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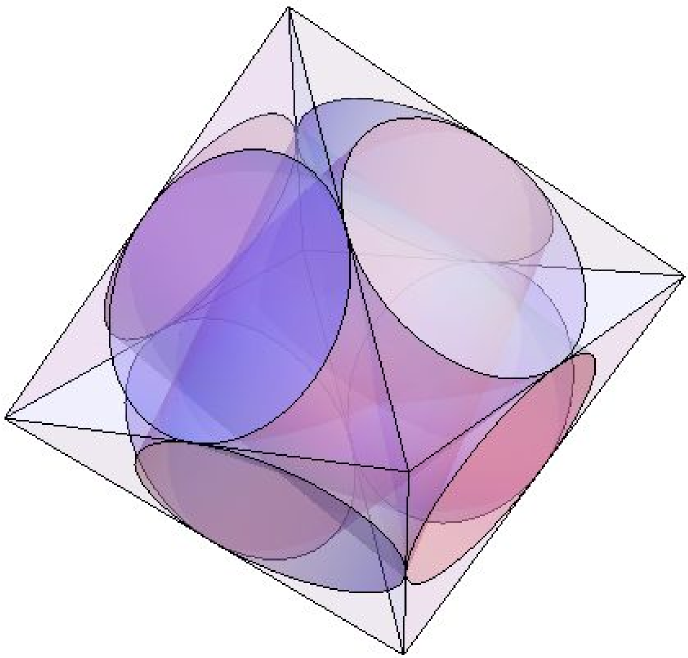
M500 245



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, send a stamped, addressed envelope to Diana Maxwell, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation.

Solution 240.1 – Two tins of biscuits

> There are two tins, each containing $n > 0$ biscuits. Take a biscuit from a tin chosen at random. Keep doing this until one tin is empty. What is the expected number of biscuits that remain in the other tin?

Ken Greatrix

I'm not sure exactly how I arrived at the following solution but it probably involved a lot of guesswork and trial and error with just a small amount of logic thrown in for good measure. It was when I restated the problem in terms of tossing a coin to choose a tin at random that my previous studies of probability forced themselves back into memory. How many coin tosses do you need to ensure n heads? More to the point—what is the expected number of coin tosses in total when n heads have occurred?

After furtively searching through my course materials (being the hoarder that I am I didn't dispose of them—much to my wife's disgust!), I initially thought it would be some sort of Poisson process but finally I realized that I could use a modified version of the negative binomial distribution to solve the problem:

$$\mu = 2 \sum_{x=n}^{2n-1} (2n - x) \binom{x-1}{n-1} q^{x-n} p^n.$$

The multiplier digit, 2, is there because it's 'double-sided'. There are two similar processes occurring at the same time: one counting heads and the other counting tails. They are two dependent variables, but their individual means can be added to obtain the total mean.

The limits of the summation are n (in the unlikely situation where one tin has been chosen continually) and $2n - 1$, ensuring that a minimum of 1 biscuit remains in the other tin. The term $(2n - x)$ multiplying the terms of the summation is the number of biscuits remaining in the other tin. The rest of the formula is taken directly from the *M245 Handbook* (The Open University, 1984) and represents the probability that x biscuits in total need to be taken in order to remove all n biscuits from one of the tins.

Extending the problem. If you have two tins of biscuits with different numbers of biscuits in each, then the average number remaining in the other tin when one is emptied is given by

$$\mu = \sum_{x=m}^{m+n-1} (m+n-x) \binom{x-1}{m-1} q^{x-m} p^m + \sum_{y=n}^{m+n-1} (m+n-y) \binom{y-1}{n-1} q^{y-n} p^n.$$

With three tins and any (different) start values, the formula becomes

$$\begin{aligned}\mu \;=\; & \sum_{x=a}^{a+b+c-2} (a+b+c-x)\binom{x-1}{a-1} q^{x-a} p^{a} \\ & + \sum_{y=b}^{a+b+c-2} (a+b+c-y)\binom{y-1}{b-1} q^{y-b} p^{b} \\ & + \sum_{z=c}^{a+b+c-2} (a+b+c-z)\binom{z-1}{c-1} q^{z-c} p^{c}\end{aligned}$$

to calculate the expected remainder in two tins when the third is emptied.

I have a bit of a concern here. There must be at least one biscuit remaining in each of two tins when the third is empty, hence the upper limit of summation shown as $(a+b+c-2)$. But how do I know that these two biscuits aren't both in the same tin, with two tins being empty?

Challenge problems

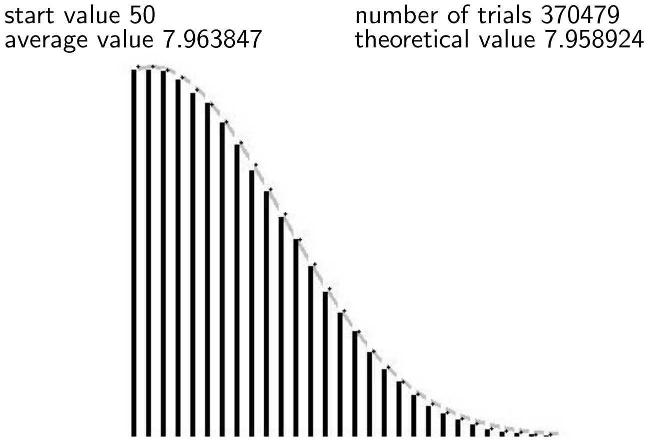
1. What is the expected number of biscuits remaining in the third tin when the other two tins have been emptied? The situation here is whether you remove a tin from the game when it becomes empty (and then reduce the problem to two tins), or whether you continue to count random choices of the empty tin.

2. Using either of the constraints described in the first challenge, what is the formula for any number of tins with any number of biscuits in each?

3. Put me out of my misery—show that the upper limit of summation in the three-tin version should be $(a+b+c-2)$.

By way of encouragement to Dick Boardman (and others in a similar situation), I also have a background in engineering. Serving an apprenticeship with an electrical company, I gained an 'ONC' and a 'Full-Tech C&G', both of which are now obsolete. Having found the maths components of these qualifications to my liking, I decided to follow up by enrolling with the OU. I don't feel that I have yet made the full transition to mathematical thinking, but from what you say you're only one step away from this. Your approach seems to be one of specialized thinking, by which I mean using examples, models and particular values. The next step is to put your ideas into a generalized formula. But actually you seem to have got to that situation—your program must use a formula to calculate your particular examples.

Initially I also followed the computer simulation route. I tried various values of starters, but when I used higher values, I noticed that the probabilities seemed to follow a normal distribution curve, as shown in the diagram for 50 biscuits initially in each tin.



Probability bar-chart of remaining biscuits

I compiled it as a free-running simulation, only stopping it when the values settled to a steady state. The dots to the right of each bar are the expected (calculated) value, and even without a goodness-of-fit test it would seem to be correct. The grey dotted line is a representation of the positive half of a normal distribution but as you can see, although it's close, I couldn't make it fit exactly showing that the probabilities don't follow this format. I would suggest that the distribution of the probabilities of biscuits remaining does indeed tend towards a normal distribution as $n \to \infty$.

Problem 245.1 – Birthday dinner

Tony Forbes

Four members of a dinner club dine out four times a year, each time to celebrate the birthday of one of their number. At these events the agreement is that the birthday person's meal is free, the entire cost being met equitably by the other three members of the group. This year, however, in order to economize and because the relevant birthdays are close to each other, the third and fourth meals are combined into one. How should the bill be settled?

Solution 242.4 – Two sums

Prove that

$$\sum_{r=1}^{n} \binom{2n-r-1}{n-r} 2^r \;=\; 2^{2n-1}$$

and

$$\sum_{r=1}^{n} \binom{2n-r-1}{n-r} 2^r r \;=\; 2n\binom{2n-1}{n}.$$

Tommy Moorhouse

The first step in proving this result is a lemma.

Lemma 1

$$\sum_{k=n}^{2n-1} \binom{k-1}{k-n} 2^{-k} \;=\; \frac{1}{2}.$$

Proof This seems to be done most easily as follows. Denote the sum in the lemma by S_n. Then

$$\begin{aligned} S_{n+1} &= \sum_{k=n+1}^{2n+1} \binom{k-1}{k-(n+1)} 2^{-k} \\ &= \sum_{l=n}^{2n} \binom{l}{l-n} 2^{-(l+1)} \quad = \sum_{l=n}^{2n} \binom{l}{n} 2^{-(l+1)} & (1) \\ &= \sum_{l=n}^{2n} 2^{-(l+1)} \left\{ \binom{l-1}{l-n-1} + \binom{l-1}{l-n} \right\}, & (2) \end{aligned}$$

where (1) follows by renumbering ($l = k-1$) and (2) is Pascal's triangle identity. We will refer to the second version of the sum in (1) below.

Renumbering in the first term of (2) ($m = l-1$) gives (remembering that the $m = n-1$ binomial coefficient must vanish)

$$\begin{aligned} S_{n+1} &= \sum_{m=n}^{2n-1} 2^{-(m+2)} \binom{m}{n} + \sum_{l=n}^{2n} 2^{-(l+1)} \binom{l-1}{n-1} \\ &= \frac{1}{2}\left(S_{n+1} - 2^{-(2n+1)}\binom{2n}{n}\right) + \frac{1}{2}\left(S_n + 2^{-2n}\binom{2n-1}{n-1}\right). & (3) \end{aligned}$$

But

$$\binom{2n}{n} \;=\; 2\binom{2n-1}{n-1}$$

so the last terms in the brackets in (3) cancel and we find

$$S_{n+1} \;=\; \frac{1}{2}(S_{n+1}+S_n).$$

Thus $S_{n+1} = S_n$ for all $n > 0$ and, since $S_1 = 1/2$, the lemma is proved. □

Now we prove the result

$$\sum_{r=1}^{n} \binom{2n-r-1}{n-r} 2^r \;=\; 2^{2n-1}.$$

This follows by renumbering ($k = n - r$) to get

$$\sum_{k=0}^{n-1} \binom{n+k-1}{k} 2^{n-k}.$$

Now take 2^n outside the sum and let $l = n + k$. The sum becomes

$$2^n \sum_{k=0}^{n-1} 2^{-k} \binom{n-1+k}{k} \;=\; 2^n \sum_{l=n}^{2n-1} \binom{l-1}{l-n} 2^{n-l} \;=\; 2^{2n} \sum_{l=n}^{2n-1} \binom{l-1}{l-n} 2^{-l}.$$

By Lemma 1 we have

$$\sum_{r=1}^{n} \binom{2n-r-1}{n-r} 2^r \;=\; 2^{2n-1}.$$

The second identity can be proved along similar lines. First we prove another lemma.

Lemma 2 If

$$T_n \;=\; \sum_{k=n}^{2n-1} 2^{-k} k \binom{k-1}{k-n}$$

then

$$T_{n+1} \;=\; T_n + 1 - 2^{-(2n+1)} \binom{2n}{n}.$$

Proof

$$\begin{aligned} T_{n+1} \;&=\; \sum_{k=n+1}^{2n+1} \binom{k-1}{k-(n+1)} 2^{-k} k \qquad \left(= \sum_{k=n+1}^{2n+1} \binom{k-1}{n} 2^{-k} k \right) \\ &=\; \sum_{l=n}^{2n} 2^{-(l+1)} (l+1) \left\{ \binom{l-1}{n-1} + \binom{l-1}{n} \right\}. \end{aligned}$$

The first group of terms can be rewritten by expanding and renumbering

$$\frac{1}{2}\sum_{l=n}^{2n} 2^{-l} l \binom{l-1}{n-1} + \frac{1}{2}\sum_{l=n}^{2n} 2^{-l}\binom{l-1}{n-1}.$$

As before, we notice that this can be expressed in familiar terms using the result of the first part:

$$\frac{1}{2}\left(T_n + 2^{-2n}\cdot 2n\binom{2n-1}{n-1}\right) + \frac{1}{2}\left(\frac{1}{2} + 2^{-2n}\binom{2n-1}{n-1}\right).$$

Similarly, the second group of terms is

$$\frac{1}{2}\sum_{l=n}^{2n} 2^{-l} l \binom{l-1}{n} + \frac{1}{2}\sum_{l=n}^{2n} 2^{-l}\binom{l-1}{n}$$

and treating this as we did the first group we find it to be

$$\frac{1}{2}\left(T_{n+1} - 2^{-(2n+1)}(2n+1)\binom{2n}{n}\right) + \frac{1}{2}\left(\frac{1}{2} - 2^{-(2n+1)}\binom{2n}{n}\right).$$

Adding the groups of terms together and rearranging to get T_{n+1} on the left, we find a lot of cancellations and

$$T_{n+1} \;=\; T_n + 1 - 2^{-(2n+1)}\binom{2n}{n},$$

which proves the lemma. □

Now we let L_n denote the sum we are interested in:

$$\begin{aligned} L_n &= \sum_{r=1}^{n}\binom{2n-r-1}{n-r}2^r r \\ &= \sum_{l=n}^{2n-1}(2n-l)2^{2n-l}\binom{l-1}{l-n} \\ &= 2^{2n}\cdot 2n\cdot\frac{1}{2} - 2^{2n}T_n. \end{aligned}$$

Here we have renumbered (as will be becoming familiar!) and used the result of the first part. Now we prove the result by induction. First, $L_1 =$

$4(1-T_1)=2$ which agrees with the formula for $n=1$. Now

$$\begin{aligned}
L_{n+1} &= (n+1)2^{2(n+1)} - 2^{2(n+1)}T_{n+1} \\
&= (n+1)2^{2(n+1)} - 4\cdot 2^{2n}T_n - 2^{2(n+1)} + 2\binom{2n}{n} \qquad (4)\\
&= (n+1)2^{2(n+1)} - 4(n\cdot 2^{2n} - L_n) - 2^{2(n+1)} + 2\binom{2n}{n} \qquad (5)\\
&= 4L_n + 2\binom{2n}{n} \\
&= 4\cdot 2n\binom{2n-1}{n} + 4\binom{2n-1}{n} \\
&= 4(2n+1)\frac{(2n-1)!}{n!(n-1)!}.
\end{aligned}$$

Line (5) follows from (4) by assuming the result to hold for L_n. The final expression is just

$$2(n+1)\binom{2n+1}{n+1}$$

and so the induction step is established, proving that

$$\sum_{r=1}^{n}\binom{2n-r-1}{n-r}2^r r \;=\; 2n\binom{2n-1}{n}.$$

Problem 245.2 – Intersecting cylinders

Determine the volume of the intersection of the cylinders

$$x^2+y^2\le 1 \quad \text{and} \quad x^2+z^2\le 1.$$

Problem 245.3 – Power residues

Paul Barnett

Let q be a prime and let m be a positive integer. For which pairs (q,m) is it the case that $n^q \equiv n \pmod{m}$ for all n?

Solution 242.1 – Interesting integrals

Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx \;=\; \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\,dx \;=\; \frac{\pi}{e}.$$

Bryan Orman

There are many ways of evaluating these but the one I like uses the Residue Theorem (what else!) with the closed contour taken as the usual semicircle in the upper half plane and the two simple functions

$$\frac{e^{iz}}{z-i} \quad \text{and} \quad \frac{e^{iz}}{z+i}.$$

Their residues are simply e^{-1} and 0. They give, respectively, the sum of the two integrals to be $2\pi/e$ and the difference to be 0.

Tony Forbes

The following argument also works. We have seen that

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z-i}\,dx \;=\; \frac{2\pi i}{e} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{e^{iz}}{z+i}\,dx \;=\; 0.$$

Adding gives

$$\int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2+1}\,dx \;=\; \int_{-\infty}^{\infty} \frac{z(\cos z + i \sin z)}{z^2+1}\,dx \;=\; \frac{\pi i}{e}$$

and subtracting gives

$$\int_{-\infty}^{\infty} \frac{ie^{iz}}{z^2+1}\,dx \;=\; \int_{-\infty}^{\infty} \frac{(i\cos z + \sin z)}{z^2+1}\,dx \;=\; \frac{\pi i}{e}.$$

In each case we take the imaginary part to yield the second and first integrals respectively. Incidentally we have shown that

$$\int_{-\infty}^{\infty} \frac{z\cos z}{z^2+1}\,dx \;=\; \int_{-\infty}^{\infty} \frac{\sin z}{z^2+1}\,dx \;=\; 0.$$

Vincent Lynch

I did the M337 exam last month so this is right up my street. Let

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx.$$

There is a very useful theorem in the handbook of M337 on page 29 which I quote.

> Let p and q be polynomial functions such that:
> 1. The degree of q exceeds that of p by at least one;
> 2. Any poles of p/q on the real axis are simple.
>
> Then, if $k > 0$,
>
> $$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} e^{ikt}\,dt = 2\pi i S + \pi i T,$$
>
> where S is the sum of the residues of the function $z \mapsto \big(p(z)/q(z)\big)e^{ikz}$ at those poles in the upper half plane and T is the sum of the residues of the same function at those poles on the real axis.

All conditions are satisfied with $p(x) = 1$, $q(x) = x^2+1$ and $k = 1$; S is the residue of $\dfrac{e^{iz}}{z^2+1}$ at the point $z = i$. Using the g/h rule, S is the value of $\dfrac{g}{h'} = \dfrac{e^{iz}}{2z}$ at $z = i$, which is $\dfrac{e^{-1}}{2i}$, and $2\pi i S = \dfrac{\pi}{e}$. Then, since there are no poles on the real axis, $I = \mathrm{Re}(2\pi i S + \pi i T) = \pi/e$.

For the second integral, we require

$$\mathrm{Im}\int_{-\infty}^{\infty} \frac{z\cos z}{z^2+1}\,dz,$$

and using the same rules it is

$$\mathrm{Im}\left(2\pi i\left(\text{value of } \frac{ze^{iz}}{2z} \text{ at } z = i\right)\right) = \mathrm{Im}\left(2\pi i \cdot \frac{1}{2e}\right) = \frac{\pi}{e}.$$

I have n marbles. All but six are red. Similarly for orange, yellow, green, blue, indigo and violet. What is n?

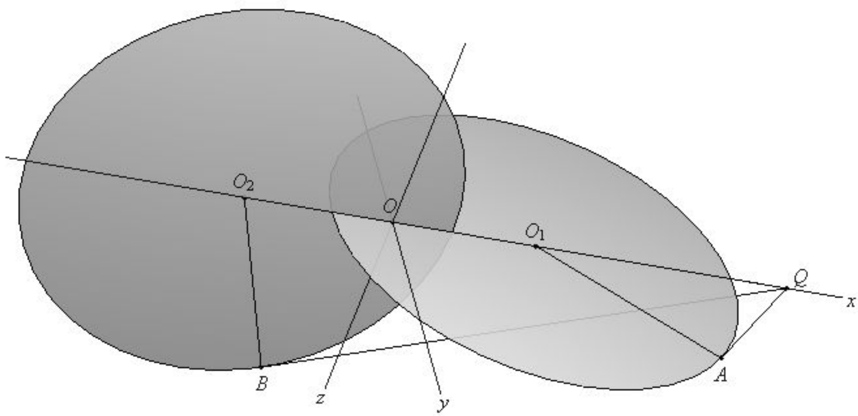
Two discs

Recall that in M500 **228** under the title 'Mathematics in the kitchen – VI' we discussed the construction of a remarkable object made from two orthogonal unit discs fixed together such that their centres are $\sqrt{2}$ radius units apart. We asserted that the thing will roll around on a perfectly flat kitchen worktop, with no unique stable position. In other words, the centre of gravity is always at the same height above the surface.

Dick Boardman

Consider a solid made up of two unit discs at right angles, with their centres $\sqrt{2}$ apart. Choose the x-axis to be the line joining the centres and the origin to be halfway between the centres. Let disc 1 be in the (x, y)-plane and let disc 2 be in the (x, z)-plane. Let $s = \sqrt{2}/2$ be half the distance between the centres. Then O_1, the centre of disc 1, will have coordinates $(s, 0, 0)$ and O_2, the centre of disc 2, will have coordinates $(-s, 0, 0)$.

Suppose the solid is resting on a flat table with disc 1 touching the table at point A and disc 2 touching the table at B. Let Q be the point where the x-axis meets the table and let the coordinates of Q be $(q, 0, 0)$, where q is a parameter to be varied. In the (x, y)-plane, the points O_1, A and Q form a right-angled triangle with side O_1A of length 1 and hypotenuse O_1Q of length $q - s$. Similarly, in the (x, z)-plane the points O_2, Q and B form a right-angled triangle with side O_2B of length 1 and hypotenuse O_2Q of length $q + s$.



Let $\phi = \angle O_1QA$ and $\theta = \angle O_2QB$. Then the coordinates of A and B are

$$A: \quad (s+\sin\phi,\ \cos\phi,\ 0) \quad = \quad \left(\frac{1}{q-s}+s,\ \frac{\sqrt{(q-s)^2-1}}{q-s},\ 0\right)$$

and

$$B: \quad (-s+\sin\theta,\ 0,\ \cos\theta) \quad = \quad \left(\frac{1}{q+s}-s,\ 0,\ \frac{\sqrt{(q+s)^2-1}}{q+s}\right).$$

Let the equation of the plane containing A, B and Q be $fx+gy+hz=1$. We find f, g and h by substituting the coordinates of A, B and Q into this equation and solving the three simultaneous equations obtained. The distance of the plane from the origin is $1/\sqrt{f^2+g^2+h^2}$. Thus

$$\begin{aligned} f &= \frac{1}{q}, \\ g &= \frac{q-s-\sin\phi}{q\cos\phi} = \frac{\sqrt{(q-s)^2-1}}{q}, \\ h &= \frac{q+s-\sin\theta}{q\cos\theta} = \frac{\sqrt{(q+s)^2-1}}{q}. \end{aligned}$$

It is easily verified that $(f,g,h)\cdot A = (f,g,h)\cdot B = (f,g,h)\cdot Q = 1$ and

$$f^2+g^2+h^2 \quad = \quad \frac{2q^2+2s^2-1}{q^2},$$

which is equal to 2 on substituting $s=\sqrt{2}/2$. Therefore the centre of gravity, O, is always at distance $\sqrt{2}/2$ from the table top, the plane defined by A, B and Q. For this method to work, A, B and Q must be distinct and not collinear. There are two orientations of the discs where this test fails. But then the calculations are much easier and can be left to the reader.

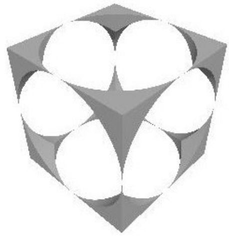
Problem 245.4 – GCSE question

Compute

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}+\sqrt{k-1}} \quad \text{and} \quad \sum_{k=1}^{n} \frac{(-1)^k}{\sqrt{k}-\sqrt{k-1}}.$$

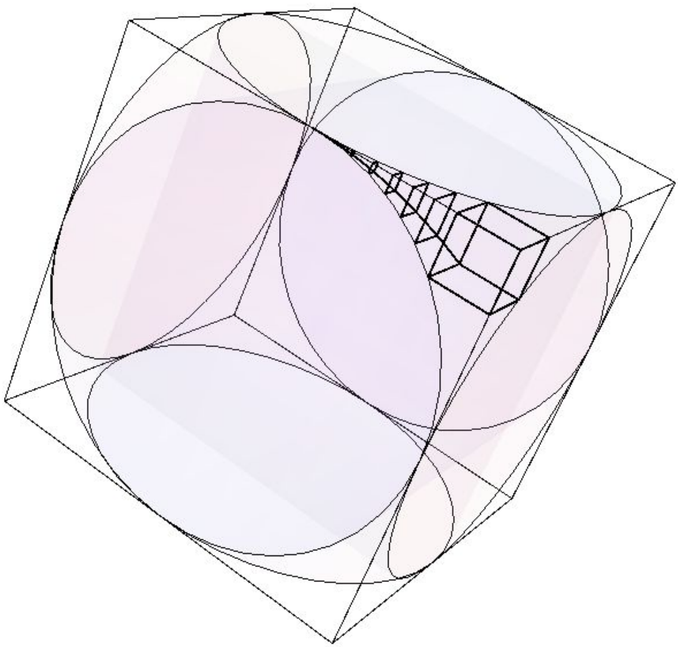
Solution 242.6 – Three cylinders

Start with a $1\,\mathrm{m}^3$ cube. Take out three mutually orthogonal cylinders of length $1\,\mathrm{m}$ and diameter $1\,\mathrm{m}$. What is the volume that remains? The cylinders should of course fit snugly inside the cube along its main axes, as suggested by the picture on the right.



Tamsin Forbes

Let the cube's vertices have coordinates $(\pm 1, \pm 1, \pm 1)$. So my cube is $2 \times 2 \times 2$ and I must remember to divide by eight to answer the actual problem.



Let the cylinders be

$$x^2 + y^2 \leq 1, \quad x^2 + z^2 \leq 1, \quad y^2 + z^2 \leq 1.$$

From the diagram it can be seen that the volume of the material left behind is of two types.

First consider the little cube in the corner. If its outer vertex has coordinates $(1, 1, 1)$, then its inner vertex, as a root of $x^2 + y^2 = x^2 + z^2 =$

$y^2 + z^2 = 1$, must have coordinates $(\sqrt{2}/2, \sqrt{2}/2, \sqrt{2}/2)$. Hence the volume of the cube is $(1 - \sqrt{2}/2)^3$.

The other contribution comes from the 24 squiggly bits in the half-edges of the main cube. Consider the square cross-section at distance x from the mid-point of the edge. Its outer vertex is at $(x, 1, 1)$ and we want to find the coordinates of the innermost vertex, which lies on the indicated curve that goes from $(0, 1, 1)$ to $(\sqrt{2}/2, \sqrt{2}/2, \sqrt{2}/2)$. But the coordinates of this vertex are (x, y, z), where y and z are given by $x^2 + y^2 = x^2 + z^2 = 1$. Hence $y = z = \sqrt{1 - x^2}$ and the length of the side of the little square is $1 - \sqrt{1 - x^2}$. We can now compute the volume of one of the half-edge pieces by integrating the area of the square from $x = 1$ to $x = \sqrt{2}/2$:

$$\int_1^{\sqrt{2}/2} \left(1 - \sqrt{1 - x^2}\right)^2 dx \;=\; \frac{11\sqrt{2}}{12} - \frac{1}{2} - \frac{\pi}{4}.$$

Multiplying by 24, adding the contribution from the eight little cubes, and finally dividing by eight gives

$$3\left(\frac{11\sqrt{2}}{12} - \frac{1}{2} - \frac{\pi}{4}\right) + \left(1 - \frac{\sqrt{2}}{2}\right)^3 \;=\; 1 + \sqrt{2} - \frac{3\pi}{4} \;\approx\; 0.0580191.$$

Tony Forbes

Working with a cube of volume 8, let $V_1 = 2\pi$ be the volume of a single cylinder. For $i = 2, 3$, let V_i be the volume of the intersection of i mutually orthogonal cylinders. In M500 **192** Dick Boardman and David Kerr calculated $V_3 = 8(2 - \sqrt{2})$. To get the other volume, V_2, we can perform an analysis similar to Dick's solution in M500 **192** to obtain

$$V_2 \;=\; 8 \int_0^1 \int_0^{\sqrt{1-z^2}} \sqrt{1 - u^2}\, du\, dz \;=\; \frac{16}{3}.$$

But from the rational nature of 16/3 I cannot help wondering if there is an easier way to get this result; so I shall set it as a problem. Applying the inclusion–exclusion principle gives

$$V \;=\; 8 - 3V_1 + 3V_2 - V_3 \;=\; 8 - 6\pi + 16 - 8(2 - \sqrt{2}) \;=\; 8 + 8\sqrt{2} - 6\pi$$

for the volume of remaining material—in agreement with Tamsin's solution.

Tracks

Ralph Hancock

Tony's question in 'Tracks' on page 13 [M500 **242**] is rather vaguely expressed. To get the first bit out of the way, if you are on the ground looking at the tank going past at 20 mph, you see the bottom of the track as stationary and the top as going forwards at 40 mph. Hardly worth asking.

But what does 'the best place to put the driving wheels' mean? Drive from either end will move the tracks, and the total friction acting over the whole length of the tracks will be the same. However, the Achilles heel of tanks is track breakage, and I assume that this is caused mostly by fatigue fractures in the links or link pins by repeated tension and release, and that fatigue breaks more tracks than sudden high stress does.

In this case, it's better to have the driving wheels at the back. The friction between undriven wheels and tracks is much greater at the bottom of the track where the ground is pressing the tracks against the wheels. Rear driving wheels pull the lower part of the tracks along steadily in an almost straight line, and the tracks return over the idler wheels at the top with relatively little friction or stress. This minimizes the stress on the tracks at the vulnerable point where they go around the front undriven wheels.

If the driving wheels are at the front, they are not actually pushing the tracks, since these are flexible and can't transmit a compressive force. They are pulling the tracks along the top edge, round the undriven rear wheels, and along the high-friction bottom run. The stress on the tracks will be highest where they go around the undriven rear wheels, much higher than anywhere in the other arrangement, and will cause a fatigue failure sooner.

All this assumes that the tank is travelling on a relatively flat surface. However, front drive has an advantage when the tank encounters an obstacle that brings the front edge of the tracks into contact with the ground and the tank has to lift its front over the obstacle, so that stress on the tracks is briefly at its highest level. Here, having a positive drive to the front of the tracks will reduce the stress at the front, where at that moment the tension is very high. But presumably this doesn't happen all that often, as opposed to the constant punishment that the tracks get when travelling from place to place.

Problem 245.5 – Numbers

Tony Forbes

You and your opponent play a game. You start by choosing a positive integer X_0. Thereafter your opponent and you take turns to choose positive integers X_1, X_2, ..., such that X_0,X_1, X_2, ... are distinct and $X_{n+1} = X_n - 2$, $X_n - 1$ or $X_n + 1$. If not possible, the player whose turn it is loses.

Assuming you both play with perfect intelligence, classify the starting numbers X_0 as either (i) you win, (ii) you lose, or (iii) draw (infinitely long game).

Here is an example. You choose 3. If he chooses 2 or 1, you respond with 1 or 2 respectively and win. So he chooses 4. But then you lose if you choose 2; so you go for 5 and force a draw. Hence $X_0 = 3$ is a draw.

Solution 241.2 – Irrational numbers

> If πe is irrational, prove that at most one of $\pi + e$, $\pi - e$, $\pi^2 + e^2$, $\pi^2 - e^2$ is rational.

Ian Adamson

The number e^2 is irrational since otherwise $x^2 - e^2$ is a polynomial with rational coefficients so e is algebraic, a contradiction.

At most one of $\{\pi + e,\ \pi - e\}$ and at most one of $\{\pi^2 + e^2,\ \pi^2 - e^2\}$ are rational by irrationality of e and e^2. (1)

Assume $\pi + e$ to be rational; then $\pi - e$ is irrational by (1) and $\pi^2 + e^2$ is irrational by the irrationality of πe. Also $(\pi + e)(\pi - e)$ is irrational; so $\pi^2 - e^2$ is irrational. Assume $\pi - e$ to be rational; then $\pi + e$ is irrational by (1), $\pi^2 + e^2$ is irrational by the irrationality of πe, and $\pi^2 - e^2$ is again irrational.

Assume $\pi^2 + e^2$ rational; then $\pi + e$ and $\pi - e$ are irrational since otherwise their squares are rational which contradicts irrationality of πe, and $\pi^2 - e^2$ is irrational by (1). Assume $\pi^2 - e^2$ rational; then $\pi + e$, $\pi - e$ are both rational or irrational, so are both irrational by (1), and $\pi^2 + e^2$ is irrational by (1).

Problem 245.6 – Quintic

Solve $x^4 + x^5 = e^6$.

Is piphobia just an irrational fear?

Problem 245.7 – Triangle division

Dick Boardman

Divide an equilateral triangle into three parts of the same shape such that

(i) the three pieces have the same size, or

(ii) the three pieces have different sizes, or

(iii) two pieces have the same size and one has a different size.

There are three problems to attack; one is easy, one moderately difficult and one fiendish.

Problem 245.8 – Four cylinders

Tony Forbes

Having seen how to compute the volume left behind when you remove three mutually orthogonal unit-diameter cylinders from a unit cube, I am now wondering what the solution is for the other four Platonic solids. There is no corresponding problem for the tetrahedron because it does not have sets of parallel faces. However, the octahedron, the dodecahedron and the icosahedron do. But let's do one at a time. So for this problem we tackle the next one up from the cube.

Take a solid regular octahedron of side length 1. For each pair (f_1, f_2) of opposite faces of the octahedron, remove all material from the finite solid cylinder bounded by the in-circles of the equilateral triangles f_1 and f_2. What is the volume that remains?

Problem 245.9 – Transcendental numbers

Tony Forbes

Suppose $\pi+e$ is rational. Let r and s be rational numbers with $(r, s) \neq (0, 0)$, $(1, 0)$, $(0, -1)$. Prove that

$$\frac{\pi^r}{e^s} + \frac{e^r}{\pi^s}$$

is transcendental. This is like Problem 241.2 – Irrational numbers, except that a different number is hypothetically rational. Observe that problems 241.2 and 245.9 are inconsistent—at least one of $\pi + e$ and πe must be transcendental.

The bank job

Jeremy Humphries

This is a money game show, the first edition of which ran for six consecutive nights in January on C4. You can look up general details on line. Essentially each of the first five nights produces a winner, who brings a bag of winnings to the pool for the final night. The pool was nearly half a million pounds. I watched a couple of the shows, including the last one.

On the final night the five played the standard game until three were eliminated. Then the final two were given a kind of 'prisoner's dilemma' scenario. Each had two suitcases, one of half the money and the other of trash. Each knew which was which of his own cases. The deal was that each would give one case to the opponent. If they both give money, then they share the pool equally (quarter of a million pounds each). If one gives trash and one gives money, then the scoundrel who has now got all the money keeps it all. If they both give trash then they both get nothing and the pool is shared without further ado among the three eliminated players (∼£160,000 each in this case).

Each finalist made a convincing and emotional statement to the other about how he was trusting the other to give money, and he himself would give money, so they would get half each. Then both gave the other trash, so that they ended up with nothing, and the three 'losers' got £160,000 each.

It seems to me that the two finalists are always likely to trash each other, so the best strategy is to lose deliberately on the final night, and hope consequently to get a one-third share of the pool. It was clear, or seemed to be, that nobody employed that strategy, but this was the first run of the show. Maybe that strategy was employed in future shows. Or something else. Tricky business, this game theory. We would be interested in readers' thoughts on the matter.

Problem 245.10 – Every other day

Tony Forbes

Can anyone come up with a really simple function F, say, that maps a date to either 0 or 1 such that $F(\text{today}) = 1 - F(\text{yesterday})$.

This is not just an academic exercise. Such a function will be very useful in those situations where the label on the packet says, 'Take 1 tablet every 2 days.'

Contents — M500 245 – April 2012

Front cover: Octahedron and four cylinders.