*

ISSN 1350-8539

M500 247

2

1448 07307 27259

4344 21921 81779

125 81493 74372 42812 88504 63819

15829 39848 23364 12158 73582 71559 28812 60379 53113 10917 17859

2505 69856 31259 43085 53751 66358 43781 90483 72646 25081 04712 46355 25215
93240 83917 13622 21259 85395 95893 80491 42499 29339

62 78525 28925 14157 85918 14469 48394 27454 87517 25159 66206 97526 04468
68657 49429 50413 84228 77932 20315 66337 01279 03461 42657 70942 49337 02083
93843 87194 01060 51278 68920 92328 27006 94331 52057 26124 55707 57543 89430
10283 27254 23242 27045 14419

3941 98798 07769 57426 09449 72998 42990 45036 65623 84516 09460 41473 73058
06652 56705 27458 70088 71215 58506 44647 11995 81110 61806 14995 18883 27034
67763 98976 08998 05558 38964 35358 58957 33252 85064 44412 31238 23157 16013
37854 66626 85451 37950 90749 06075 30886 92385 07249 86513 44528 47944 70438
20267 38513 57414 39912 22104 43666 44344 57301 16022 94548 11887 52036 56509
93365 71596 35089 98801 81991 65323 36033 76339 39355 60032 53381 70999 70768
04192 73686 10324 19616 57208 27412 80565 61732 15979

155 39269 24058 99940 69876 15624 22104 34718 90840 08959 61552 26866 32000
35282 59666 22033 58691 95174 44206 18400 31569 61418 81963 90784 19642 11092
59241 11492 65855 52892 99207 17060 46680 89761 32561 19273 03433 68244 72513
22161 15796 82705 46921 95176 47215 80818 28088 75589 15225 65192 12166 72110
23855 81060 79436 67206 43504 97600 77239 81390 38853 64375 27182 00201 50118
84548 45438 24332 13840 87836 87185 34193 55746 93009 30553 08916 18835 45830
63347 65588 09847 26773 44230 76385 65704 36275 53077 61183 02920 53906 84005
74749 88500 38865 20132 02985 25410 39266 64712 69492 52380 75192 86015 30719
79461 28996 49455 16306 14449 19974 54940 72060 79310 67408 49237 14837 64253
32909 82048 48064 08323 98679 81389 19739 24393 56567 08347 77721 02148 11092
30300 33209 03335 80586 10835 52666 58523 96357 70605 86370 52450 97080 83104
06141 74358 83111 87267 19688 47445 87925 96692 31860 54219 77854 75370 37595
40613 30920 47653 75266 11328 82659 26859 71528 90581 49636 08456 07152 08393
18541 55714 56756 17926 59139

The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, send a stamped, addressed envelope to Diana Maxwell, below.

Advice to authors We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation.

The Ancient Egyptian number system

Sebastian Hayes

The Egyptian Number System and Egyptian mathematics in general have traditionally held a 'poor cousin' status compared to the more celebrated Babylonian and Greek systems[1]. The Egyptian system is, like ours, a base ten system but arranged in ascending, rather than descending, order of size by our reckoning since the Egyptians, like the Arabs still do, wrote from right to left. It does not use positional notation but instead has a different picture-sign for each power: an upright stick for the unit, a bent stick for ten, a coiled rope for a hundred, a lotus for a thousand, a snake for ten thousand, a tadpole for a hundred thousand and a seated man holding up his hands in amazement for a million. There was no sign for zero and, since the system, like the Greek, did not depend on place value, none was needed.

Numerals less than ten are repeated upright strokes, as many strokes as there are objects; i.e. | | | is our '3' and if we have 'three hundreds' the hundred sign will be repeated[2]. This means a simple glance will show the approximate size of the quantity being represented, since a collection in the thousands will have thousand ideograms which are readily distinguishable from hundred or ten thousand ideograms. With our positional notation, one has to look closely to distinguish between say 10000 and 1000 especially since it has become the fashion to leave out the comma for the thousand. Admittedly, Egyptian notation does make it difficult to distinguish between quantities less than ten but some Egyptian texts get round this by arranging the units or other repeated signs in two rows with a maximum of five in the top row, e.g. writing 'seven' as | | | | | with | | in the row below. This makes instantaneous assessment much easier and in effect means having a sub-base of five.

Multiplication is incredibly easy using Egyptian methods since it depends wholly on doubling and then adding rather like Russian multiplication (see my earlier article in M500 **243**).

Division for the Egyptians was simply multiplication in reverse. Instead of dividing our 134 by 7 the Egyptian scribe would lay out his powers of two on one side and 7 repeatedly doubled on the other.

1	7
2	14
4	28
8	56
16	112

He would stop here because he would see that the next doubling would take him well beyond his goal 134. He would then get as close as he could to 134 using the entries on the right-hand side, namely 112, 14 and 7. He would then add the corresponding powers of two, namely 16, 2 and 1 giving as quotient 19 with remainder 1 since the nearest combination of 7s fell one short.

Such a procedure is, once you have got the hang of it, no lengthier than our 'long division' which children at school (and beyond) often have a lot of difficulty with. In the Egyptian system the powers of 2 form a framework within which every number can be situated, and with practice one can juggle them around to pin down any number in one's head, at least to a fair degree of accuracy. The method works, of course, only because every number can be expressed as a combination of powers of 2. The Egyptian scribes must have realized this though they do not say so specifically[3].

What of fractions? Here the Egyptians ran into difficulties because they did not have our stroke notation 1/4, 5/6 &c. They got round the problem by using reciprocals of numbers, noting this by a bar placed over the top. One advantage of this was that it was not necessary to invent any new signs for quantities less than the unit, and this seems to have been an important consideration. However, it meant reducing every proper fraction to a sum of unit fractions—with the important exception of 2/3, which had a special sign of its own. For some reason, a scribe performing a calculation would not write down a reciprocal more than once in succession: he would not for example put a bar over the sign for our 7 and repeat this twice to express our 3/7 but transform it into a series of unit fractions. It is not clear why the Egyptian scribes did this.

Conceptually, the Egyptian scribes were apparently unable to make the giant leap in thought involved in extending the base-10 system backwards to represent quantities smaller than 1. This said, the scribes showed quite remarkable ingenuity and fluency in reducing proper fractions to brief lists of unit fractions, never, according to Gilling, *Mathematics in the Time of the Pharaohs*, using more than four terms. This implies that every proper fraction can be reduced to at most four unit fractions. I have been unable to prove this or test the claim extensively with the very limited computer power I have at my disposal. (Proving this could be an interesting problem for M500 readers.)

Suppose a rational number a/b with a, b positive integers, $a < b$ and $a \neq 1$ (since otherwise we already have a 'series' of unit fractions). We can also assume without loss of generality that $\gcd(a, b) = 1$, i.e. a and b do not have a common divisor other than unity. Now,

$$\frac{a}{b} = \frac{1}{n} + \frac{na-b}{nb}; \quad \text{e.g.} \quad \frac{7}{24} = \frac{1}{5} + \frac{35-24}{5 \cdot 24} = \frac{1}{5} + \frac{11}{120}.$$

This leaves 11/120 to be reduced and it might seem that the series would carry on indefinitely with smaller and smaller terms. However, if we take $n = 4$ we obtain

$$\frac{7}{24} = \frac{1}{4} + \frac{28-24}{4 \cdot 24} = \frac{1}{4} + \frac{1}{24}.$$

This turns out to be a lucky fluke but it enables us to deduce the rule

$$\frac{a}{b} = \frac{1}{n} + \frac{1}{b} \ \text{ if } \ an - b = n, \quad \text{or} \quad \frac{b}{a-1} = n.$$

i.e. b is an exact multiple of $a - 1$. By looking out for such cases and ones where the numerator can be split to provide such a case, we can often spot ways to rapidly reduce a fraction. For example,

$$\frac{7}{15} = \frac{2+5}{15} = \frac{1}{3} + \frac{2}{15}.$$

Here, we can use the fact that $16 = 2 \cdot 8$ i.e. $b + 1 = ma$, which means that $an - b = 1$ or $m = (b + 1)/a$. So

$$\frac{2}{15} = \frac{1}{8} + \frac{1}{120}.$$

Using these two cases, $b = n(a-1)$ and $ma = (b+1)$ we can reduce a certain percentage of fractions immediately. For example, if we have a numerator of 2 and an odd denominator, the fraction will reduce at once.

An extension of these two rules enables us to at least reduce the number of terms in the expansion. With $n = (b + r)/a$ we have

$$\frac{a}{(b+r)-r} = \frac{1}{n} + \frac{r}{bn}; \quad \text{e.g.} \quad \frac{7}{17} = \frac{1}{3} + \frac{4}{51}.$$

Leonardo of Pisa mentions these methods in his *Liber Abaci* though I stumbled on them myself messing around with unit fractions. He also cites the artifice of splitting the numerator into a sum of divisors of the denominator where this is possible. Thus

$$\frac{7}{12} = \frac{1}{3} + \frac{1}{4}.$$

There is, as a last resort, one foolproof general method (also mentioned by Fibonacci), which consists in setting $n = \lceil b/a \rceil$, i.e. the first integer $> b/a$. Since b/a is situated between $\lceil b/a \rceil - 1$ and $\lceil b/a \rceil$,

$$1 > \left\lceil \frac{b}{a} \right\rceil - \frac{b}{a} = \frac{r}{a}, \quad \text{or} \quad a > \left\lceil \frac{b}{a} \right\rceil a - b = r.$$

Thus, the numerator of the non-unit fraction in an expansion is always less than the numerator of the original fraction. Continuing in this way, we choose $m = \lceil nb/r \rceil$:

$$\frac{r}{nb} = \frac{1}{m} + \frac{mr - nb}{mnb}.$$

We see that the denominators rapidly increase since

$$b < nb < mnb < tmnb < \ldots.$$

Also, setting $mr - nb = c$, and so on, $a > r > c > d > \ldots$.

Since all numerators are positive integers (or zero) this means that we will eventually arrive at either a numerator of 1 or 0 (in the case where $b = na$) This method, also mentioned by Leonardo of Pisa, does not, however, always provide us with the shortest expansion.

In the problems dealt with in the Rhind papyrus we find that the Egyptian scribes used somewhat arcane criteria for selecting their expansions. Brevity is one consideration but it is not the only or always the principal one. The scribe showed a marked preference for smaller numbers (i.e. smaller denominators) though he would accept 'a slightly larger first number, if it will greatly reduce the last number' (Gillings). Also, according to Gillings, even numbers are always preferred to odd numbers.

A computer has been put to work evaluating a range of possible unit-fraction expressions and the scribe's choice of expansions, according to his criteria, has been shown to be, at least in the vast majority of cases, optimal. Fractions were probably first invented by the Egyptians in order to equitably divide up portions of bread and beer since temple personnel were remunerated in kind, gold and silver being reserved for large-scale State expenditure. However, many of the complicated unit fraction expansions could not possibly have had any practical use: the Egyptian scribes, like Leonardo of Pisa (Fibonacci), seem to have become fascinated by unit fractions for their own sake as indeed I am in danger of becoming myself to judge by the time I have spent on them!

Notes

1. But McLeish in his book, *Number*, rightly praises it highly. Egyptian mathematics is a good deal simpler than Greek or Babylonian and this is precisely what makes it more user-friendly. The whole of Egyptian calculation was essentially based on only three elementary procedures: doubling, distinguishing between odd and even, adding. One would only need to know the two-times table and become fluent in unit fractions (partly doubtless through using tables) to carry out quite complicated calculations. This simplicity strikes me as a plus rather than a minus.

2. This is the so-called hieroglyphic numeral system used for State documents and tomb paintings. For everyday calculations, scribes used the much faster 'hieratic' system which, because signs were run together in freehand script, they often got modified in the process. The hieratic numerals may thus be described as 'semi-ciphered'. For example, 6 in hieratic remains as two rows of three 'sticks' | | | but '7' has become a single character. I have experimented with hieratic Egyptian numerals and, with one or two natural simplifications (natural to me) I find that writing down numbers and performing calculations the Egyptian way is hardly more cumbersome than with our present system.

3. Had we Europeans taken the Egyptian system as our starting point rather than the Greek, we would have realized much sooner that all numbers could be written in base 2 using just two signs. Leibnitz seems to have been the first to see this but we had to wait until the mid-twentieth century and the advent of computers before this insight was put to any practical use.

Problem 247.1 – 39/163

Applying the 'last resort' method as explained in Sebastian Hayes's article (see page 4) gives

$$\begin{aligned}\frac{39}{163} &= \frac{1}{5}+\frac{1}{26}+\frac{1}{1247}+\frac{1}{2935993}\\ &\quad+\frac{1}{11082924787499}+\frac{1}{28660618430582834379078 7504}\\ &\quad+\frac{1}{1232146573235196678590495661410921941724665869330 37520}.\end{aligned}$$

Find a simpler expression for 39/163 as a sum of distinct unit-numerator fractions.

Solution 241.3 – Four integrals

For brevity, write $\alpha = \log(1+\sqrt{2})$ and $\beta = \log(2+\sqrt{3})$. Let r be a real number. Show that

$$\int_0^{\alpha} \cosh^r x\,dx \;=\; \int_0^{\pi/4} \frac{dx}{\cos^{r+1} x}.$$

Let s be a non-negative real number. Show that

$$\int_0^{\beta} \sinh^s x\,dx \;=\; \int_0^{\pi/3} \frac{\sin^s x}{\cos^{s+1} x}\,dx.$$

Basil Thompson

We have

$$\begin{aligned}\int_0^{\alpha} \cosh^r x\,dx &= \left[\frac{1}{r}\sinh x \cosh^{r-1} x\right]_0^{\alpha} + \frac{r-1}{r}\int_0^{\alpha} \cosh^{r-2} x\,dx \\ &= \frac{(\sqrt{2})^{r-1}}{r} + \frac{r-1}{r}\int_0^{\alpha} \cosh^{r-2} x\,dx.\end{aligned}$$

Also

$$\begin{aligned}\int_0^{\pi/4} \frac{dx}{\cos^{r+1} x} &= \left[\frac{\sin x}{r\cos^r x}\right]_0^{\pi/4} + \frac{r-1}{r}\int_0^{\pi/4} \frac{dx}{\cos^{r-1} x} \\ &= \frac{(\sqrt{2})^{r-1}}{r} + \frac{r-1}{r}\int_0^{\pi/4} \frac{dx}{\cos^{r-1} x}.\end{aligned}$$

Both integrals are of the same form; hence we can conclude that

$$\int_0^{\alpha} \cosh^r x\,dx \;=\; \int_0^{\pi/4} \frac{dx}{\cos^{r+1} x} \tag{1}$$

for all integers if we can show that (1) is true for $r = 0$ and $r = 1$. For $r = 0$, we have

$$\int_0^{\alpha} dx - \int_0^{\pi/4} \frac{dx}{\cos x} \;=\; \log(1+\sqrt{2}) - 2\,\mathrm{arctanh}\left(\tan\frac{\pi}{8}\right) \;=\; 0,$$

and when $r = 1$ the previous analysis shows that both integrals are obviously equal to 1.

For the other equality,

$$\begin{aligned}\int_0^\beta \sinh^s x\,dx &= \left[\frac{\sinh^{s-1} x \cosh x}{s}\right]_0^\beta - \frac{s-1}{s}\int_0^\beta \sinh^{s-2} x\,dx \\ &= \frac{2(\sqrt{3})^{s-1}}{s} - \frac{s-1}{s}\int_0^\beta \sinh^{s-2} x\,dx\end{aligned}$$

and

$$\begin{aligned}\int_0^{\pi/3} \frac{\sinh^s x}{\cos^{s+1}}\,dx &= \left[\frac{\sin^{s-1} x}{s\cos^s x}\right]_0^{\pi/3} - \frac{s-1}{s}\int_0^{\pi/3} \frac{\sinh^{s-2} x}{\cos^{s-1} x}\,dx \\ &= \frac{2(\sqrt{3})^{s-1}}{s} - \frac{s-1}{s}\int_0^{\pi/3} \frac{\sinh^{s-2} x}{\cos^{s-1} x}\,dx\end{aligned}$$

when s is non-negative. The same observation as before applies;

$$\int_0^\beta \sinh^s x dx = \int_0^{\pi/3} \frac{\sinh^s x}{\cos^{s+1}}\,dx$$

for integer $s \geq 0$ since it obviously holds for $s = 1$, and when $s = 0$ we have

$$\int_0^\beta dx - \int_0^{\pi/3} \frac{dx}{\cos x} = \log(2+\sqrt{3}) - 2\,\mathrm{arctanh}\left(\tan\frac{\pi}{6}\right) = 0.$$

Steve Moon

For the first inequality, make the substitution $\cosh x = 1/(\cos y)$, which is easily checked to be 'allowable'. When $x = 0$, $y = \arccos(1/(\cosh 0)) = 0$ and when $x = \alpha$, we have

$$\cosh\log(1+\sqrt{2}) = \frac{1}{2}\left(1+\sqrt{2}+\frac{1}{1+\sqrt{2}}\right) = \sqrt{2}$$

and hence $y = \arccos(1/\sqrt{2}) = \pi/4$.

Differentiating $\cosh x = 1/(\cos y)$ implicitly gives

$$\sinh x = \frac{\sin y}{\cos^2 y}\frac{dy}{dx}$$

and hence

$$dx = \frac{\sin y}{\cos^2 y}\frac{dy}{\sinh x} = \frac{\sin y}{\cos^2 y}\frac{dy}{\sqrt{\cosh^2 x - 1}} = \frac{\sin y}{\cos y}\frac{dy}{\sqrt{1-\cos y}} = \frac{dy}{\cos y}.$$

Now substituting for $\cosh^r xdx$ in the original integral gives

$$\int_0^{\alpha} \cosh^r x\, dx \;=\; \int_0^{\pi/4} \frac{1}{\cos^r y}\frac{dy}{\cos y} \;=\; \int_0^{\pi/4} \frac{dy}{\cos^{r+1} y}.$$

For the second equality, we use the substitution $\sinh x = \tan y$. But $x = 0 \Rightarrow \sinh x = 0 \Rightarrow \tan y = 0$ and so $y = 0$. When $x = \beta$,

$$\sinh x \;=\; \frac{1}{2}\left(2+\sqrt{3}+\frac{1}{2+\sqrt{3}}\right) \;=\; \sqrt{3}.$$

So $x = \beta \Rightarrow y = \arctan\sqrt{3} = \pi/3$. Again, it is easily checked that the substitition is valid.

Differentiating $\sinh x = \tan y$ with respect to x gives $\cosh x = \sec^2 y\, dy/dx$. Therefore

$$dx \;=\; \frac{\sec^2 y}{\cosh x}\, dy \;=\; \frac{\sec^2 y}{\sqrt{1+\sinh^2 x}}\, dy \;=\; \frac{\sec^2 y}{\sqrt{1+\tan^2 y}}\, dy \;=\; \sec y\, dy$$

and

$$\int_0^{\beta} \sinh^s x\, dx \;=\; \int_0^{\pi/3} (\tan^s y)(\sec y)\, dy \;=\; \int_0^{\pi/3} \frac{\sin^s y}{\cos^{s+1} y}\, dy.$$

Tony Forbes

The problem actually originated from an observation of my daughter Tamsin. Suppose you want to compute $\int_0^1 (x^2+1)^t\, dx$ by substitution. You seem to have have a choice. You can put $x = \sinh y$, in which case the integral becomes $\int_0^{\alpha} \cosh^{2t+1} y\, dy$, or you can put $x = \tan y$ to get $\int_0^{\pi/4} \sec^{2t+2}\, dy$.

The second equality came from a similar integral, $\int_1^2 (x^2-1)^t\, dx$. Here there is a choice between $x = \cosh y$ and $x = \sec y$ to get $\int_0^{\beta} \sinh^{2t+1} y\, dy$ and $\int_0^{\pi/3} \tan^{2t+1} y \sec y\, dy$ respectively.

Solution 243.8 – Pentadecagon

> Devise a nice ruler-and-compasses construction for the regular 15-gon.

Tony Forbes

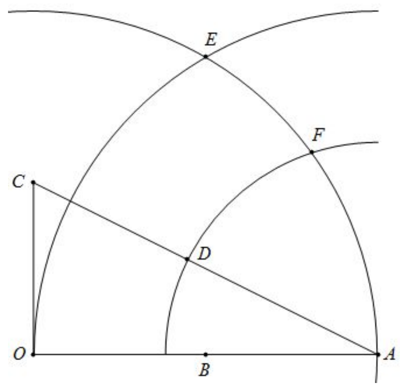
No (valid) examples were submitted but I have had a small amount of feedback, which I shall attempt to answer.

Firstly, the regular pentadecagon *is* constructable with ruler and compasses. In Section VII of *Disquisitiones Arithmeticae* (or at least in the English translation thereof by A. A. Clarke), Gauss proved that for prime p, a regular p-gon is constructible if $p = 2^{2^k} + 1$ for some integer k; that is, if p is a *Fermat prime*. Also he claimed to have a rigorous proof that a construction is impossible for any n divisible by an odd non-Fermat prime or by the square of any odd prime. Moreover, one has the fact (surely known to Euclid) that if an a-gon and a b-gon are constructible, then so is an $\mathrm{lcm}(a, b)$-gon. In particular, 3, 5, $15 = \mathrm{lcm}(3, 5)$ and 17 are possible but 7, 9, 11 and 13 are not.

Secondly, the only attempt I received used the edge of a regular 16-gon inscribed in a circle of radius 16 to inscribe a regular 15-gon in a circle of radius 15. Unfortunately this works only when $16 \sin 11.25° = 15 \sin 12°$.

Thirdly, I'm sorry, I don't know what 'nice' means. After a bit of doodling I came up with this construction. See if you can find a nicer one.

Draw a circle with centre O and radius OA of length 1. The other points are constructed in alphabetical order with $|CD| = |OC| = |OB| = |AB| = 1/2$, $\angle AOC = 90°$, $|AE| = 1$ and $|AF| = |AD|$. The required edge is EF.



Clearly $|AC| = \sqrt{5}/2$. So $|AF| = |AD| = (\sqrt{5} - 1)/2 = 2 \sin \pi/10$, and therefore $\angle AOF = 2\pi/10 = 36°$. Obviously $\angle AOE = 60°$. Hence $\angle EOF = 24°$.

Dog: I heard you'd lost your voice.

Cat: No—I've just got a ν μ.

Solution 242.1 – Interesting integrals

Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx \;=\; \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\,dx \;=\; \frac{\pi}{e},$$

two interesting integrals both of which evaluate to that possibly rational number π/e.

Basil Thompson

The solution I found involves integrating from 0 to ∞ and the more general case:

$$\int_0^{\infty} \frac{\cos rx}{x^2+a^2}\,dx \;=\; \frac{\pi}{2a}\,e^{-ar}, \qquad a>0,\ r \ge 0. \tag{1}$$

The result is due to Laplace, published in 1811.

Let

$$I \;=\; \int_0^{\infty} \frac{\cos rx}{x^2+a^2}\,dx$$

and introduce the variable z by replacing $1/(x^2+a^2)$ thus:

$$\frac{1}{x^2+a^2} \;=\; \int_0^{\infty} 2z\,e^{-(a^2+x^2)z^2}\,dz.$$

Then

$$\begin{aligned}
I &= \int_0^{\infty}\int_0^{\infty} (\cos rx)\,2z\,e^{-(a^2+x^2)z^2}\,dx\,dz \\
&= \int_0^{\infty} 2z\,e^{-a^2z^2}\int_0^{\infty} e^{-x^2z^2}(\cos rx)\,dx\,dz \\
&= \int_0^{\infty} 2z\,e^{-a^2z^2}\left(\frac{\sqrt{\pi}}{2z}\,e^{-r^2/(4z^2)}\right)dz \\
&= \sqrt{\pi}\int_0^{\infty} e^{-(a^2z^2+r^2/(4z^2))}dz \;=\; \sqrt{\pi}\cdot\frac{\sqrt{\pi}}{2a}\,e^{-ar} \;=\; \frac{\pi}{2a}\,e^{-ar},
\end{aligned}$$

thus proving (1). Putting $a=r=1$ gives

$$\int_0^{\infty} \frac{\cos x}{x^2+1}\,dx \;=\; \frac{\pi}{2e}; \qquad \text{hence} \qquad \int_{\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx \;=\; \frac{\pi}{e}$$

as $(\cos x)/(1+x^2)$ is symmetrical about the y axis.

To find $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\, dx$ we go back to the general case. Differentiating (1) under the integral sign and the right-hand side with respect to r,

$$\int_0^{\infty} \frac{-x \sin rx}{x^2+a^2}\, dx \;=\; -\frac{\pi}{2}\, e^{-ar}.$$

Putting $a = r = 1$ then gives

$$\int_0^{\infty} \frac{x \sin x}{x^2+1}\, dx \;=\; \frac{\pi}{2e}; \qquad \text{hence} \qquad \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\, dx \;=\; \frac{\pi}{e}$$

since, as before, the integrand is symmetric about the y axis.

Tommy Moorhouse

One approach to this problem is to integrate over a suitable contour in the complex plane and deduce the value of the real integral. In this case we integrate

$$\int_C \frac{e^{itz} dz}{1+z^2}.$$

The contour C runs along the real axis from $-R$ to R, where $R > 1$, then back to $-R$ along the semicircle of radius R in the upper half plane. The integral is $2\pi i$ times the residue at $z = i$ which, by the cover-up rule, is πe^{-t}. Taking $t = 1$ gives πe^{-1}. To get

$$\int_{-\infty}^{\infty} \frac{x(\sin x) dx}{1+x^2}$$

we can differentiate the expression above with respect to t. The details (and questions about interchanging the order of differentiation and integration) are left to the reader.

The next stage is to estimate the integral over the semicircular arc. On the arc take $z = Re^{i\theta}$ so that $|e^{izt} dz| = Re^{-Rt \sin\theta} < R$ for suitably large R. Hence the integral over the semicircle of radius R tends to zero as $R \to \infty$.

Problem 247.2 – Integral

For $n > 1$, show that

$$\int_0^{\infty} \frac{dx}{x^n+1} \;=\; \frac{\pi}{n\,(\sin \pi/n)}.$$

Solution 244.4 – Another product

Show that $$\prod_{k=1}^{\infty} \frac{k^2}{k^2+1} \;=\; \Gamma(1-i)\Gamma(1+i) \;=\; |i!|^2 \;=\; \frac{\pi}{\sinh \pi}.$$

Tommy Moorhouse

The infinite product can be expressed as

$$P_\infty \;=\; \prod_{k=1}^{\infty} \frac{k}{k+i}.\frac{k}{k-i}$$

and in this form we see that it can be written (see for example Whittaker and Watson, *Modern Analysis*, section 12.13)

$$P_\infty \;=\; \Gamma(i)\Gamma(-i) \;=\; \frac{-\pi}{i \sin i\pi} \;=\; \frac{\pi}{\sinh \pi}.$$

So far this is just an exercise in looking up function definitions and properties, but we can find another expression that can be extended to the finite sum. Consider the product written as

$$\prod_{k=1}^{\infty} \frac{1}{1+1/k^2}.$$

Taking logarithms we find

$$\log P_\infty \;=\; -\sum_{k=1}^{\infty} \log\left(1+\frac{1}{k^2}\right).$$

Expanding the logarithms in the sum using

$$\log(1+x) \;=\; x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$

we find

$$\log P_\infty \;=\; \sum_{k=1}^{\infty}\left(-\frac{1}{k^2} + \frac{1}{2k^4} - \frac{1}{3k^6} + \cdots + \frac{(-1)^n}{n\,k^{2n}} + \cdots\right).$$

The sum (not considering issues of convergence) can be expressed as a sum of zeta functions, and we have

$$P_\infty \;=\; \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n \zeta(2n)}{n}\right).$$

This hints at how we might explore the sums P_m formed by taking the range of k from 1 to m. These sums are twice those denoted $g(n)$ in M500 **244**. Now

$$\sum_{k=1}^{m} k^{-s} \;=\; \zeta(s) - \zeta(s, m+1),$$

where

$$\zeta(s, N) \;=\; \sum_{k=0}^{\infty} (N+k)^{-s}$$

is the Hurwitz zeta function (note the limits in the sum). We can now write

$$P_m \;=\; \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n(\zeta(2n) - \zeta(2n, m+1))}{n}\right).$$

The series converges very slowly but it gives the correct values (although the rational expressions cannot be readily obtained this way).

We have also shown in a roundabout way that

$$\sum_{n=1}^{\infty} \frac{(-1)^n \zeta(2n)}{n} \;=\; \log\left(\frac{\pi}{\sinh \pi}\right).$$

Solution 241.7 – Multiplicative function

> Let f be an increasing, multiplicative function that maps positive integers to positive integers. Suppose also $f(2) = 2$. Show that f must be the identity function.

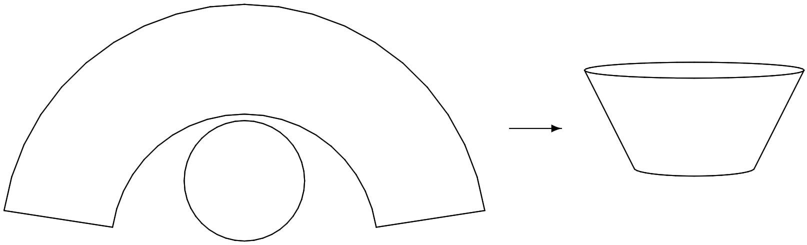
David Marchant

Since f is multiplicative, $f(1) = 1$, and we are given $f(2) = 2$. Assume that f is not the identity function, so let n be the first integer such that $f(n) \neq n$. Since $f(n-1) = n-1$, we must have $f(n) > n$. As f is multiplicative, n must be prime, and since $f(2) = 2$, n must also be odd. Now consider $f(n+1)$; $n+1$ must be even, and hence composite, and so by multiplicity $f(n+1) = n+1$. But now we have $f(n) > n$ and $f(n) < f(n+1) = n+1$, which is a contradiction, hence f must be the identity function.

That's six sentences. We previously printed proofs by David Wild (11 sentences, M500 246) and myself (14 sentences, M500 243). — TF

Solution 242.5 – Coffee cup

A coffee cup in the form of a truncated cone closed at its thin end is made from plastic sheeting. There are two parts. A section of an annulus of radii r and R, $R > r$, subtending an angle of θ, and a disc of radius $r\theta/(2\pi)$. Assuming that the total surface area is 1 unit, choose the parameters to maximize the volume of the cup.



Steve Moon

With the parameters as in the statement of the problem, the volume of the cup is the volume of a cone of radius $b = R\theta/(2\pi)$ and height $H = \sqrt{R^2 - b^2}$ minus the volume of a cone of radius $a = r\theta/(2\pi)$ and height $h = \sqrt{r^2 - a^2}$. Thus

$$V \;=\; \frac{\pi}{3}\,b^2 H - \frac{\pi}{3}\,a^2 h \;=\; \frac{\theta^2(R^3 - r^3)}{12\pi}\sqrt{1 - \frac{\theta^2}{4\pi^2}}. \tag{1}$$

And for the surface area we have

$$A \;=\; \pi(R^2 - r^2)\frac{\theta}{2\pi} + \pi a^2 \;=\; \frac{R^2\theta}{2} + \frac{r^2\theta^2}{4\pi} - \frac{r^2\theta}{2} \;=\; 1. \tag{2}$$

We need to maximize V given by (1) subject to the constraint on A given by (2). The method is Lagrange's undetermined multipliers (λ).

Differentiate (1) and (2) with respect to r, R and θ and combine using λ. Differentiating with respect to R:

$$\frac{\theta^2 R^2}{4\pi}\sqrt{1 - \frac{\theta^2}{4\pi^2}} - \lambda R\theta \;=\; 0$$

and hence

$$\frac{\theta R}{4\pi}\sqrt{1 - \frac{\theta^2}{4\pi^2}} \;=\; \lambda. \tag{3}$$

Differentiating with respect to r:

$$\frac{\theta r}{4\pi}\sqrt{1-\frac{\theta^2}{4\pi^2}}+\lambda\left(\frac{\theta}{2\pi}-1\right) \;=\; 0. \tag{4}$$

Differentiating with respect to θ:

$$\frac{R^3-r^3}{12\pi}\cdot\frac{2\theta\left(1-\dfrac{\theta^2}{4\pi^2}\right)-\dfrac{\theta^3}{4\pi^2}}{\sqrt{1-\dfrac{\theta^2}{4\pi^2}}}-\lambda\left(\frac{R^2}{2}-\frac{r^2}{2}+\frac{r^2\theta}{2\pi}\right) \;=\; 0. \tag{5}$$

Eliminating λ between (3) and (4),

$$\frac{\theta r}{4\pi}\sqrt{1-\frac{\theta^2}{4\pi^2}}+\frac{\theta R}{4\pi}\sqrt{1-\frac{\theta^2}{4\pi^2}}\left(\frac{\theta}{2\pi}-1\right)r \;=\; 0,$$

and simplifying gives

$$r+R\left(\frac{\theta}{2\pi}-1\right)=0, \quad \text{or} \quad \frac{r}{R}=1-\frac{\theta}{2\pi}, \quad \text{or} \quad R-r=\frac{R\theta}{2\pi}. \tag{6}$$

Now eliminate λ between (3) and (5) to obtain

$$\frac{R^3-r^3}{12\pi}\left(2\theta-\frac{\theta^3}{2\pi^2}-\frac{\theta^3}{4\pi^2}\right)-\frac{\theta R}{4\pi}\left(1-\frac{\theta^2}{4\pi^2}\right)\left(\frac{R^2-r^2}{2}+\frac{r^2\theta}{2\pi}\right) \;=\; 0,$$

and after a bit of work during which we make use of $R\theta/(2\pi)=R-r$ from (6) this simplifies to

$$2\left(\frac{r}{R}\right)^2+\left(\frac{3\theta^2}{4\pi^2}-1\right)\frac{r}{R}+\frac{3\theta^2}{4\pi^2}-1 \;=\; 0.$$

Solve the quadratic and take the positive root to get

$$\frac{r}{R} \;=\; \frac{1}{4}\left(1-\frac{3\theta^2}{4\pi^2}\right)\left(1+\sqrt{\frac{36\pi^2-3\theta^2}{4\pi^2-3\theta^2}}\right)$$

and recall from (6) that $r/R=1-\theta/(2\pi)$. Therefore

$$1-\frac{\theta}{2\pi} \;=\; \frac{1}{4}\left(1-\frac{3\theta^2}{4\pi^2}\right)\left(1+\sqrt{\frac{36\pi^2-3\theta^2}{4\pi^2-3\theta^2}}\right).$$

Put $k = \theta/\pi$ and simplify to get

$$12 - 8k + 3k^2 \;=\; \sqrt{(36 - 3k^2)(4 - 3k^2)}.$$

Therefore

$$(12 - 8k + 3k^2)^2 \;=\; (36 - 3k^2)(4 - 3k^2),$$

and luckily this apparent quartic in k actually reduces to a quadratic:

$$3k^2 - 16k + 12 \;=\; 0$$

with solution $k = \bigl(8 \pm 2\sqrt{7}\bigr)/3$. But $\theta < 2\pi \Rightarrow k < 2$; so we take the negative root. Hence

$$\theta \;=\; \frac{2\pi}{3}\bigl(4 - \sqrt{7}\bigr).$$

So now we have from (6)

$$\frac{r}{R} \;=\; \frac{\sqrt{7} - 1}{3}.$$

Using the constraint equation (2) in the form

$$\frac{R^2\theta}{2}\left(1 + \frac{r^2}{R^2} \cdot \frac{\theta}{2\pi} - \frac{r^2}{R^2}\right) \;=\; 1,$$

and substituting for r/R and θ we have

$$\frac{R^2}{2} \cdot \frac{\bigl(8 - 2\sqrt{7}\bigr)\pi}{3}\left(1 + \left(\frac{\sqrt{7} - 1}{3}\right)^2\left(\frac{8 - 2\sqrt{7}}{6}\right) - \left(\frac{\sqrt{7} - 1}{3}\right)^2\right) \;=\; 1.$$

Multiplying by $81/\pi$ and cancelling the 2 on the left-hand side gives

$$R^2\bigl(4 - \sqrt{7}\bigr)\left(27 + \bigl(\sqrt{7} - 1\bigr)^2\bigl(4 - \sqrt{7}\bigr) - 3\bigl(\sqrt{7} - 1\bigr)^2\right) \;=\; \frac{81}{\pi},$$

and after simplification,

$$R^2\bigl(4 - \sqrt{7}\bigr)\bigl(49 - 10\sqrt{7}\bigr) \;=\; \frac{81}{\pi}.$$

Therefore

$$R^2 \;=\; \frac{81}{\bigl(4 - \sqrt{7}\bigr)\bigl(49 - 10\sqrt{7}\bigr)\pi} \;=\; \frac{81}{\bigl(266 - 89\sqrt{7}\bigr)\pi} \;=\; \frac{266 + 89\sqrt{7}}{189\pi}$$

and

$$R \;=\; \frac{1}{3}\sqrt{\frac{266+89\sqrt{7}}{21\pi}}.$$

So for surface area $A = 1$, the volume of the coffee cup is maximized for

$$\begin{aligned}
\theta &= \frac{2\pi}{3}\,(4-\sqrt{7}) \;\approx\; 2.8363 \;\approx\; 162.51^\circ, \\
R &= \frac{1}{3}\sqrt{\frac{266+89\sqrt{7}}{21\pi}} \;\approx\; 0.91900, \\
r &= \frac{\sqrt{7}-1}{9}\sqrt{\frac{266+89\sqrt{7}}{21\pi}} \;\approx\; 0.50415,
\end{aligned}$$

giving $r/R \approx 0.54858$. As a check, you can plug the expressions for θ, R and r into equation (2) to confirm that they yield $A = 1$ exactly. The values for θ and r/R which I measured from the diagram lead me to conclude that they are in the right area. It's harder to check r and R individually without determining the scaling for A. Finally we have this expression for the maximum volume:

$$V_{\max} \;=\; \frac{4-\sqrt{7}}{81}\sqrt{\frac{60+42\sqrt{7}}{\pi}} \;\approx\; 0.12339.$$

Tony Forbes

I might as well use the rest of this page for a brief explanation of the method in case you haven't seen it before. The problem is to maximize $V(\theta, r, R)$ subject to the constraint $A(\theta, r, R) = 1$. We introduce a new variable λ, the *Lagrangian multiplier*, and consider the *Lagrangian function*

$$\Lambda(\theta, r, R) \;=\; V(\theta, r, R) - \lambda\,(A(\theta, r, R) - 1).$$

The theory says that if $V(\theta, r, R)$ is a maximum subject to $A(\theta, r, R) = 1$, then there exists a λ such that (θ, r, R) is a stationary point of $\Lambda(\theta, r, R)$. We find the stationary points in the usual way by setting

$$\frac{\partial \Lambda}{\partial \theta} \;=\; \frac{\partial \Lambda}{\partial r} \;=\; \frac{\partial \Lambda}{\partial R} \;=\; 0$$

being careful to select the one that is relevant to the problem.

Observe, by the way, that the length of the cup's side is the same as the radius of the open end.

Two times five equals ten revisited

Tony Forbes

Let n and x be positive integers and let $k = n - x^2$. Define positive integer sequences X_i and N_i by

$$\begin{aligned} X_1 &= x, & \\ X_2 &= x+1, & \\ N_i &= X_i^2 + k, & i = 1, 2, \ldots, \\ X_{j+1} &= N_j - X_j + 1, & j = 2, 3, \ldots. \end{aligned}$$

As explained in Bryan Orman's article [1], the numbers N_i have the remarkable property that not only does each one satisfy $N_i = X_i^2 + k$ but for any positive integer r the product $N_1 N_2 \ldots N_r$ has the same form: square + k.

Take for instance $n = 2$ and $x = 1$. Then $k = 1$ and after computing the first few elements of the sequence N_i,

$$2, 5, 17, 197, 33857, 1133938277, 1285739650972396817, \ldots,$$

we see (as in [1]) that $2 \times 5 = 10 = 3^2 + 1$, $2 \times 5 \times 17 = 170 = 13^2 + 1$, $2 \times 5 \times 17 \times 197 = 33490 = 183^2 + 1, \ldots$. If we put $x = 2$ (with n still equal to 2), then $k = -2$ and we obtain the other sequence in [1],

$$2, 7, 23, 359, 116279, 13441851719, 180680260806215679959, \ldots,$$

with a similar property: $2 \times 7 = 14 = 4^2 - 2$, $2 \times 7 \times 23 = 322 = 18^2 - 2$, etc. Furthermore, the first five numbers in each sequence are prime but not the sixth. Bryan's challenge to readers (at least to this particular reader) implicit in the concluding remarks of [1] was to find n and x such that the sequence N_i generated by these parameters begins with at least six primes.

Let us fix $n = 2$ from now on (otherwise at least one of N_1 and N_2 will be non-prime). We can now regard the N_i as functions of just the variable x, and it makes sense to write them as $N_i(x)$. Clearly $N_1(x) = 2$ and from the defining recursion we obtain these polynomials:

$$\begin{aligned} N_2(x) &= 2x + 3, \\ N_3(x) &= 6x + 11, \\ N_4(x) &= 24x^2 + 90x + 83, \\ N_5(x) &= 576x^4 + 4080x^3 + 10824x^2 + 12750x + 5627, \end{aligned}$$

but thereafter they get horribly complicated with the degree doubling at each further step.

Our task is to find a positive integer x which makes $N_i(x)$ prime for $i = 1, 2, \ldots, r$, say. As we have seen, we obtain five primes with $x = 1$ or $x = 2$. To get six primes we can write a fairly crude program that checks the polynomials $N_i(1)$, $N_i(2)$, ..., $i = 2, 3, \ldots, 6$. Unless my computer has made a mistake the first occurrence is at $x = 16850$, $k = -2839\,22498$, and you can verify that the first six numbers in the sequence $N_i(16850)$, namely 2, 33703, 101111, 68156 56583, 46452 02611 70219 83127 and 21577 90729 74329 81321 20726 73807 93111 82311, are prime. For seven primes, we have to go a little further, to $x = 4\,67453$ ($k = -21\,85123\,07207$).

The crude program doesn't work very well for eight primes. So instead we go for a more sophisticated approach. We make a long list, $\mathcal{L}$, of values of x to test. We notice that $N_2(3t)$ is divisible by 3; so we remove all multiples of 3 from $\mathcal{L}$. Similarly, $N_2(5t+1)$ and $N_3(5t+4)$ are divisible by 5; so we remove numbers congruent to 1 or 4 (mod 5) from $\mathcal{L}$. In general, for each prime q up to some limit, $q_{\max}$, we determine those residue classes $r \pmod q$ such that $N_2(r)N_3(r)\ldots N_8(r) \equiv 0 \pmod q$ and remove them from $\mathcal{L}$. We are sieving $\mathcal{L}$ by the primes $q = 3, 5, \ldots, q_{\max}$. The survivors from the sieve are checked using a standard test: if $2^N \equiv 2 \pmod N$, then N is probably prime. The test is not watertight—for example, composite Mersenne and Fermat numbers are probably prime—but it is good enough for our purpose so long as it is backed up by some rigorous primality proof.

The amazing thing is that we succeed; $x = 2891\,01265$ (giving $k = -83\,57954\,14246\,00223$) produces 8 primes. But when $x = 724\,03653\,63628$ ($k = -5\,24228\,90598\,82402\,06653\,22382$) we get *nine* primes. Obviously you want to see them written out in full, so I have put the primes on the cover of this magazine. Proving the primality of $N_9(x)$ was not a trivial exercise.

[1] Bryan Orman, Two times five equals ten, M500 **246**, 10–13.

Problem 247.3 – Balls

Cannon balls are stacked in the usual square pyramid structure, which is stable so long as there is some device to prevent the bottom layer from dispersing. What is remarkable about this arrangement is that the number of balls in each layer is a square. What is even more remarkable is that when there are 24 layers the total number of balls is also a square. Furthermore, the total can sometimes be a square even if the pyramid is truncated. This suggests a problem. When is it possible to stack a square number of cannon balls to make a truncated square pyramid?

Letters

Biscuits

This problem [Problem 240.1 – two tins of biscuits. There are two tins, each containing $n > 0$ biscuits. Take a biscuit from a tin chosen at random. Keep doing this until one tin is empty. What is the expected number of biscuits that remain in the other tin?] is a variant of one that is quite well known under the name 'Banach's matchbox problem' — there is an article in *Wikipedia*. Banach's problem is to find the probability that there are exactly r biscuits in the other tin (or rather matches in the other box).

Ken Greatrix's solution does not take account of the tins being chosen at random, which would mean that his solution could be simplified slightly by taking $p = q = 0.5$.

In my favourite book, *An Introduction to Probability Theory and Its Applications*, William Feller gives a numerical example. He also generalizes the problem to $p \neq q$ and applies this to the game of table tennis. Feller also pours cold water on the assumption that Banach was responsible for the problem. He says that Banach inspired the problem only to the extent that he smoked and kept a box of matches in his right pocket and another in his left pocket.

Roger Dennis

How strange that Ken Greatrix [M500 **245**] should suppose that one would use a Poisson process to calculate the number of biscuits remaining in the tin. Anyone can see that that would only work for tins of sardines.

Of course numbers of biscuits do not have to be integers, on account of crumbs. And I am sure that some kinds, such as Bourbons, can exist in irrational quantities because of the squidgy nature of the filling.

Ralph Hancock

Problem 247.4 – Modular equation

Suppose $0 < q < 1$ and let

$$u = \frac{(1+q)(1+q^3)(1+q^5)\ldots}{\sqrt[24]{64q}}, \qquad v = \frac{(1+q^5)(1+q^{15})(1+q^{25})\ldots}{\sqrt[24]{64q^5}}.$$

Show that

$$\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3 = 2\left(u^2v^2 - \frac{1}{u^2v^2}\right).$$

Problem 247.5 – Sums of powers

Tony Forbes

One of the exciting things you can do in elementary arithmetic is discover a pair of n^{th} powers of non-integer rationals that add up to an integer. For instance if you were to add $\left(1\frac{8}{17}\right)^4$ and $\left(8\frac{12}{17}\right)^4$, you not unexpectedly get a fraction with a large denominator, 480175841/83521. But change the second term slightly and the surprising result is $\left(1\frac{8}{17}\right)^4 + \left(8\frac{13}{17}\right)^4 = 5906$, an integer. Even more surprising is the pair $\left(15288\frac{3}{5}\right)^{14}$ and $\left(3224\frac{4}{5}\right)^{14}$ found by Seiji Tomita in 2009. However, after a bit of experimentation one realizes that there is a simple method for generating infinitely many examples. A problem is suggested.

Let a and b be integers greater than 1. Determine those integers $n \geq b$ for which

$$\left(a^b + \frac{1}{a}\right)^n + \left(a^b - 1 + \frac{a-1}{a}\right)^n \ \in \ \mathbb{Z}.$$

M500 Winter Weekend 2013

Join with fellow mathematicians for a weekend of fun. If you want a fantastic weekend and are interested in things mathematical, then this is for you, accessible to anyone who has studied mathematics—even if you are just starting. The **thirty-second M500 Society Winter Weekend** will be held at

Florence Boot Hall, **Nottingham University**

Friday 4$^{\text{th}}$ – Sunday 6$^{\text{th}}$ January 2013.

The theme is to be decided. Cost: £195 to M500 members, £200 to non-members. You can obtain a booking form from the M500 site.

http://www.m500.org.uk/winter/booking.pdf

If you have no access to the internet, send a stamped addressed envelope to

Diana Maxwell

Please note that the address has changed from last year.

We will have the usual extras. On Friday we will be running a pub quiz with Valuable Prizes, and for the ceilidh on Saturday night we urge you to bring your favourite musical instrument (and your voice). Hope to see you there.

Front cover: nine primes—see pages 18–19.