} \approx 77.028.$$

Therefore the ball will have left the system after the 80^{th} bounce.





Tony Forbes

Actually the problem was inspired by a practical application.

If you ever watch the National Lottery draw on a Wednesday or Saturday evening, you will notice that the randomizing device involves table-tennis balls bouncing off each other in a manner that can be modelled by something along the lines of the preceding analysis. As we have seen, it took only the minutest nudge to knock the system off balance. Indeed the actual lottery machine is much worse. Instead of two fixed balls and one bouncing between them, we have 49 balls rebounding of each other and off the six containing walls. Of course, it is the nonlinear surfaces of the balls (and not the linear planes) that cause the system to behave in such a chaotic manner.

So here is some advice if you play the lottery with a fixed set of numbers and you feel trapped in the system. Perhaps you imagine the day when your favourite numbers are drawn but you haven't bought a ticket. Disaster! You have missed out on a prize worth millions of pounds!

Well, almost certainly not. Buying a ticket changes the initial conditions of the draw, and, as explained above, the system is extremely sensitive to initial conditions. If in the original, ticketless state your numbers were drawn, then in the new state the gravitational force of the ink on your ticket would disturb the system enough to produce a totally different set of numbers. If you see your numbers drawn, therefore, you are probably £1 better off than if you had spent the money on a ticket.

Now that we have mentioned the National Lottery, we offer another piece of useful advice. When you next play, try not checking the result. Just leave it. While the ticket remains unverified your life will continue in a quantum superposition of states, one for every possible outcome, including at least one state in which you are extremely wealthy. And while the result is outstanding you will not be tempted to spend money on any more lottery tickets. Of course, once the deadline for unclaimed prizes expires your quantum superposition will collapse into a state where your wealth is diminished by £1.

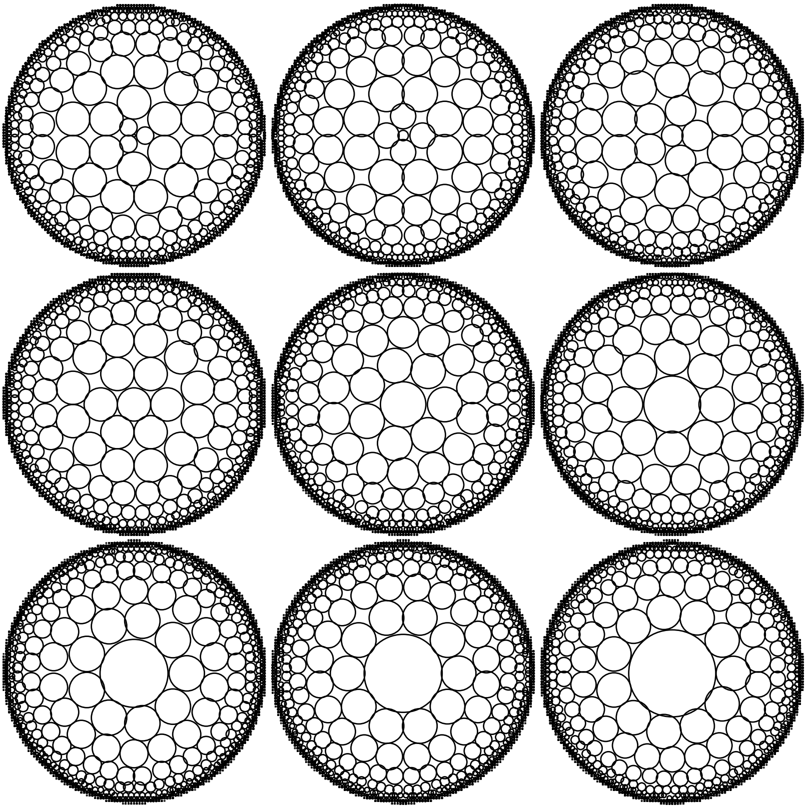
And while we're on the subject of bouncing balls, see if you can answer this simple question from Jeremy Humphries:

> During a legitimate snooker game how can you score 162 on a single visit to the table?

See page 20 for another snooker problem.

Problem 203.1 – Circles

Generalizing the thing on the cover of M500 **198** (top row, third item, below), compute the radius of the central circle assuming that the limiting circle has radius 1. We illustrate for $n = 3, 4, \ldots, 11$ but note that the central circle for $n = 3$ is too small to be seen.



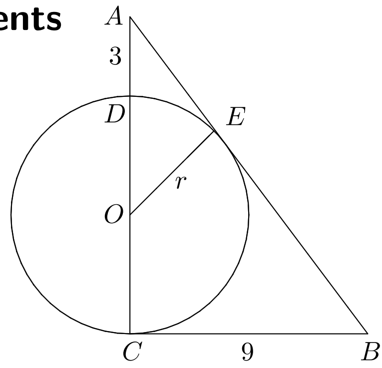
Problem 203.2 – Rotating digits

Find all integer solutions $(b, n, a_0, a_1, \ldots, a_n)$ of the equation

$$b\,(10^n a_n + \cdots + 10a_1 + a_0) \;=\; 10^n a_0 + 10^{n-1} a_n + \cdots + 10a_2 + a_1$$

subject to the conditions that $b, n \geq 2$, $1 \leq a_n \leq 9$ and $0 \leq a_i \leq 9$ for $i = 0, 1, \ldots, n-1$.

Solution 199.3 – Two tangents



> Lines AB and BC are tangents to the circle. The centre of the circle, O, lies on AC. If $|BC| = 9$ and $|AD| = 3$, what is the value of r, the radius of the circle?

Keith Drever

Since the problem was published in a newspaper we can assume you do not have to be a rocket scientist to solve it.

Tangents BC and BE meet the circle at right angles and $|BE| = |BC| = 9$. So $\angle AEO = \angle BCA = 90°$. Since $\triangle ACB$ is a right-angled triangle and $|BC| = 9$, we can guess that it is a $3 : 4 : 5$ triangle with BC representing the '3' side. We can immediately see that the radius r is $9/2$. To prove this, we find side AE of the right-angled triangle AEO that should be 6. Indeed $|AE| = \sqrt{|AO|^2 - |OE|^2} = \sqrt{56.25 - 20.25} = 6$, as expected.

For a more mathematical solution, let $x = |AE|$. Then we have

$$x^2 = (r+3)^2 - r^2 \quad \text{and} \quad (9+x)^2 = 9^2 + (3+2r)^2,$$

which simplify to

$$x^2 = 3(2r+3) \quad \text{and} \quad 9x = r(2r+3).$$

Squaring the second equation and dividing by the first yields $2r^3 + 3r^2 - 243 = 0$. Mathcad 'polyroots' gives the only real solution as $r = 4.5$.

Patrick Lee

Let $\angle BAC = \theta$. Then $|AE| = r \cot\theta$. It is clear that $|BE| = |BC| = 9$, so $|AB| = r\cot\theta + 9$. By similar triangles, $(r+3)/r = (r\cot\theta + 9)/9$; i.e. $r^2 \cot\theta = 27$. Substituting $\cot\theta = (2r+3)/9$ in the last equation and rearranging gives $2r^3 + 3r^2 = 243$.

We also had solutions from A. J. Moulder, Tony Huntington, Simon Geard, Steve Moon, Ian Bruce Adamson and John Spencer. A. J. Moulder, who solved the cubic $2r^3 + 3r^2 - 243 = 0$ using a method described in the third edition of *Perry's Chemical Engineers' Handbook*, says that this problem was left with him by the chairman of his local U3A mathematics group.

Problem 203.3 – Counting out

Tony Forbes

There are n objects arranged in a circle. They can be anything. In a typical application they could be n schoolchildren who are using the counting out process described below to select one of their number to perform an unpleasant task of some kind. However, we shall treat the objects as if they are just $0, 1, \ldots, n-1$ arranged anti-clockwise around the circle.

n	cycle length	fixed points
3	2	0
4	3	0
6	5	0
7	6	0
8	3	0 3 4 6 7
9	7	0 6
10	9	0
11	8	0 7 10
14	13	0
15	14	0
17	15	0 12
23	20	0 8 15
35	34	0
101	99	0 94
127	126	0
128	127	0
130	129	0
151	148	0 75 91
399	396	0 70 353
491	488	0 236 375
644	641	0 160 633

We perform the following process in n stages.

Stage 1: Remove 0.

Stage k: Remove the kth number, counting anti-clockwise starting from the number adjacent to the one that was removed in stage $k-1$.

Record the numbers as they are removed. Then you will have a permutation of $0, 1, \ldots, n-1$. Convert the permutation to standard cyclic form, omitting all fixed points.

For example, when $n = 8$ you should end up with the permutation $(0, 1, \ldots, 7) \mapsto (0, 2, 5, 3, 4, 1, 6, 7)$, which is $(1, 2, 5)$ in cyclic notation.

The table on the right shows all those $n \le 1000$ where there is a single cycle.

We ask: (i) Which n result in a single cycle. (ii) Clearly 0 is always a fixed point of the permutation. What are the other fixed points, if any.

GCSE question: Last year 204 Japanese cars were imported by a garage. This year the number of cars imported has increased by five twelfths. How many cars have been imported this year?

Editor's answer: 204.41666. . . .

An alternative solution to the cups problem

Andrew Colin

The problem is this: you have m cups in a row, some the right way up and some the wrong way up.

$$\bigcup\bigcup\cdots\bigcup\bigcap\bigcap\cdots\bigcap$$

Can you get all the cups the right way up if you are allowed to invert the cups only n at a time? The problem appeared in M500 **184** as 'Cups and downs' by Paul Garcia.

Useful ideas. 1. Many moves are equivalent.

The problem definition does not state that the cups turned over must be adjacent. It follows that if the number of cups turned per move is n, then all moves which turn a cups from up to down, and b cups from down to up where $a + b = n$ are equivalent. Therefore the number of possible distinct moves at any stage is at most $n + 1$.

2. Parity of an arrangement.

Consider an arrangement where z cups are facing up. Then we define p, the *parity* of this arrangement as $z \pmod 2$. If z is even, $p = 0$. If z is odd, $p = 1$.

Clearly any move which involves an even number of cups cannot change the parity of the arrangement. Therefore it is impossible, by any sequence of such moves, to arrive at an arrangement which has a different parity from the starting position. In particular, it is impossible to change $\bigcap\bigcap\bigcap\bigcup$ into $\bigcap\bigcap\bigcap\bigcap$ by any sequence that involves turning two cups at a time.

Since moves that involve an even number of cups won't work for half the initial arrangements, we shan't consider them further. Instead consider moves that turn an odd number of cups: 1, 3, 5, etc. Call this n.

A general algorithm for solving the cup problem when n is odd. It is convenient to pretend that at the end we want all the cups to face down. If you are looking for a configuration with k cups facing up, just find any k cups in the initial configuration, stick labels on them and pretend they are facing the other way.

We suppose that we are starting with a collection of m cups, of which x are initially facing up, and $m - x$ are already down; n is the number of cups to be switched for each move.

If $n = 1$, the rule is trivial—just keep turning cups until they are all facing down.

For $n > 2$, there are three possible cases.

Case 1: $n > m$. You can't apply the rule at all—there are not enough cups to make a move.

Case 2: $n = m$. There is only one possible move. The problem can be solved only if all the cups are initially facing up.

Case 3: $n < m$. All problems can be solved, as follows.

Rule 1. As long as x, the number of cups facing up, exceeds n, keep turning down these cups in groups of n at a time. Then apply rule 2.

Rule 2. If x, the number of cups facing up, is now odd but less than n, turn over a group which contains $(x+1)/2$ cups facing up, and $n-(x+1)/2$ facing down. This will bring the number of cups facing up to $n-1$, which is an even number. Apply rule 3.

Rule 3. If x, the number of cups facing up, is now even, turn over a group that contains $x/2$ cups facing up, and $n-x/2$ cups facing down. This will bring the number of cups facing up to n exactly. Apply rule 4.

Rule 4. Turn over the remaining n cups.

Example. Consider $n = 5$, with a starting group which has 24 cups of which 18 are facing down.

UUUUUU∩∩∩∩∩∩∩∩∩∩∩∩∩∩∩∩∩∩

Apply rule 1. Turn over five facing-down cups three times. This leaves three cups facing down.

UUUUUUUUUUUUUUUUUUUUU∩∩∩

Apply rule 2. Turn over a group with two cups facing down and three facing up. This leaves four cups facing down.

UUUUUUUUUUUUUUUUUUUU∩∩∩∩

Apply rule 3. Turn over a group with two cups facing down and three facing up. This leaves five cups facing down.

UUUUUUUUUUUUUUUUUUU∩∩∩∩∩

Apply rule 4. Turn over the five cups facing down. The problem is solved.

UUUUUUUUUUUUUUUUUUUUUUUU

Crossnumber 202 solution

Across 1. 135^2, 4. 3^6, 6. 939, 7. 288^2, 9. 23^2, 10. 3^{18}, 13. 135, 14. 14^4, 17. 282, 18. 459, 19. 5^6; Down 1. 198, 2. 279, 3. 29^6, 4. 282^2, 5. 313^2, 8. 288, 10. 182^2, 11. 273^2, 12. 29^2, 15. 466, 16. 5^4

Problem 203.4 – Cyclic quadrilateral

A quadrilateral is inscribed in a circle. In other words, it is a *cyclic quadrilateral*—all four vertices lie on the circle. The sides are a, b, c and d. Show that R, the radius of the circle, is independent of the shape of the quadrilateral and is a function of a, b, c and d.

Another cyclic quadrilateral (again with side lengths a, b, c and d) has a circle inside it which is tangent to all four sides. Show that r, the radius of the inner circle, is also a function of just a, b, c and d.

What are R and r?

Problem 203.5 – 50p in a corner

This is like 'Problem 200.4 – Circle in a box' except that the coin has a different shape and there is one less dimension.

What is the locus of the centre of a 50p piece lying flat in the (x, y)-plane such that its edge is in contact with both the positive x-axis and the positive y-axis?

To make a British 50p piece, draw a regular heptagon and round off each side with an arc that has the opposite vertex as its centre. Then cut the shape out of suitable sheet metal. (Or you could change a small amount of your local currency to Sterling.)

Problem 203.6 – Loops

There are n pieces of string lying in a heap on the table. Choose two ends at random and tie them together. Choose two unused ends from the $2n-2$ remaining and tie them together. And so on until there are no free ends left.

What is the probability of creating a single loop of string?

Problem 203.7 – Rhombus

A snooker table has a playing area of sides $a \times b$, $a > b$, and its cushions have coefficient of elasticity e, $0 < e \leq 1$. A ball, initially placed in contact with the a side, is struck so that it leaves at angle θ to the side. The ball then follows a rhombus-shaped path and returns to its starting point.

Show that $t = \tan\theta$ satisfies the quadratic

$$2\,a\,b\,t^2 + (a^2 - b^2)(1 + e)\,t - 2\,a\,b\,e \;=\; 0.$$

Mathematics in the kitchen – IV

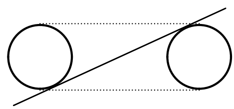
ADF

For this experiment you will need some doughnuts and a sharp knife. By 'doughnut' I mean one of those ring-shaped cakes which you can purchase, cooked while you wait, at around 10 for £1 from a specialist stall such as the one opposite the Argos store in Kingston-upon-Thames. Each doughnut should be a reasonable approximation to a torus.

Take a doughnut and (using the knife) make a single plane cut that passes through the doughnut's centre of symmetry. Try various angles.

Observe first that if the cut is made horizontally, the boundary of the resulting cross-section consists of two concentric circles—one corresponds to the outer rim of the torus and the other to the inner rim. Let us call the radii of the circles $R + r$ and $R - r$, repsectively. At the other extreme, a vertical cut produces two circles of radius r each with centres separated by distance $2R$.

Now make cuts at various intermediate angles. Before trying these out, you might like to speculate as to how the two large concentric circles (resulting from a cut at 0° to the horizontal) become two widely separated small circles as the angle varies continuously from 0° to 90°.



Finally, assuming that you still have one doughnut left, make a cut at the specific angle $\alpha = \arcsin(r/R)$. You do not need precise measurements but in practice this is a little tricky. Ensure that, as well as passing through the centre of symmetry, the knife *makes a tangent to the torus surface* at the start of the cut and also when it comes out of the torus at the other end. Try to guess what the boundary of the cross-section will look like before you perform the cut.

For the enlightenment of readers, especially those for whom knives and doughnuts are unavailable, there will be a full explanation in M500 **204**. Meanwhile, I would like mention that this topic was inspired by a short talk by **Ian Harrison** at the start of the 2005 M500 Winter Weekend. Judging from the reaction of Ian, myself and the rest of the audience, you will surely be utterly amazed at the result of the tangential cut!

Warning: sharp kitchen implements should always be used with care. And if you intend to eat the doughnuts after the experiment, please wash your hands (and the knife) before you start.

Letters to the Editor

Useless statistics

Dear Tony,

I was reminded of the article 'Odds and ends' in M500 **201** when we were told:

> BBC London news, 6.30 14 January, 2005 – £12.3 billion is spent by Britons on fast food, that is, £204 per person.

If I divide 12.3 billion by 60 million, I get 205, so obviously our BBC statistician wanted to be more precise than that and used 60.294118.

I have to report, however, that last year my great-aunt Matilda was on holiday in Switzerland for two weeks; my grandfather fell off his ladder and couldn't get to the kebab for six weeks; my eldest son broke his jaw whilst playing football and can only take liquids, and ....

A statistic such as this may have meaning to another statistician but to the average TV viewer it is pointless.

Regards,

Ron Potkin

Toilet paper

Something similar to our 'Problem 197.5 – Toilet paper' came up on one of Marcus du Sautoy's BBC4 programmes. The question (set by some DJ) was:

> A vinyl record is 31 cm in diameter. The outside non-playable bit is 1 cm wide. The inside non-playable bit is 2 cm, the label is 5 cm diameter. There are 30 grooves per cm. How far does the needle travel when you play the music from beginning to end?

One contestant pointed out that the circles get smaller as you go further in, so you will have some kind of diminishing series. Another came up with a solution of 6000π — I didn't get how she got that figure because I was busy marvelling at how four bright people could be so stupid. None of them got it. Nor did they get the answer to a subsidiary question after they were told the answer, which was, 'Why was the answer approximate (as in "A little more than")?'

Eddie Kent

More on -plexes

Dear Eddie and Tony,

Mindaugas has joined in and sent the following list of -plexes (or -plets, just change $-x$ to $-t$), with his explanation, below. The 21-and-up sequence is much more concise than JRH's [M500 **199** 26].

None of them except *decemplex* is attested, but their building seems to be quite straightforward. *Centuplex* again exists. '26-fold' is problematic because no word for 'sixfold' is attested, but let it be. *Vigintisimplex* (and *triginta-* etc. *-simplex*) is also a bit of a problem, because it might sound like 'twenty-simple', but '*simplex*' is exactly 'one-fold'. I think that some of these words, particularly those from 11 to 19, can't be built in any way without sounding weird.

Thus: 10 *decemplex*, 11 *undecimplex*, 12 *duodecimplex*, 13 *tredecimplex*, 14 *quattuordecimplex*, 15 *quindecimplex*, 16 *sedecimplex*, 17 *septendecimplex*, 18 *duodevigintiplex*, 19 *undevigintiplex*, 20 *vigintiplex*, 21 *vigintisimplex*, 22 *vigintiduplex*, 23 *vigintitriplex*, 24 *vigintiquadruplex*, 25 *vigintiquintuplex*, 26 *vigintisextuplex*, 27 *vigintiseptemplex*, 28 *duodetrigintaplex*, 29 *undetrigintaplex*, 30 *trigintaplex*, 31 *trigintasimplex*, etc., 40 *quadragintaplex*, 50 *quinquagintaplex*, 60 *sexagintaplex*, 70 *septuagintaplex*, 80 *octogintaplex*, 90 *nonagintaplex*, 98 *nonagintanovemplex* (not *duodecentuplex*), 99 *undecentuplex*, 100 *centuplex.*

Best wishes,

Ralph Hancock

Means on a circle

John Bull's article, 'On the average' [M500 **201** 1] made me think of the following. We have a regular polygon with n vertices, $n > 2$, inscribed in a circle of unit radius. Consider the chords from any one vertex to each of the others. Then for the *squares* of the chord lengths, we have

$$\begin{aligned}\text{arithmetic mean} &= \frac{2n}{n-1},\\ \text{geometric mean} &= n^{2/(n-1)},\\ \text{harmonic mean} &= \frac{12}{n+1}.\end{aligned}$$

Sebastian Hayes

Solution 190.6 – Triangle

David Porter

A rather late suggestion for the ruler-and-compasses construction required by this problem but possibly different enough from those given in M500 **193** to still be of interest. The problem is to locate P, given the equilateral triangle GHC, as in the diagram opposite. Likewise for any regular polygon.

Consider the general case where we have a regular polygon with internal angle β. In the diagram, let C be one corner of the polygon and let A and B be the centre points of the two sides that meet at C. We thus require a construction that gives the point P such that $PA = 2PB$ and the angles θ and ϕ are equal. First, if $\theta = \phi$ the opposite angles of the quadrilateral $ACBP$ are supplementary and hence $ACBP$ is a cyclic quadrilateral and so the point P lies on the circumscribing circle of triangle ABC.

My construction is as follows.

(1) Draw the circumscribing circle of triangle ABC.

(2) Drop the perpendicular from A onto BC at D.

(3) Extend CB to the left and mark E on it such that $ED = 2AC$.

(4) Draw the perpendicular at E and mark F on it such that $EF = DA$.

(5) Draw the straight line connecting F to C; then the point P at which this cuts the circle is one corner of the required internal polygon.

Proof. Let α be the angle between CB and CF; then by the sine rule on triangles CPB and CPA we have $BP/\sin\alpha = PC/\sin(\pi-\theta) = PC/\sin\theta$ and $AP/\sin(\beta-\alpha) = PC/\sin\phi = PC/\sin\theta$. Hence

$$BP/\sin\alpha \;=\; AP/\sin(\beta-\alpha).$$

Rearranging and expanding the sine of the composite angle gives us

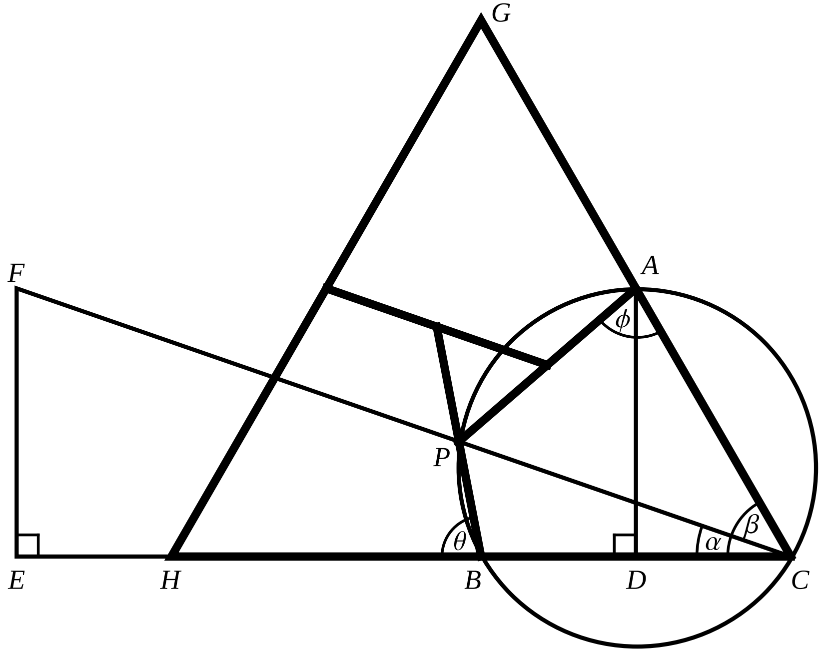
$$AP\sin\alpha \;=\; BP(\sin\beta\cos\alpha - \cos\beta\sin\alpha)$$

which, providing α is not zero, we can divide through by $\sin\alpha$ to give $AP = BP(\sin\beta/\tan\alpha - \cos\beta)$. So

$$AP \;=\; BP((DA/AC)/(EF/EC) - DC/AC).$$

But $EF = DA$; so $AP = BP(EC - DC)/AC$ and since $EC - DC = 2AC$ this yields $AP = 2BP$ as required.

Though this proof is based on a diagram in which β is acute, the construction as described also works for larger angles whilst the proof remains unchanged provided we assume DC becomes negative when D falls to the right of C.



Mathematics Revision Weekend 2005

Preliminary announcement

The **31st M500 Society Mathematics Revision Weekend** will be held at **Aston University**, **Birmingham** over **9–11 September 2005**.

The Weekend is designed to help with revision and exam preparation, and is open to all OU students. Tutorial sessions start at 19.30 on the Friday and finish at 17.00 on the Sunday. We plan to present most Open University mathematics courses.

The cost, including standard accommodation and all meals from bed and breakfast Friday to lunch Sunday, will be £150. Add £22 for *en-suite* facilities. The cost for non-residents will be £75. M500 members get a discount of £10. See the Society's web page, www.m500.org.uk, for full details and an application form, or send a stamped, addressed envelope to

Jeremy Humphries, M500 Weekend 2005.

Contents M500 203 – April 2005