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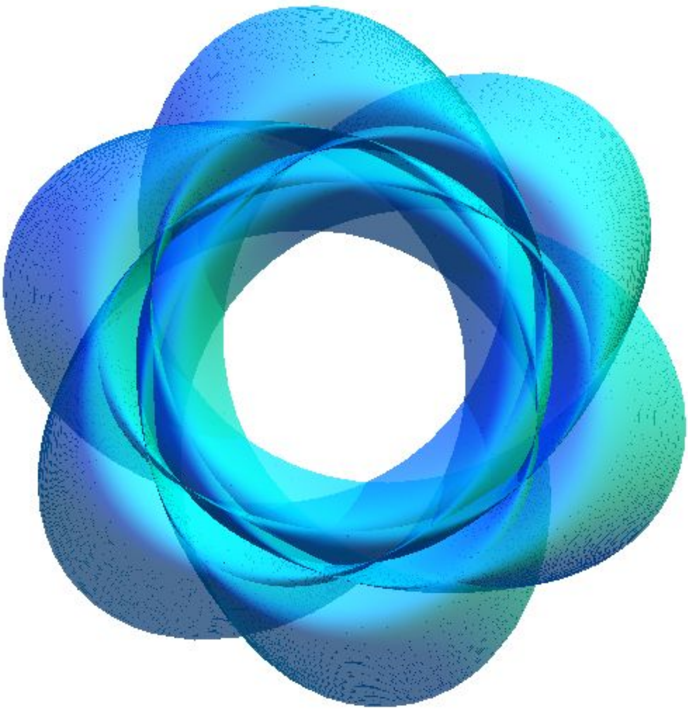

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Visualizing sections of the Hopf fibration

Tommy Moorhouse

Part I. The 3-sphere

Introduction The idea of the first part of this article is to give a straightforward and accessible account of the geometry of the sphere S^3, which 'lives' in a real four-dimensional space. Our line of attack will be to work by analogy with the 2-sphere, the surface of a ball in our familiar three-dimensional space. We will present a concrete coordinate description of S^3 and use this to project onto our three dimensional space and to take 'slices' of S^3. There are many other articles on the projection of the 3-sphere into $\mathbb{R}^3$, and the main reason for including this is to make the article self-contained.

Although S^3 is a natural generalization of S^2 it has a richer structure than one might expect: it isn't all spheres! We will look at the intriguing Hopf fibration, one of the simplest non-trivial fibre bundles—if you don't know what this means we hope to give you an easy-to-digest and easy-to-visualize explicit picture. If you do know something about bundles we hope you will find the elementary treatment interesting.

The second part of the article explores a visual approach to studying a map from S^3 to S^2 known as the Hopf fibration. We will look, in particular, at some beautiful projections of 'sections' of this fibration to $\mathbb{R}^3$.

Definitions Almost all our work will be done in real three- and four-dimensional spaces, where the distance function is the intuitive one. A sphere is then the set of points a fixed distance from some fixed point. We normally label spheres by their dimension, which may be taken as the number of independent coordinates needed to identify any point on the sphere. Thus the 1-sphere (the circle) needs just one angular coordinate, for example the angle measured anticlockwise from the positive x-axis. The point $(\cos\theta, \sin\theta)$ identifies uniquely any point on the unit circle. The 1-sphere lives in a plane, which has two dimensions. We will take the viewpoint that in general an n-sphere is embedded in ('lives in') an $(n+1)$-dimensional space, which may lie in a space of higher dimension, just as straight lines lie in a plane: this point will be made clear later. We also include the 0-sphere, which is just a pair of points, for example the points $x = \pm 1$ on the x-axis.

Coordinates and projections To identify the points on a 2-sphere (e.g. the surface of a globe) we need points x, y, z such that $x^2 + y^2 + z^2 = 1$. If we choose a fairly standard coordinate system using latitude and longitude θ and ϕ with θ ranging from 0 (North pole) to π (South pole) and ϕ ranging from 0 to 2π (Greenwich meridian!) we find that the set $(x, y, z) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$ fits the bill.

Note that the coordinates at the North and South poles (N and S) are a little odd: these points are not identified uniquely, since ϕ can take any value at these points. This is not a massive flaw: we can always choose coordinates that are 'bad' at a different pair of distinguished points if we are interested in what is going on at N and S. For all other points the coordinates are good.

On the 3-sphere we use coordinates

$$(x, y, z, w) \;=\; (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta\cos\psi, \sin\theta\sin\psi).$$

It is easily seen that these cover the unit 3-sphere and are good coordinates for all but two points. We will call the intersection of S^3 with any 3-plane through the origin of $\mathbb{R}^4$ a 'slice' of S^3 to distinguish it from any projection that we will soon describe. These slices are analogous to the circles cut from a 2-sphere by a plane (i.e. a 2-plane) through the origin. Clearly slices correspond to the great circles on the 2-sphere. For example if we intersect S^3 with the 3-plane $w = 0$ (that is, we look for all the points on S^3 with last coordinate zero) we get the 2-sphere described in the coordinates given above, since $\sin\psi = 0$. (It will be seen later what happens when we choose $\sin\theta = 0$.)

We now want to project from the 3-sphere onto our three-dimensional world, which we take to be the 3-plane $w = 0$. We can imagine the projection to be the shadow cast by the 3-sphere when a point source of light sits at one particular point on the sphere. If we project from $(0, 0, 0, 1)$ we will see most of the 3-sphere but anything passing through this point will be projected to infinity. We will find that slices (analogous to great circles) are projected onto ellipsoidal objects. Note that we take care to distinguish between the 'true' objects living in S^3 and their projections onto $\mathbb{R}^3$.

Explicitly, consider a line parameterized by a time t passing through $(0, 0, 0, 1)$ at time $t = 1$ and through $(\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ \sin\theta\cos\psi,$ $\sin\theta\sin\psi)$ at $t = 0$. This line intersects the plane $w = 0$ when the vector

$$(0, 0, 0, 1)\,t + (1 - t)(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta\cos\psi, \sin\theta\sin\psi)$$

has zero in the last place. This happens when

$$t \;=\; \frac{-\sin\theta\sin\psi}{1 - \sin\theta\sin\psi}$$

and here we have coordinates

$$(1-t)(x,y,z,w) \;=\; \frac{(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta\cos\psi, 0)}{(1-\sin\theta\sin\psi)}.$$

To see what this means, note that different values of ψ give different slices of S^3. These all intersect in the common equator (a 1-sphere) $x^2+y^2=1$, which corresponds to taking $\theta=0$. The great sphere corresponding to $\psi=0$ is seen to project to a 2-sphere in our world. Other values of ψ still give spheres on S^3 but these project to ellipsoidal shapes passing through the common equator.

There is another map from S^3 to S^2 of quite different character. The coordinate description of S^3 suggests another type of structure: if we fix θ and let ϕ and ψ vary independently the points so described have the character of a torus, the 'product' of two circles, often visualized as the surface of a doughnut! However, these tori lie in $\mathbb{R}^4$ so we need to project to $w=0$ (for example). We will look at this more closely in Part II.

Part II: The Hopf fibration

The Hopf fibration seems to crop up almost everywhere in recent physics, from twistor theory [1] and the asymptotic structure of space–time to magnetic monopoles [2] and representations of rotations in space [3]. The motivation behind the study presented here was originally to get a simple visual impression of this ubiquitous structure. The surprising images that emerged lead to a more detailed consideration of the topology that constrains and dictates the form of the images. Some elementary topological ideas are used to explore this further.

Consider the (Hopf) map

$$\pi : (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta\cos\psi, \sin\theta\sin\psi)$$
$$\mapsto (\cos\theta\cos(\phi-\psi), \cos\theta\sin(\phi-\psi), \sin\theta).$$

The image is clearly a 2-sphere (lying in a different copy of $\mathbb{R}^3$) and every point of S^3 is mapped to a point of S^2. The map is not one-to-one, however: we can check that

$$(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta\cos\psi, \sin\theta\sin\psi)$$

and

$$(\cos\theta\cos(\phi+\alpha), \cos\theta\sin(\phi+\alpha), \sin\theta\cos(\psi+\alpha), \sin\theta\sin(\psi+\alpha))$$

map to the same point for all α in $[0, 2\pi]$. Letting α vary over this range we obtain, for fixed θ and ϕ, a circle that wraps once around the torus in S^3. The reader might like to draw this circle to see what it would look like on an ordinary torus (doughnut) in our world. In the language of fibre bundles this circle is a fibre of the Hopf map. It is said to lie over the image point in S^2. (It is perhaps easier, for the reader familiar with projective geometry, to deduce the above from the map $(z_1, z_2) \in S^3 \to [z_1, z_2] \in \mathbb{CP}^1$ defining the Hopf fibration: here S^3 is the subset of $\mathbb{C}^2$ defined by $z_1\bar{z_1} + z_2\bar{z_2} = 1$.)

Projecting to $z = 0$ we find that the tori go to

$$\frac{1}{1 - \sin\theta \sin\psi}(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta\cos\psi).$$

If we fix θ we effectively set the radii of the torus in $\mathbb{R}^4$. The projection of any of these tori is a genuine 2-torus in our world. These tori fill our 3-space, with each successive torus surrounding the previous one. There are two 'degenerate' tori, corresponding to $\theta = 0$, which is the unit circle in $z = w = 0$; and to $\theta = \pi/2$, the z-axis. A recent description of the situation can be found in [3]. One finds that when the Hopf fibration is described in the literature the four-dimensional situation is often not distinguished from the three-dimensional projection.

Sections of the Hopf fibration In the language of fibre bundles, of which the Hopf fibration is a simple example, a section of π is a map $\sigma : S^2 \to S^3$ such that for any point p of S^2 we have $\pi(\sigma(p)) = p$. We see that a section maps p to a point in the fibre lying 'above' p. Suppose we fix θ. Then we can think of all the points $\sigma(p)$ as lying on the same torus (see the previous paragraph to see what this means). Take the map in coordinates to be

$$(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta) \mapsto (\cos\theta\cos\omega, \cos\theta\sin\omega, \sin\theta\cos\psi, \sin\theta\sin\psi).$$

Since σ is a section of π we must have $\psi(p) - \omega(p) = \phi$. Apart from this essential restriction we are free to choose ω to be a continuous function of θ and ϕ to define a section of the Hopf fibration. We will see that the ϕ-dependence dictates the topology of the section in a well-defined way, while the θ-dependence introduces what we might call a twist.

To get a feeling for the reason behind the condition $\psi - \omega = \phi$, fix θ and note that the condition means that whatever curve our function marks out on the surface of the torus defined by θ it winds at least once around the torus (because ϕ grows from 0 to 2π), and cannot be shrunk to a point on the torus. Thus it passes through each of the fibres of the Hopf map, and does indeed define a section if it passes once through each fibre.

The simplest way to see this is to use the model of a torus as a rectangle with opposite sides identified [Figure T1]. Thus when a curve runs off one side of the rectangle it reappears on the opposite side: one could imagine rolling the rectangle up to give you a model of the torus in three dimensions. Using this picture [Figure T2] it is easy to convince oneself that any fibre (parallel to the diagonal from lower left to upper right) meets a section (that is, a curve winding n times around the horizontal (ω-) and $n+1$ times around the vertical (ψ-) direction) in exactly one point. Some elementary topology will consolidate this and is left as a challenge. A curve of this type is said to be a (representative of a) homotopy class and cannot be continuously deformed into a curve of a different class. Each integer n defines a homotopy class, and these classes form a subset of the complete set of homotopy classes of the torus. There is much more information about homotopy in the literature.

In fact, the sections restricted to constant θ are knots (called 'torus knots' in, for example, [5]). One can use this fact to produce images of the torus knots by restricting the range of θ [Figure S1].

The simplest section arises for $\omega = 0, \psi = \phi$. In this case as ϕ varies for fixed θ we see that the image of a small circle on the base 2-sphere winds once around the torus. Varying θ and remembering that our section is continuous (so points $\lambda(p)$ and $\lambda(q)$ are close together whenever p and q are) we see that the projected section is a plane. If we allow ω and ϕ to depend on θ we see that, since the restriction of any section to a torus defined by fixed θ belongs to a fixed homotopy class and this cannot be changed by a continuous map, the projected section is still a deformed (or twisted) plane with the same topology.

Another simple section, independent of θ, arises when we take $\omega(p) = 2\phi$, $\psi(p) = \phi$. The small circles on the base 2-sphere map to circles wrapping once round the 'big' circle and twice round the 'small' circle of the torus. Putting these together by varying θ we now get what looks like a plane with a twist near the origin. Figure T3 shows what is happening in a model using a stack of rectangular representations of the torus to give a tetrahedron. In this figure the top and bottom surfaces of the tetrahedron are identified, as are the sides. The section appears as a set of fins twisting from the vertical to the horizontal: detailed interpretation is left to the interested reader.

We can play this game to produce surprisingly complex surfaces in $\mathbb{R}^3$, choosing $\omega(\theta,\phi) = (n+1)\phi, \psi(\theta,\phi) = n\phi$. In this case we get a swirling

n-leafed figure close to the origin, untangling to give n symmetrically distributed planes as we move away from the origin. We can also let the sections depend on θ, giving more intricate flower-like surfaces [Figures S2, S3].

We might wonder what happens 'at infinity' in $\mathbb{R}^3$. In fact it can be shown that every ray from the origin can be considered to pass through a single point at infinity when $\mathbb{R}^3$ is compactified to give S^3 (much as the 2-plane is compactified to S^2 by adding a single point at infinity). It turns out that a transformation (inversion) interchanging the origin and infinity in $\mathbb{R}^3$ preserves the form of all the sections, so the picture around infinity is in a sense essentially the same as that close to the origin.

Conclusion We have looked at what might seem at first a rather daunting mathematical construction, the Hopf fibration, and extracted a very explict picture of the sections projected to three dimensions. I personally was surprised by the intricacy of the projected sections, the connection with torus knots, and the application of elementary topology. (An introduction to elementary topology can be found in [4] or many of the articles to be found on the internet.) I am sure there is scope for further investigation.

References

[1] S. A. Huggett and K. P. Tod, *An Introduction to Twistor Theory* (second edition), CUP 1994, page 60ff.

[2] M. Gockeler and T. Schucker, *Differential Geometry, Gauge Theories and Gravity*, CUP 1989, page 140ff.

[3] D. W. Lyons, *An elementary introduction to the Hopf fibration*, Lebanon Valley College preprint 2004.

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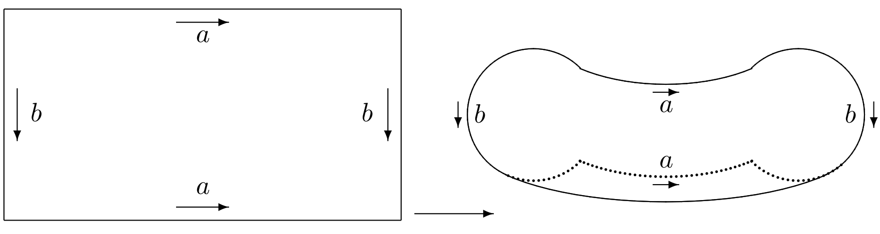


Figure T1: rolling up a torus

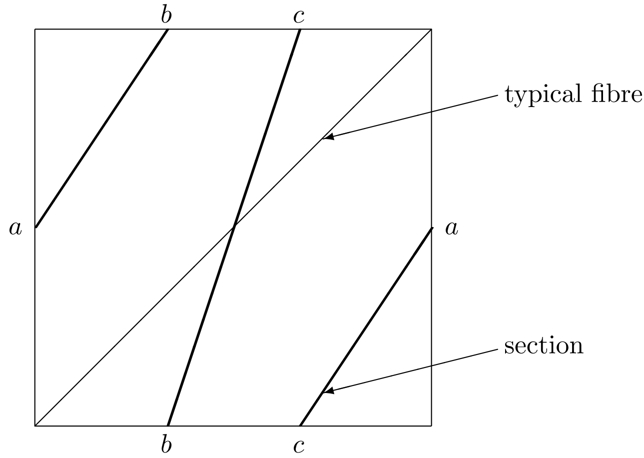


Figure T2: torus

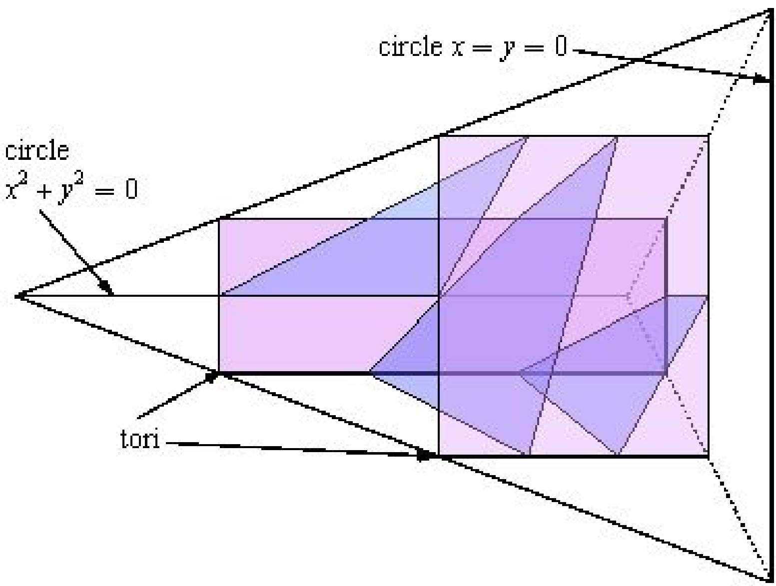


Figure T3: stacked tori

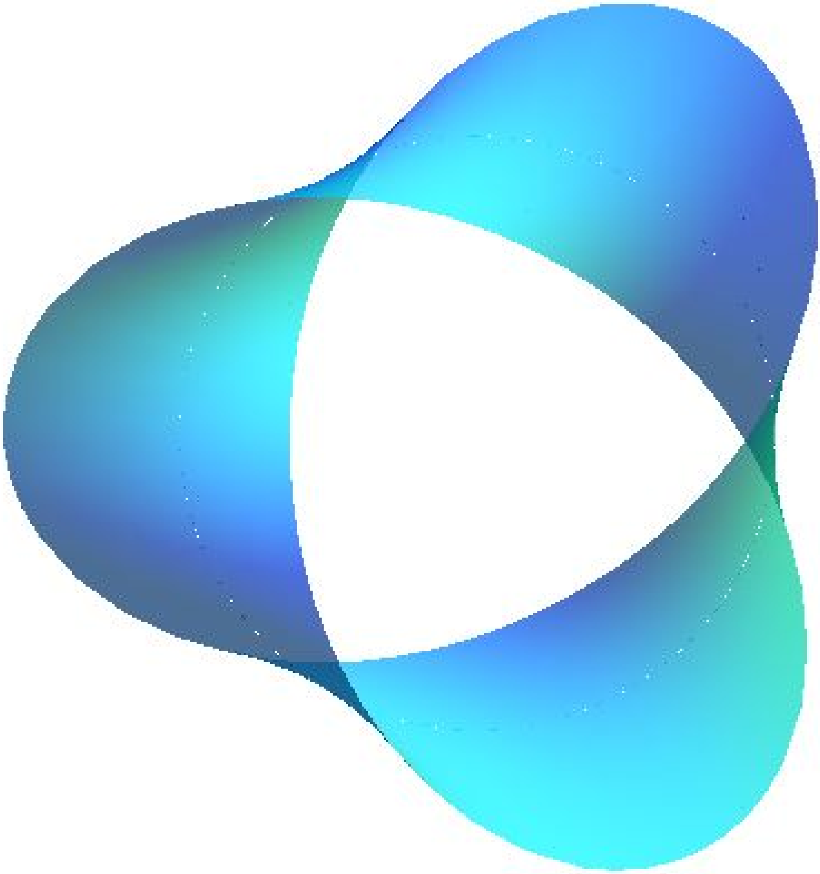


Figure S1

$$\left(\frac{\cos 2s\cos t}{1-\sin t\sin 3s}, \frac{\sin 2s\cos t}{1-\sin t\sin 3s}, \frac{\sin t\cos 3s}{1-\sin t\sin 3s}\right),$$

$$0 \le s \le 2\pi,\ 0 \le t \le \frac{\pi}{7}$$

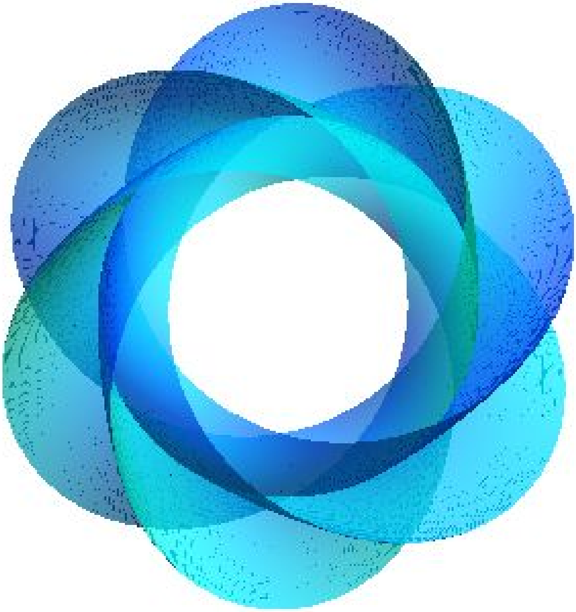


Figure S2

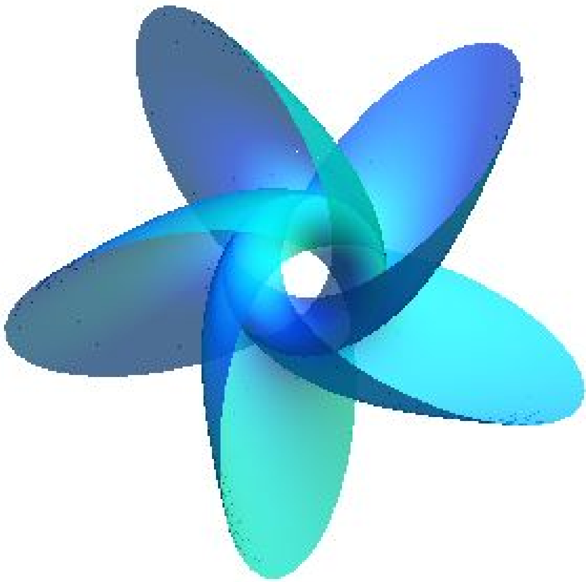


Figure S3

Problem 227.1 – 25 steps

Tony Forbes

You and your opponent play a game. You place your markers at a position called START and your objectives are to travel step by step along a path to reach a place called FINISH, which is 25 steps away from START. You take turns to create a random ordered pair of integers (n_1, n_2), $1 \le n_1, n_2 \le 6$, by throwing a pair of dice and for each of $i = 1, 2$ in that order, either move your token up to n_i steps forwards, or move your opponent's token up to $n_i + 2$ steps backwards.

An example. You go first. You throw $(3, 2)$. In the absence of any better alternative, you might as well move your token 5 places forward. Your opponent throws $(6, 1)$. He has several choices. He can move his token 7 places forward, or (using the 6) move you back 5 places to START and then himself 1 forward, or move himself 6 forward and you 3 back.

What's the expected length of such a game?

I was going to make the additional assumption that both players play sensibly, but unfortunately I'm not sure I know what 'sensibly' means.

This is based on a game marketed under the name *Enigma*. I should point out that the rules stated above correspond only to the non-deterministic parts of the game. There is also an element of skill involving the asking and answering of questions set in riddle format.

I actually did some experiments. Let us assume that both players adopt the strategy of moving their opponent's token at least n_i+1 steps backwards whenever a sensible opportunity arises to do so. This could happen whenever the opponent is at least $n_i + 1$ steps from START. It seems to me the best way to play. By using the backwards option you are maximizing the use of your throw whilst still giving yourself a chance of reaching FINISH. But remember that it is not sensible to move your opponent any number of steps backwards if you yourself are within $n_1 + n_2$ steps of winning.

However, under this strategy a typical game is going to last an awfully long time. Just to give a small sample of what might happen, here are the numbers of rounds for some typical games:

$$2621160, 1828738, 588450, 1031936, 2346550, 209041, 479808, 593392.$$

This would also apply to the real game unless there is an extremely significant difference between the abilities of the players with regard to the element of skill, where a player must answer correctly a randomly selected riddle to earn the right to throw the dice.

Problem 227.2 – Conspiracy theory

Colin Reid

A conspiracy theory is sweeping across the internet. Believers in the conspiracy theory persuade others by means of peer pressure: if at least half of a gullible person's friends believe the theory, that person will soon believe the theory him/herself, and once someone believes the theory, they will never stop believing it. Non-gullible people will never believe the theory, no matter how many of their friends do. Assume friendship is unchanging, and that nobody is born or dies over the time span under consideration.

The average gullible person has x friends who are not gullible, and initially, the average believer in the conspiracy theory has y friends who are not believers. Initially, the proportion of gullible people who believe the theory is p.

1. Suppose $py < x$. Show that at least one gullible person will never believe the conspiracy theory.

2. At some stage, Jon Ronson (who is not gullible) writes a blog entry which discredits the theory. Every gullible person reads the blog entry—the effect on them is as if Jon is now one of their friends. (The blog entry has no effect on Jon's existing friends, or on people who already believe.) At this point, there are m believers, who each have on average z non-believer friends (not counting Jon). Show that the number of believers will now never exceed $m(z + 1)$.

Kiwi fruit

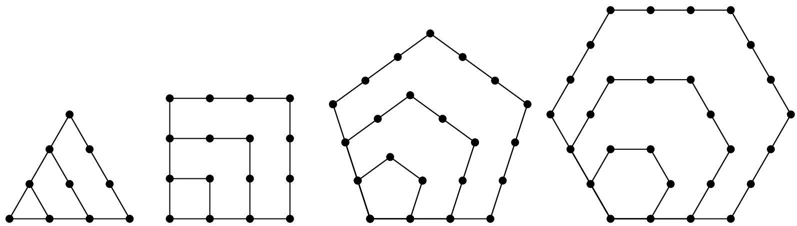
Tony Forbes

I had an interesting experience in a supermarket involving apple juice. I purchased (amongst other things) seven litres at 84p each, buy one get one free. So they charged me for seven, refunding the cost of three. A day or two later I went back to claim the eighth. After I explained the state of my senility the sales assistant was very co-operative. First she told me to get a litre of the same brand of apple juice. She put it through her cash register, placed it in a bag and handed it back to me. Then she gave me 84 pence! I protested, of course, arguing that all I wanted was the free eighth litre. But she argued more forcefully that by acquiring the eighth carton I was entitled to a refund of 84p. A queue was developing and I became reluctant to continue the debate. So I took the money and left. Who was right?

Problem 227.3 – Pentagonal numbers

Tony Forbes

We want to consider numbers $(3r^2 - r)/2$, where r is an integer. These take on two forms depending on the sign of r. When r is positive you get the truly *pentagonal numbers*. You can see by looking at the following picture how these objects fit into the scheme of things—the rth pentagonal number is the number of dots in the rth iteration of the third diagram from the left: 1, 5, 12, 22, ...



On the other hand, if r is negative, the resulting numbers are nearly but not quite pentagonal. However you can make a pentagon if you agree to double one of the sides by drawing a row of dots just below the base of the main pentagon. Combining both types of numbers, together with $r = 0$, gives this sequence of pentagonal-like numbers:

$$\mathcal{P} \;=\; (0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, 57, 70, 77, 92, 100, \ldots).$$

That's the background. Now for the problem.

Which arithmetical progressions avoid $\mathcal{P}$?

In other words, for which values of a and b, $a > 0$, $0 \le b < a$ does the set $\{at + b : t = 0, 1, 2, \ldots\}$ avoid all the elements of the sequence $\mathcal{P}$?

A good place to start is where a is prime. For instance, when $a = 23$ it looks as if the avoiding arithmetical progressions occur when $b = 4$, 6, 9, 10, 13, 14, 16, 18, 19, 20, and 21.

When the province of British Columbia noticed that the log tables in the provincial parks were rotting in the rain, they replaced them all with stone tables. It caused near extinction of the snake population. This was because most of the snakes were adders, and they needed log tables to multiply.

Problem 227.4 – Ten coins

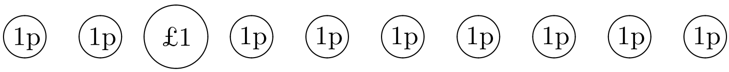
Tony Forbes

You and someone else play a game. You begin by placing ten coins in a line. Then you and your opponent take turns to remove a coin from one of the ends of the line. When all the coins are removed, the player with the most amount of money wins.

(i) Devise a general strategy which will guarantee that the player who goes first never loses.

(ii) Investigate ways of optimizing your winnings.

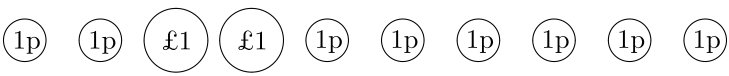
For example, suppose you lay out a pound coin and nine 1p pieces as follows.



Then if you go first, you can get the pound. You take a penny from the left. He must now take a penny from the right; otherwise the pound becomes available to you on your next turn. Thereafter you and he take pennies from the right until he is forced to expose the pound.

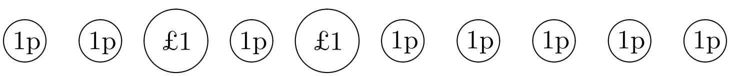
What we would be interested to see is a general strategy of some kind. By the way, there is no rule that the coins must be legal tender. Values like -163p, 0p, $\frac{32}{27}$p, πp, 1729p and so on are perfectly legitimate.

You might also like to have a go at this next specific example involving two pounds and eight pennies.



It seems to be obvious that you cannot grab both pounds for yourself. For as soon as you take one you expose the other for your opponent. Is it possible to force a draw? Is it possible to do any better than forcing a draw?

Now what if the two pound coins are separated?

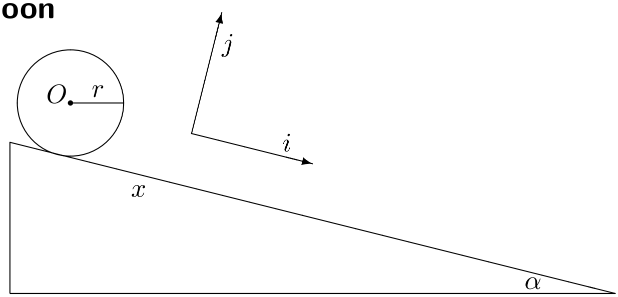


Solution 224.1 – Three rolling spheres

If the times to roll down an inclined plane are t_1 for a hollow sphere, t_2 for a solid sphere and t_3 for a 'semi-solid' sphere (solid except for a central hole of half the radius), prove that

$$t_1 : t_2 : t_3 \;=\; \sqrt{\frac{5}{3}} : \sqrt{\frac{7}{5}} : \sqrt{\frac{101}{70}} \;\approx\; 1.291 : 1.183 : 1.201.$$

Steve Moon



Establish a system of coordinates with unit vectors i and j, parallel and perpendicular to the plane respectively.

When the object is rolled a distance x down the plane it has turned through angle θ about O. Therefore $x = r\theta$ and by differentiation with respect to time, $\dot{x} = r\dot{\theta}$. When the centre of mass O lies at $xi + rj$ the velocity of the centre of mass is $\dot{x}i$ and the angular velocity about O is $\dot{\theta}$.

By conservation of energy, the object has kinetic energy (translational plus rotational) equal to its potential energy loss. hence

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 \;=\; mgx\sin\alpha.$$

Then use $\dot{x} = r\dot{\theta}$,

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{I\dot{x}^2}{r^2} \;=\; mgx\sin\alpha,$$

and differentiate with respect to time,

$$m\dot{x}\ddot{x} + \frac{I\dot{x}\ddot{x}}{r^2} \;=\; mg\dot{x}\sin\alpha,$$

and tidy for $\ddot{x}$,

$$\ddot{x} \;=\; \frac{mg\sin\alpha}{1 + I/(mr^2)}$$

down the plane.

This acceleration $\ddot{x}$ is a constant; so using '$s = ut + \frac{1}{2}at^2$', a standard equation of motion, where $u = 0$ and t is the time to travel distance s, we get

$$x = \frac{1}{2}\,\frac{g\sin\alpha}{1 + I/(mr^2)}\,t^2.$$

Therefore

$$t = \left(\frac{2x}{g\sin\alpha}\left(1 + \frac{I}{mr^2}\right)\right)^{1/2}.$$

Now $(2x/(g\sin\alpha))^{1/2}$ is the same for each type of object. Therefore

$$t = K\left(1 + \frac{I}{mr^2}\right)^{1/2}$$

for some constant K.

Introducing subscripts 1 for a hollow sphere, 2 for a solid sphere and 3 for a semi-solid sphere, we have the following moments of inertia,

$$I_1 = \frac{2}{3}m_1r^2, \qquad I_2 = \frac{2}{5}m_2r^2$$

and

$$I_3 = \frac{2}{5}m_3\left(\frac{r^5 - (r/2)^5}{r^3 - (r/2)^3}\right) = \frac{2}{5}m_3\,\frac{31r^5}{32}\,\frac{8}{7r^3} = \frac{31}{70}\,m_3r^2.$$

Hence

$$\begin{aligned} t_1 : t_2 : t_3 &= \left(1 + \frac{I_1}{m_1r^2}\right)^{1/2} : \left(1 + \frac{I_1}{m_2r^2}\right)^{1/2} : \left(1 + \frac{I_1}{m_3r^2}\right)^{1/2} \\ &= \left(1 + \frac{2}{3}\right)^{1/2} : \left(1 + \frac{2}{5}\right)^{1/2} : \left(1 + \frac{31}{70}\right)^{1/2} \\ &= \sqrt{\frac{5}{3}} : \sqrt{\frac{7}{5}} : \sqrt{\frac{101}{70}}. \end{aligned}$$

An explosive device was found in a can of Alphabetti Spaghetti. A spokesman from the bomb squad said that if it had detonated, it could have spelt disaster. [Sent by JRH]

Solution 221.4 – Eleven bottles

> Three people, A, B, C, are stuck in a lift over the weekend. They have 11 bottles of water, four supplied by A and seven by B, which are to be shared equitably. C donates £11 for the water. How is it to be divided between A and B? The same thing happens the following weekend, but this time A has three bottles and B has eight. Again, £11 is to be split between A and B.

Jimmy Mellon

An interesting problem that you have ripped out of a book. Unfortunately, the book has framed the problem in terms that are part of the continuum of wobbly logic that stretches from the *pretium justum* of medieval theologians to today's 'Fair Trade' coffee. Fairness is not (despite what mathematicians may feel about the origins of probability theory) an objective, rational principle. Your 'problem' is however easily resolved if restated in market terms.

The market is closed (it is in a lift!); supply is fixed (11 bottles which, like you, I presume to be equal). Access to the bottles is determined by agreement to be 11/3 per person (thus assuming that equitable means equal shares). It is disposed at a 'price' (£3 per bottle) which is set by C's initiative (it may be arbitrary or, e.g. it may be the actual cost of the bottles in the supermarket) and agreed by A and B. (If it is not agreed by all three, then the market has to resolve the problem by bids and we will no longer have equitable meaning equal but shares determined by willingness to purchase and to sell.)

It is an imperfect market, but it clears itself. So for the first weekend A has one-third of a bottle more than his quota. This he disposes of to C, who hands over £1; and likewise C secures three and one-third bottles for £10 from B.

The second weekend, A needs two-thirds of a bottle to complete his quota. He buys it for £2 from B. C needs three and two-thirds bottles, so he buys them from B for £11.

For each weekend then, A, B and C have each £11-worth of water, and those who have supplied it have been compensated at the 'going rate'.

Phillip Whettlock

This is how I'd approach the problem if I were C ...

C buys the bottles of water from A and B, paying a proportionate amount. So A now has £4, B has £7 and C has the 11 bottles of water.

They now decide to share the water. Having establish the price of a bottle of water as £1, C sells 11/3 of a bottle to each of A and B. Thus,

A buys 3.67 bottles worth of water and has £0.33 left.

B buys 3.67 bottles worth of water and has £3.33 left.

C has the remaining 3.66 bottles worth of water and £7.34.

In the case of the second weekend, if C again buys all the water to begin with, A does not have enough money to buy an equal share of water. A now has £3, but requires £3.67 to buy an equal share.

So, C says, then A can pay the owed money next week, when they get stuck in the lift again!

Dick Boardman

There is a long tradition of puzzle setters including useless information in their puzzles whose purpose is to confuse the solver. This has been done here. First, two extra assumptions, all the water has been consumed and A, B and C drank equal amounts. The normal principle is that each person should pay for what he has consumed and someone who pays more shall receive a refund.

Each person is assumed to have drunk 11/3 bottles of water. The cost of a bottle of water is not stated but we may reasonably assume that C knows it and has based his contribution on it, i.e. 11/3 bottles cost £11.00. That is, each bottle costs £3.00.

For the first weekend, B has contributed 7 bottles costing £21.00 and has consumed 11/3 bottles (£11.00). He is therefore due for a refund of £10.00. B has contributed 4 bottles (£12.00) and consumed 11/3 bottles (£11.00). He is due for a refund of £1.00. Thus C's contribution should be split £1.00 to A and £10.00 to B.

For the second weekend, B has contributed 8 bottles (£24.00) and consumed 11/3 bottles (£11.00) so he is due for a refund of £13.00. A has contributed 3 bottles (£9.00) and drunk 11/3 bottles (£11.00). He must therefore contribute a further £2.00. Thus B should receive £11.00 from C and £2.00 from A.

Tony Forbes

I confess that I am not at all happy with the way market forces have been allowed to creep into the previous discussions. C is merely acting charitably. Nobody is buying anything. Nobody is selling anything. If C had kept her money, there would be no problem—A and B would then share £0 equitably at no loss to either party. In the actual problem it must surely be a fundamental principle that neither A nor B should be forced to part with any money.

One could go with the recommended solution for the first week (£1 to A, £10 to B), but let B have just the entire £11 for the second week. However, this introduces a hideous discontinuity in the derivative of the amount of money received by A as the number of bottles he supplies decreases continuously from 11 to zero. What we must seek is a simple general solution which works equally well for all splits.

Let us see if we can remove the discontinuity and smooth things out. Suppose A contributes a bottles and B contributes $11 - a$ bottles. In the case $a = 0$ it is clear that B should get the whole amount of C's money, £11. Similarly, at the other extreme, where $a = 11$, A must get £11. And if A and B supplied the same number, 5.5 bottles each, who would argue against the splitting of C's money equally: £5.50 to A and £5.50 to B?

So what's the simplest function that passes through the three points $(0, 11)$, $(5.5, 5.5)$ and $(11, 0)$? Well of course, it's a straight line. Thus I am forced to conclude that the only correct answer to the problem is the 'wrong' answer. Split the money in proportion to bottles *supplied*: £4 to A and £7 to B for the first week, £3 to A and £8 to B for the second week.

This solution works for all values of a in the range zero to 11, and in every case both A and B receive non-negative amounts of money. On the other hand, this is not true if A, for instance, were to turn up with a negative number of bottles. Then the solution offered would require B to accept £$(-a)$ from A in addition to the £11 from C. A just reward for A.

Problem 227.5 – Laces

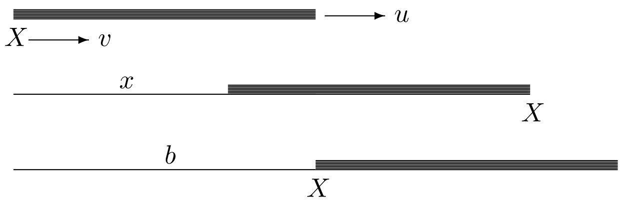
Your journey involves walking at your normal constant walking pace partly along fixed ground and partly along a moving travelator. But you must stop for a short while to tie your shoe laces. Where is the best place to do this? Surprisingly, it does make a difference—as you can confirm by working the problem. However, what we really want is an elementary observation that makes the answer obvious.

Solution 224.3 – Inspecting the column

A column of soldiers of length a is marching steadily along a road, when an officer on horseback rides at uniform speed from the rear to the front and back again while the column moves distance b. How far does the officer move?

Norman Graham

Let x be the distance the column has moved by the time the officer X reaches the front. Let the speeds be u for the soldiers and v for the officer, $v > u$.



The time for the officer to reach the front is

$$\frac{x}{u} = \frac{x+a}{v}.$$

The time to return to the rear of the column is

$$\frac{b-x}{u} = \frac{x+a-b}{v}.$$

Therefore

$$\frac{u}{v} = \frac{x}{x+a} = \frac{b-x}{x+a-b},$$

which can be solved to get

$$x = \frac{1}{2}\left(b - a \pm \sqrt{a^2+b^2}\right).$$

Hence the distance travelled by the officer is

$$(x+a) + (x+a-b) = a \pm \sqrt{a^2+b^2}.$$

Solved in a similar manner by **Steve Moon** (with identical notation!) and Tony Moulder.

Oliver Atkin

Eddie Kent

Arthur Oliver Lonsdale (A. O. L.) Atkin, who died recently, had a connection with M500. He was responsible for Problem 157.3, Binomial coefficients, and for the intelligent primality test offer in **155** 26. He was also mentioned by Tony Forbes in connection with titanic twin primes.

He was born in Liverpool in 1925 and read mathematics at Cambridge. After graduating he tried to join the army, but was recruited to Bletchley Park instead. When the war ended he acquired a doctorate and a wife, Raynor. He went on to learn programming on Atlas and to develop a technique for solving numerical problems with a computer.

As he pointed out, using a machine in mathematics can just produce lists which might be pretty and impressive but don't give much insight. His aim was to look for happy accidents. 'The computer's role,' he wrote, 'was merely to find an accident relevant to a known general theorem. However, inspecting the tables ... O'Brien and I observed the following: If $24n-1$ is divisible by 13^m and $p(n)$ [the *partition function*—the number of ways of expressing the integer n as a sum of positive integers] is divisible by 13^m, then so is $p(N)$, where

$$24N-1 = r^2(24n-1) \quad \text{and} \quad r \geq 5 \text{ is prime.}$$

This in its turn enabled us to make a more general conjecture as to a multiplicative property of $p(n)$ in relation to divisibility by powers of 13, which I have recently proved.' And thus was developed one more component of the structure Andrew Wiles was later to use to crack FLT.

During his career Atkin, together with Daniel J. Bernstein, developed the Sieve of Atkin. This is related to the sieve of Eratosthenes, but does some preliminary work and then marks off multiples of primes squared, rather than of primes.

In the 60s he went to America, eventually becoming professor emeritus at the University of Illinois at Chicago where he was equally adept at computing and playing the organ.

He died on December 28 from complications after a fall at his home; he was 83 and leaves two children and five grandchildren. His wife died in 1970.

... Meanwhile, those of us who can compute can hardly be expected to keep writing papers saying 'I can do the following useless calculation in two seconds', and indeed what editor would publish them? [Oliver Atkin]

Letters to the Editor

Morse

Your piece on palindromes in morse (M500 **224** 17) reminded me of a game I used to play with a friend at school. You would be given a word in morse, but with the dots and dashes run together with no spaces. You had to guess the word.

Our favourite was EGG, which in morse is $\cdot - - \cdot - - \cdot$.

I had another friend who was sent to Loughborough during his National Service to learn to send and receive morse at high speed. It drove him mad, literally. He was admitted to mental hospital, and then transferred to teleprinters, which were considered much less demanding.

John Reade

I was intrigued by 'Palindromes in morse'. I don't think it is valid to produce a palindrome of continuous dots and dashes that can be split into letters arbitrarily. Morse is, and has to be, sent with tiny breaks between the letters, or it would be unreadable. This makes the problem of constructing a palindrome that works in English *and* morse rather a restrictive one, since you can only use the symmetrical letters E I S H T M O K P R X. A palindrome that works only in morse is easier, because you can use the mirror-image pairs A/N B/V D/U F/L G/W Q/Y. But you can't use C J Z, which have no mirror images. The longest sequence I could get, which sort of makes sense, is

O, a tiger! Do quail wait in Goa? No waiting. Find you're wit? No.

Maybe something a bit more coherent will come to mind.

There is a discussion of morse code in songs at rateyourmusic.com/board_message/message_id_is_1383238, with quite a few examples, including The Beatles' *Strawberry Fields Forever*, which starts with the morse for JOHN.

Ralph Hancock

Problem 227.6 – Snellius's formula

Prove *Snellius's formula*: for small $|x|$,

$$\frac{3\sin 2x}{2(2+\cos 2x)} \approx x,$$

the difference being approximately $4x^5/45$. Then see if you can invent even more bizarre approximations to the identity function.

Contents — M500 227 – April 2009