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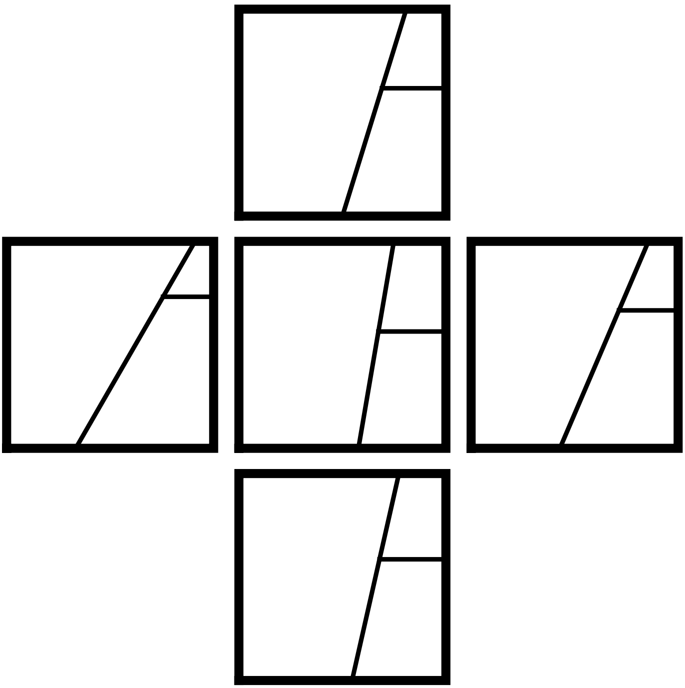
# M500 252



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# Matrix functions and polynomials

## Tommy Moorhouse

**Introduction** We will examine some of the ideas behind functions of matrices, and in particular those functions defined by series. Such functions occur for instance in quantum physics.

**Functions of matrices** Most physics texts are fairly relaxed about introducing functions of matrices defined by series. The convergence of such series is not as straightforward to deal with as the convergence of ordinary series, and divergences can lead to nonsensical conclusions, but we will make sure that all our functions are well behaved.

We will first take a function $f$ defined by a series (with possibly complex coefficients) convergent for $x$ in a suitable range:

$$f(x) \;=\; a_0 + a_1 x + \cdots + a_k x^k + \cdots .$$

Given a matrix $B$ we would like to define

$$f(B) \;=\; a_0 \mathbb{I} + a_1 B + \cdots + a_k B^k + \cdots$$

with $\mathbb{I}$ being the identity matrix. This series may converge for specific matrices $B$, but we should recognize that difficulties could arise for some matrices. As a simple example, take $B = \mathrm{diag}(x, y)$. Then it is not hard to see that $f(B)$ is the diagonal matrix $\mathrm{diag}(f(x), f(y))$. An immediate generalization, which will not be pursued here, is to allow constant matrix coefficients, when terms such as $A_i B^i C_i$ (with $A_i$ and $C_i$ being constant matrices—the labels are not vector or matrix indices) must be considered.

**Special cases** Now we will consider certain special cases where a matrix satisfies a polynomial equation. First we consider matrices $T$ satisfying $T^N = \mathbb{I}$, and we initially take $N$ to be prime. In this case we have

$$f(T) \;=\; a_0 \mathbb{I} + a_1 T + \cdots + a_{N-1} T^{N-1} + a_N \mathbb{I} + \cdots \;=\; \sum_{k=0}^{N-1} T^k \sum_{n=0}^{\infty} a_{k+Nn}.$$

With a little work we can express the inner sum more simply in terms of functions of roots of unity. Consider the root of unity $\zeta : \zeta^N = 1$. This is a complex number, not unity, also satisfying $1 + \zeta + \cdots + \zeta^{N-1} = 0$. (Can you see why?) Now

$$\begin{aligned} f(1) &= a_0 + a_1 + \cdots + a_k + \cdots\,, \\ f(\zeta) &= a_0 + a_1\zeta + \cdots + a_k\zeta^k + \cdots\,, \\ f(\zeta^2) &= a_0 + a_1\zeta^2 + \cdots + a_k\zeta^{2k} + \cdots\,, \\ \cdots &, \\ f(\zeta^{N-1}) &= a_0 + a_1\zeta^{N-1} + \cdots + a_k\zeta^{k(N-1)} + \cdots\,. \end{aligned}$$

The powers of $\zeta$ are to be reduced modulo $N$. Now we form the sums (making use of the equation satisfied by $\zeta$)

$$\rho_n(\zeta) = \frac{1}{N}\sum_{k=0}^{N-1} \zeta^{nk} f(\zeta^k)$$

so that

$$\rho_0(\zeta) = \frac{f(1) + f(\zeta) + f(\zeta^2) + \cdots + f(\zeta^{N-1})}{N} = a_0 + a_N + \cdots + a_{kN} + \cdots$$

and

$$\begin{aligned} \rho_1(\zeta) &= \frac{f(1) + \zeta f(\zeta) + \zeta^2 f(\zeta^2) + \cdots + \zeta^{N-1} f(\zeta^{N-1})}{N} \\ &= a_1 + a_{1+N} + \cdots + a_{1+kN} + \cdots\,, \end{aligned}$$

for example.

The case $T^N = a$, where $a$ is real, is solved by

$$\rho_n(\zeta) = \frac{1}{N\alpha^n}\sum_{k=0}^{N-1} \zeta^{nk} f(\alpha\zeta^k),$$

where $\alpha$ is a real positive number such that $\alpha^N = |a|$.

We now claim that

$$f(T) = \sum_{n=0}^{N-1} \rho_n(\zeta) T^n.$$

The proof is straightforward using our previous results (it helps to try a small value of $N$ at first, and write everything out longhand).

**An example** One of the simplest nontrivial examples, which is actually straightforward to calculate explicitly from the series, involves

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Clearly $T^2 = \mathbb{I}$; so we can take $N = 2$, above, and see that $\zeta = -1$. Suppose we want to find $\exp(i\theta T)$. We have $\rho_0(-1) = (e^{i\theta} + e^{-i\theta})/2 = \cos(\theta)$ and $\rho_1(-1) = (e^{i\theta} - e^{-i\theta})/2 = i\sin(\theta)$. Thus

$$\exp(i\theta T) = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}.$$

Note that $\det(\exp(i\theta T)) = 1$. We'll reproduce this result using a different method below. Matrices similar to this occur in the description of spin in quantum theory (Pauli matrices).

**A generalization** We can also consider the situation when $T$ satisfies a more general polynomial equation of degree $N$. A simple example is the quadratic

$$T^2 + bT + c\mathbb{I} = 0.$$

We can write out longhand the powers of $T$ in terms of $T$ and $\mathbb{I}$, and we soon spot that

$$T^k = (\mathbb{I},\, T)\begin{pmatrix} 0 & -c\mathbb{I} \\ \mathbb{I} & -b\mathbb{I} \end{pmatrix}^k \begin{pmatrix} \mathbb{I} \\ 0 \end{pmatrix},$$

where the matrix multiplication involves the multiplication of the blocks. We will often leave out the matrix $\mathbb{I}$, its presence being 'understood' when needed. A quick check shows that

$$\begin{pmatrix} 0 & -c \\ 1 & -b \end{pmatrix}^2 + b\begin{pmatrix} 0 & -c \\ 1 & -b \end{pmatrix} + c\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0.$$

We will use $T_0$ for both this two-by-two matrix and its four-by-four block version. It should be clear from the context which version is being used. Now, if $Q$ is any invertible matrix and $T$ satisfies $T^2 + bT + c\mathbb{I} = 0$ then $QTQ^{-1}$ satisfies the same equation. In many cases of interest we can diagonalize $T$ to find a diagonal matrix $D$ satisfying $D^2 + bD + c\mathbb{I} = 0$. Let $Q$ be the diagonalizing matrix.

Given a function $f(T)$ it is now easy to see that

$$\begin{aligned} f(T) &= \sum_{k=0}^{\infty} a_k \,(\mathbb{I},\, T) \begin{pmatrix} 0 & -c \\ 1 & -b \end{pmatrix}^k \begin{pmatrix} \mathbb{I} \\ 0 \end{pmatrix} \\ &= (\mathbb{I},\, T)\, f(T_0) \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix} \\ &= (\mathbb{I},\, T)\, Q f(D) Q^{-1} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}. \end{aligned}$$

The key advantage here is that $f(D)$ is the diagonal matrix found by applying $f$ to the corresponding elements of $D$. A familiar example will fix these ideas. Take $c = -1$, $b = 0$ in $T_0$. Then

$$T_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of $T_0$ are $\pm 1$ and the matrix $Q$ is given by

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad Q^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Let $f(x) = \exp(i\theta x)$, and we find that

$$\begin{aligned} \exp(i\theta T) &= (\mathbb{I},\, T) Q \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} Q^{-1} \begin{pmatrix} \mathbb{I} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}. \end{aligned}$$

The generalization to arbitrary polynomial matrix equations (with scalar coefficients) is surprisingly straightforward. Given the equation

$$T^N + a_{N-1} T^{N-1} + \cdots + a_k T^k + \cdots + a_0 \mathbb{I} = 0$$

there is always an $N \times N$ matrix

$$T_0 = \begin{pmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & -a_1 \\ 0 & 1 & \cdots & -a_2 \\ \cdots & \cdots & \cdots & -a_k \\ 0 & \cdots & 1 & -a_{N-1} \end{pmatrix}$$

satisfying the equation (and of course lots of other solutions of the form $QT_0Q^{-1}$). Also $\det(T_0) = \pm a_0$ and we take $a_0 \neq 0$, otherwise $T_0$ is a factor of the equation.

There are some questions the reader may like to think about. How many different solutions are there? (This may depend on the type of coefficients you allow.) Are all the $N \times N$ solutions of the form $QT_0Q^{-1}$? What does the diagonal matrix $D$ look like? What happens if the equation has repeated factors? Are there solutions of smaller dimension? It might also be interesting to consider how the series expansions generalize from the $2 \times 2$ case.

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# Problem 252.1 – Three pieces

## Dick Boardman

Divide a square into three pieces of the same shape but different sizes.

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# Problem 252.2 – Can

A tin can has radius $r$, height $h$ and surface area 1. Choose $r$ and $h$ to maximize its volume.

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# Problem 252.3 – Quadratic triangles

## Tommy Moorhouse

Suppose we have a quadratic curve $C$ given by $y = ax^2 + bx + c$ with discriminant $D = b^2 - 4ac > 0$. There are two points on the line $y = 0$ corresponding to the solutions $x = \alpha$ and $x = \beta$ with $\alpha < \beta$. For convenience we take $a > 0$, but this is not essential.

Show that the gradients of the tangent lines at $\alpha$ and at $\beta$ are equal and opposite, and that the gradient at $\beta$ is $\sqrt{D}$. Find the area of the triangle defined by the point of intersection of the tangent lines, say $\gamma$, and the points $\alpha$ and $\beta$, in terms of $D$ and $a$. Find the perimeter of this triangle in terms of $D$ and $a$. To check your results assign dimensions to $a, b$ and $c$ assuming that $x$ and $y$ have the dimensions of length, and confirm that the area and perimeter have the expected dimensions.

Deduce that all quadratic curves are symmetric about the line $x = (\alpha + \beta)/2$ and that quadratic curves are uniquely determined by the three corners $\alpha, \beta$ and $\gamma$, of their associated quadratic triangle.

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# Solution 235.3 – Odd pairs

Show that

$$S \;=\; \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} \frac{1}{(2i+1)^4} \frac{1}{(2j+1)^4} \;=\; \frac{\pi^8}{16 \cdot 8!}.$$

## Steve Moon

Consider the sum

$$T \;=\; \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(2i+1)^4} \frac{1}{(2j+1)^4}.$$

Then

$$T \;=\; \left( \sum_{i=0}^{\infty} \frac{1}{(2i+1)^4} \right)^2 \;=\; \left( \zeta(4) - \frac{\zeta(4)}{2^4} \right)^2 \;=\; \left( \frac{15}{16}\, \zeta(4) \right)^2 \;=\; \frac{225}{256}\, \zeta(4)^2.$$

On the other hand,

$$\begin{aligned} T \;&=\; \sum_{i=0}^{\infty} \sum_{j=0,\; j \neq i}^{\infty} \frac{1}{(2i+1)^4} \frac{1}{(2j+1)^4} + \sum_{i=0}^{\infty} \frac{1}{(2i+1)^8} \\ &=\; 2 \sum_{i=0}^{\infty} \sum_{j>i} \frac{1}{(2i+1)^4} \frac{1}{(2j+1)^4} + \sum_{i=0}^{\infty} \frac{1}{(2i+1)^8} \\ &=\; 2S + \zeta(8) - \frac{\zeta(8)}{2^8} \end{aligned}$$

and hence

$$S \;=\; \frac{1}{2} \left( T - \frac{2^8 - 1}{2^8}\, \zeta(8) \right) \;=\; \frac{1}{2} \left( \frac{225}{256}\, \zeta(4)^2 - \frac{255}{256}\, \zeta(8) \right).$$

Now, for positive integer $k$, we have

$$\zeta(2k) \;=\; (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!}\, B_{2k},$$

where $B_{2k}$ is the $2k^{\text{th}}$ Bernoulli number. See, for example, T. Apostol, *Introduction to Analytic Number Theory*, Theorem 12.17. Also $B_4 = B_8 = -1/30$. Therefore

$$S \;=\; \frac{1}{2} \left( \frac{225}{256} \cdot \left( \frac{(2\pi)^4}{2 \cdot 4!} \cdot \frac{1}{30} \right)^2 - \frac{255}{256} \cdot \frac{(2\pi)^8}{2 \cdot 8!} \cdot \frac{1}{30} \right) \;=\; \frac{\pi^8}{16 \cdot 8!}.$$

## Tony Forbes

Obviously Steve Moon's analysis generalizes to yield an interesting formula, which holds for any positive integer $n$:

$$\begin{aligned} S_{2n} &= \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} \frac{1}{(2i+1)^{2n}} \frac{1}{(2j+1)^{2n}} \\ &= \frac{1}{2}\left(\left(\frac{2^{2n}-1}{2^{2n}}\right)^2 \zeta(2n)^2 - \left(\frac{2^{4n}-1}{2^{4n}}\right)\zeta(4n)\right) \\ &= \frac{\pi^{4n}}{16(4n)!}\left(2\big(2^{2n}-1\big)^2 \binom{4n}{2n} B_{2n}^2 - 4\big(2^{4n}-1\big)|B_{4n}|\right) \\ &= \frac{\pi^{4n}}{16(4n)!}\, U_{2n}, \end{aligned}$$

say. Whilst playing about with this expression using MATHEMATICA I noticed with a certain amount of amazement that the factor $U_{2n}$ is always an integer, at least for the values of $n$ I tested. Thus $U_2 = 1$, as we have seen, and $U_6 = 12$, $U_4 = 577$, $U_8 = 577$, $U_{10} = 75640$, $U_{12} = 21914700$, $U_{14} = 12266203306$, $U_{16} = 12069208859489, \ldots$.

Then it occurred to me that I might actually be able to prove something. We use the theorem of von Staudt and Clausen, which states that

$$\textit{for positive integer } n, \quad B_{2n} + \sum_{p \text{ prime},\ (p-1)|2n} \frac{1}{p} \quad \textit{is an integer.}$$

A DIY proof can be found in Apostol's *Introduction to Analytic Number Theory*, Chapter 12, Exercise 12.

Now suppose $p$ is a prime factor of the denominator of $B_{2n}$. Then from the theorem we can deduce two things: (i) $p^2$ cannot divide the denominator of $B_{2n}$; and (ii) $(p-1)$ must divide $2n$. For odd $p$, it follows from (ii) that $2^{2n} \equiv 1 \pmod{p}$. Hence we can explain why the denominator of $U_{2n}$ contains no odd primes. Moreover (i) and (ii) imply that the 4 in $4B_{4n}$ cancels the 2 in the denominator of $B_{4n}$ and to eliminate the 2 in the denominator of $2B_{2n}^2$ we only need to recall the fact that $\binom{4n}{2n}$ is even.

Another consequence of the von Staudt–Clausen theorem is that

$$\frac{4(2n)!(2^{2n}-1)}{(2\pi)^{2n}}\, \zeta(2n)$$

is an integer for positive integer $n$.

# Solution 222.4 – Eleven

Find all solutions in positive integers $x$ and $n$ of $x^2 = 3^n - 11$.

## Tony Forbes

This problem seems to be rather tricky. Steve Moon sent me a contribution, which, on his own admission, amounted only to some observations to reduce the size of the possible solution space. However, I was then persuaded to attack the problem myself. Surprisingly I think I have managed to solve it, and without using anything more than elementary number theory.

If $n$ is even, $3^n - 11$ cannot be a square because it will be congruent to 2 (mod 4). So $n = 2m + 1$, say. Let $y = 3^m$. Then the equation becomes

$$3y^2 - x^2 = 11. \tag{1}$$

Forgetting about $y$ being a power of 3, we can completely solve (1) for positive integers to obtain two separate sequences of solutions,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^n \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^n \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad n = 0, 1, 2, \ldots, \tag{2}$$

namely

$(1, 2)$, $(8, 5)$, $(31, 18)$, $(116, 67)$, $(433, 250)$, $(1616, 933)$, $(6031, 3482)$, $(22508, 12995)$, $(84001, 48498)$, $(313496, 180997)$, $(1169983, 675490)$, $(4366436, 2520963)$, $(16295761, 9408362)$, $(60816608, 35112485)$, ...

and

$(4, 3)$, $(17, 10)$, $(64, 37)$, $(239, 138)$, $(892, 515)$, $(3329, 1922)$, $(12424, 7173)$, $(46367, 26770)$, $(173044, 99907)$, $(645809, 372858)$, $(2410192, 1391525)$, $(8994959, 5193242)$, $(33569644, 19381443)$, ....

To prove completeness we argue as follows. Suppose there exists a new solution, $(x, y)$, $x > 0$, not included in (2) and with $x$ as small as possible. By brute force we can show that $x$ is not very small; $x > 300$, say. We then compute

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - 3y \\ -x + 2y \end{bmatrix}$$

to show that $(2x - 3y, -x + 2y)$ is also a new solution. But, using (1) and the inequality $x > 300$, we have $x < \sqrt{3}y < x + 1$. Hence $0 < 2x - 3y < x$, contradicting the smallestness of $x$. Therefore the solution given by (2) really is complete.

Modulo 9 the $y$ values in (2) form these two sequences:

2, 5, 0, 4, 7, 6, 8, 8, 6, 7, 4, 0, 5, 2, 3, 1, 1, 3, 2, 5, 0, 4, 7, 6, 8, 8, ...

and

3, 1, 1, 3, 2, 5, 0, 4, 7, 6, 8, 8, 6, 7, 4, 0, 5, 2, 3, 1, 1, 3, 2, 5, 0, 4, ...

each repeating every 18 elements.

We now restrict $y$ to multiples of 9, which is reasonable because all sufficiently large powers of 3 also have this property. So we pick out those elements of the original sequences corresponding to where a 0 occurs in the reduced $y$ sequences. Therefore the only solutions occur in the two subsets of the original sequences defined by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{9n} \begin{bmatrix} 31 \\ 18 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{9n} \begin{bmatrix} 12424 \\ 7173 \end{bmatrix}, \quad n = 0, 1, 2, \ldots .$$

Working modulo 85 these solution sequences become

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}^{n} \begin{bmatrix} 31 \\ 18 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}^{n} \begin{bmatrix} 14 \\ 33 \end{bmatrix}, \quad n = 0, 1, 2, \ldots ,$$

namely

$$(31, 18),\ (71, 33),\ (31, 18),\ (71, 33),\ \ldots \qquad (\text{mod } 85),$$

and

$$(14, 33),\ (54, 18),\ (14, 33),\ (54, 18),\ \ldots \qquad (\text{mod } 85).$$

So $y$ takes just the two values 18 and 33 (mod 85). But the only powers of 3 modulo 85 are 1, 3, 9, 27, 81, 73, 49, 62, 16, 48, 59, 7, 21, 63, 19 and 57. Therefore (1) has no solution with $y = 3^m$, $m \geq 2$. The one possibility left is $y = 3$, which yields the only solution of the original equation,

$$4^2 \ = \ 3^3 - 11.$$

Where did 85 come from? If you calculate the 9th power of the basic two-by-two matrix, you get

$$H \ = \ \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{9} \ = \ \begin{bmatrix} 70226 & 3 \cdot 40545 \\ 40545 & 70226 \end{bmatrix},$$

and therefore we can expect much simplification in the computation of $H^n$ if we eliminate the top-right and bottom-left entries. So it makes good sense to work modulo some factor of $40545 = 3^2 \cdot 5 \cdot 17 \cdot 53$ where there are not too many powers of 3. I chose $85 = 5 \cdot 17$. It worked!

# Solution 250.9 – Product

Show that

$$\prod_{n=2}^{\infty} \frac{n^3+1}{n^3-1} \;=\; \frac{3}{2}.$$

## Tommy Moorhouse

We will consider two approaches to this problem, both involving factorization of the numerator and denominator. The first method is quite basic, but has the disadvantage that it cannot be extended to other products. We write

$$\prod_{n=2}^{\infty} \frac{n^3+1}{n^3-1} \;=\; \prod_{n=2}^{\infty} \frac{n+1}{n-1} \prod_{n=2}^{\infty} \frac{n^2-n+1}{(n+1)^2-(n+1)+1}$$

and, writing out the first few terms, we spot that those after the first few in each sub-product all cancel, and the product is 3/2.

A widely applicable alternative method can be used. It involves the identity

$$\prod_{n=1}^{\infty} \frac{(n-a_1)(n-a_2)\cdots(n-a_k)}{(n-b_1)(n-b_2)\cdots(n-b_k)} \;=\; \prod_{i=1}^{k} \frac{\Gamma(1-b_i)}{\Gamma(1-a_i)}$$

found in Whitaker and Watson. The 'missing' term in $n=1$ has to be dealt with and this will be considered below.

Let $\epsilon^3 = -1, \epsilon \neq -1$. We do not need to know the explicit form of the complex number $\epsilon$. If we take $\omega = \epsilon^2$ we have $\omega^3 = 1$. Then

$$\begin{aligned} n^3+1 &= (n-\epsilon)(n-\epsilon\omega)(n-\epsilon\omega^2), \\ n^3-1 &= (n-1)(n-\omega)(n-\omega^2). \end{aligned}$$

Thus

$$\prod_{n=2}^{\infty} \frac{n^3+1}{n^3-1} \;=\; \prod_{n=2}^{\infty} \frac{(n-\epsilon)(n-\epsilon\omega)(n-\epsilon\omega^2)}{(n-1)(n-\omega)(n-\omega^2)}.$$

To resolve the problematic $n=1$ term and apply the identity we write

$$\lim_{\delta\to 0} \delta \prod_{n=1}^{\infty} \frac{(n-\epsilon)(n-\epsilon\omega)(n-\epsilon\omega^2)}{(n-1+\delta)(n-\omega)(n-\omega^2)}$$

$$= \prod_{n=2}^{\infty} \frac{(n-\epsilon)(n-\epsilon\omega)(n-\epsilon\omega^2)}{(n-1)(n-\omega)(n-\omega^2)} \frac{(1-\epsilon)(1-\epsilon\omega)(1-\epsilon\omega^2)}{(1-\omega)(1-\omega^2)}.$$

The left-hand side is

$$\lim_{\delta\to 0} \frac{\delta\Gamma(\delta)\Gamma(1-\omega)\Gamma(1-\omega^2)}{\Gamma(1-\epsilon)\Gamma(1-\epsilon\omega)\Gamma(1-\epsilon\omega^2)}$$

and the limit $\delta\Gamma(\delta)$ is $\Gamma(1) = 1$. On the right-hand side the final group of factors evaluates to 2/3. Now $\epsilon\omega = -1, \epsilon\omega^2 = -\epsilon^2$ and we have

$$P \equiv \prod_{n=2}^{\infty} \frac{n^3+1}{n^3-1} = \frac{3}{2}\frac{\Gamma(1-\epsilon^2)\Gamma(1+\epsilon)}{\Gamma(1-\epsilon)\Gamma(1+\epsilon^2)\Gamma(2)}.$$

Happily we will not have to calculate the values of the $\Gamma$ function, relying instead on the well-known properties of the $\Gamma$ function, such as

$$z\Gamma(z) = \Gamma(z+1) \quad \text{and} \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

These properties and lots of others can be found in standard textbooks, with Whitaker and Watson being one of the best known.

We need the identity $\epsilon - \epsilon^2 = 1$. To prove this write $\epsilon - \epsilon^2 = C$, equation (1). Multiply (1) by $\epsilon$ to get a second equation (2) and add (1) and (2) to find $C = 1$. Writing out our expression for the product, noting that $\Gamma(2) = 1$, we find

$$P = \frac{3}{2}\frac{\Gamma(1-\epsilon^2)\Gamma(1+\epsilon)}{\Gamma(1-\epsilon)\Gamma(1+\epsilon^2)}.$$

Now $1+\epsilon = 2+\epsilon^2$ so

$$\Gamma(1+\epsilon) = \Gamma(2+\epsilon^2) = (1+\epsilon^2)\Gamma(1+\epsilon^2) = \epsilon^2(1+\epsilon^2)\Gamma(\epsilon^2) = -\Gamma(\epsilon^2).$$

This gives for the functions in the numerator of $P$

$$\Gamma(1-\epsilon^2)\Gamma(1+\epsilon) = -\Gamma(1-\epsilon^2)\Gamma(\epsilon^2) = \frac{-\pi}{\sin \pi\epsilon^2}$$

using the standard result. The denominator is simpler:

$$\Gamma(1-\epsilon)\Gamma(1+\epsilon^2) = \Gamma(1-\epsilon)\Gamma(\epsilon) = \frac{\pi}{\sin \pi\epsilon}.$$

Now we note that $\sin \pi\epsilon = \sin \pi(1+\epsilon^2) = -\sin \pi\epsilon^2$. Putting this all together we have

$$P = \frac{3 \sin \pi\epsilon^2}{2 \sin \pi\epsilon^2} = \frac{3}{2}.$$

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**Reference** E. T. Whitaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press (4th Ed., 1992 reprint).

# Solution 250.1 – Quadratic sum

Show that

$$\sum_{n=0}^{\infty} \frac{1}{an^2+n+1} \;=\; \gamma - \log a + O(a)$$

as $a \to 0$. Here, $\gamma \approx 0.5772156649$ is Euler's constant.

## Tommy Moorhouse

Given a positive real number $a < 1$ there is an integer $M$ such that $M+1 > 1/a > M$. We recall the definition of $\gamma$:

$$\gamma \;=\; \lim_{N\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} - \log N\right).$$

Now consider the sum

$$\sum_{n=0}^{\infty} \frac{1}{an^2+n+1} \;=\; \sum_{n=0}^{M} \frac{1}{(n+1)\left(1+\frac{an^2}{n+1}\right)} + \sum_{n=M+1}^{\infty} \frac{1}{an^2+n+1}.$$

We will show in Lemma 1 below that the second term tends to $\log 2$ as $M \to \infty$. Since $an^2/(n+1) < 1$ for $n \le 1/M$ we can expand the first term

$$\sum_{n=0}^{M} \frac{1}{(n+1)\left(1+\frac{an^2}{n+1}\right)} \;=\; \sum_{n=0}^{M} \frac{1}{(n+1)} \left(1 - \frac{an^2}{n+1} + \cdots\right).$$

Consider the terms involving finite sums (from $n = 0$ to $M$) after the first. It is tempting to assume that these terms vanish as $a$ vanishes, but we must recall that the number of terms in each sum depends on $a$. In Lemma 2 below we will show that their sum tends to $-\log 2$ as $a \to 0$. These terms therefore cancel from the sum.

The remaining term is

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{M} \;=\; 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{M} - \log M + \log M$$

where we have added and subtracted $\log M$. Now we let $a \to 0$ so that $M \to \infty$. The sum tends to $\gamma + \log M$. We can write $M = 1/a + \varepsilon$ where $\varepsilon < 1$. Then

$$\log M \;=\; \log\left(\frac{1}{a} + \varepsilon\right) \;=\; \log\frac{1}{a} + \log(1 + a\varepsilon) \;=\; -\log a + a\varepsilon + \cdots$$

so that $\log M$ tends to $-\log a$ as $a \to 0$.

Now we turn to the assertions made above.

**Lemma 1**
$$\sum_{M+1}^{\infty} \frac{1}{an^2+n+1} \to \log 2 \text{ as } M \to \infty.$$

**Proof** We will factorize the numerator into $a(n-\alpha)(n-\beta)$ where, in an approximation that has vanishing error as $a \to 0$,

$$\alpha = \frac{-M+\sqrt{M^2-4M}}{2}, \quad \beta = \frac{-M-\sqrt{M^2-4M}}{2}.$$

Now we have the partial fraction decomposition

$$\frac{1}{(n-\alpha)(n-\beta)} = \frac{1}{\alpha-\beta}\left(\frac{1}{n-\alpha}-\frac{1}{n-\beta}\right).$$

We can sum each term separately if we insert a 'cut-off' $L$ which we later allow to go to infinity. We use Abel's identity (see Apostol, Theorem 4.2) with $a(n) = u(n) = 1$.

$$\begin{aligned}
\sum_{n=M+1}^{L} \frac{1}{n-\alpha} &= \frac{L}{L-\alpha} - \frac{M+1}{M+1-\alpha} + \int_{M+1}^{L} \frac{tdt}{(t-\alpha)^2} + O(M^{-1}) \\
&= \frac{L}{L-\alpha} - \frac{M+1}{M+1-\alpha} + \int_{M+1-\alpha}^{L-\alpha} \left(\frac{1}{s}+\frac{\alpha}{s^2}\right) ds + O(M^{-1}) \\
&= \frac{L}{L-\alpha} - \frac{M+1}{M+1-\alpha} + \log\left(\frac{L-\alpha}{M+1-\alpha}\right) - \frac{\alpha}{L-\alpha} \\
&\quad + \frac{\alpha}{M+1-\alpha} + O(M^{-1}) \\
&\to \log\left(\frac{L-\alpha}{M+1-\alpha}\right) \text{ as } L, M \to \infty.
\end{aligned}$$

Repeating the calculation for the other fraction, using $aM \to 1$ and $\alpha-\beta \to M$ as $M \to \infty$, and letting $L \to \infty$, we find

$$\begin{aligned}
\frac{1}{a(\alpha-\beta)} \sum_{M+1}^{\infty} \left(\frac{1}{n-\alpha}-\frac{1}{n-\beta}\right) &= \frac{1}{a(\alpha-\beta)} \log\left(\frac{M+1-\beta}{M+1-\alpha}\right) \\
&\to \log\left(1+\frac{\alpha-\beta}{M+1}\right) \\
&\to \log 2.
\end{aligned}$$

□

**Lemma 2**

$$\frac{1}{M^k}\sum_{n=1}^{M}\frac{n^{2k}}{(n+1)^{k+1}} \;\to\; \frac{1}{k}+O\left(\frac{1}{M}\right) \text{ as } M\to\infty.$$

**Proof** The key to the proof is, once again, the use of Abel's identity (Apostol, Theorem 4.2). In the notation of Apostol we have $a(n) = u(n) = 1, A(x) = [x]$ and $f(n) = n^{2k}/(n+1)^{k+1}$. Now shifting the variable to $y = x+1$ we have

$$f'(x) \;=\; f'(y-1) \;=\; \frac{(y-1)^{2k-1}((k+1)(y-1)-2ky)}{y^{k+2}}.$$

We will only need the leading term of $[y]f'(y)$, which is $(1-k)y^{k-2}[y]$. Since $(x-1) < [x] \le x$ the leading term of the integral can, to order $1/M$, be written without the square brackets. Abel's identity now gives

$$\begin{aligned}\frac{1}{M^k}\sum_{n=0}^{M}\frac{n^{2k}}{(n+1)^{k+1}} \;&=\; \frac{M^{k+1}}{(M+1)^{k+1}}+\frac{(1-k)}{M^k}\int_1^{M+1}(y^{k-1}+\cdots)dy \\ &\to\; 1+\frac{1-k}{k}+O\left(\frac{1}{M}\right) \;=\; \frac{1}{k}+O\left(\frac{1}{M}\right) \text{ as } M\to\infty.\end{aligned}$$

From this lemma, and noting that the terms of order $1/M$ appear with alternating signs in the sum, and so sum to a term of order $1/M$, we see that

$$\begin{aligned}\sum_{n=0}^{M}\left(-\frac{an^2}{n+1^2}+\cdots+(-1)^k a^k\frac{n^{2k}}{(n+1)^{k+1}}\right) \;&\to\; -1+\frac{1}{2}-\frac{1}{3}+\cdots \\ &=\; -\log(1-(-1)) \;=\; -\log 2,\end{aligned}$$

as required. □

Having established these lemmas we have shown that

$$\sum_{n=0}^{\infty}\frac{1}{an^2+n+1} \;\to\; \gamma-\log a$$

as $a\to 0$.

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**Reference** T. M. Apostol, *Introduction to Analytic Number Theory*, Springer 1976.

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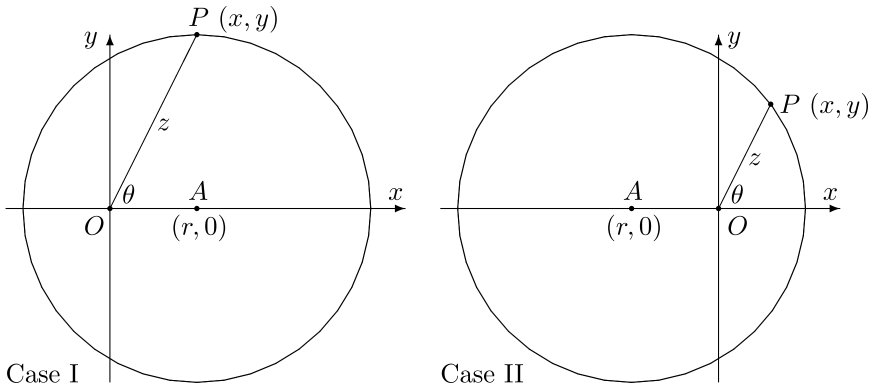
# Solution 238.1 – Disc

> Choose a point at random in the unit disc. Choose a direction at random. What is the expected distance from the point to the unit circle in the chosen direction?

## Steve Moon

Wherever the randomly chosen point lies on the unit disc, we can make the point the origin of a set of $(x, y)$ axes and place the centre of the unit disc on the positive or negative $x$-axis, as shown.




In each case, $A$ is the centre of the disc, $P$ is a general point $(x, y)$ on the unit circle with centre $A$, distance $z$ from random point $O$. Also $P$ lies on direction $\theta$ from the positive $x$-axis. Point $O$ is at distance $r$ from $A$, $0 \le r \le 1$. In each case,

$$x = z\cos\theta, \qquad y = z\sin\theta, \qquad z = \sqrt{x^2 + y^2} \ge 0.$$

By symmetry, we can establish an expected value for $z$ by considering only the situations where $y \ge 0$ and $0 \le \theta \le \pi/2$ for each of cases I and II.

**Case I** The equation of the circle is $(x - r)^2 + y^2 = 1$. Substituting for $x$ and $y$, we have

$$(z\cos\theta - r)^2 + (z\sin\theta)^2 = 1 \quad \Rightarrow \quad z = r\cos\theta \pm \sqrt{1 - r^2\sin^2\theta}$$

and to ensure $z \ge 0$ we take the positive option,

$$z = r\cos\theta + \sqrt{1 - r^2\sin^2\theta}. \tag{1}$$

**Case II** A similar analysis for the circle $(x+r)^2+y^2=1$ gives

$$(z\cos\theta+r)^2+(z\sin\theta)^2 \;=\; 1,$$

leading to

$$z \;=\; -r\cos\theta+\sqrt{1-r^2\sin^2\theta}. \tag{2}$$

This is non-negative for $0\le r\le 1$, $0\le\theta\le\pi/2$ because

$$\sqrt{1-r^2\sin^2\theta} \;\ge\; \sqrt{r^2-r^2\sin^2\theta} \;\ge\; r\cos\theta \;\ge\; 0.$$

**Distribution** For a random distribution of points over the unit disc, we would expect the number of points in a thin annulus of radius $r$ and thickness $\delta r$ to increase as $s$ increases from 0 to 1.

Consider the whole disc. The probability of finding a point in some thin annulus in the interval $[r, r+\delta r]$ is given by (area of annulus)/(area of disc) $=2\pi r\,\delta r/\pi = 2r\,\delta r$. For the first quadrant, the probability is $\frac{1}{2}\pi r\,\delta r/(\frac{1}{4}\pi)=2r\,\delta r$, the same.

The probability of choosing a random direction in the interval $[\theta,\theta+\delta\theta$, $0\le\theta\le 2\pi$ is $\delta\theta/(2\pi)$. But if $\theta$ is restricted to the first quadrant, the probability becomes $\delta\theta/(\frac{1}{2}\pi)=2\delta\theta/\pi$.

Now $r$ and $\theta$ are independent random variables; so the probability of selecting a point in $[r,r+\delta r]$, $[\theta,\theta+\delta\theta]$ in the first quadrant is

$$P(r,\theta) \;=\; 2r\,\delta r\cdot\frac{2\delta\theta}{\pi} \;=\; \frac{4r}{\pi}\,\delta r\,\delta\,\theta. \tag{3}$$

As a check we can integrate $P(r,\theta)$ over the first quadrant to get

$$\int_0^{\pi/2}\int_0^1 \frac{4r}{\pi}\,dr\,d\theta \;=\; \int_0^{\pi/2}\left[\frac{2r^2}{\pi}\right]_0^1 d\theta \;=\; \left[\frac{2\theta}{\pi}\right]_0^{\pi/2} \;=\; 1.$$

We use (3) to find the expected value of $z$ for each case. For Case I, from (1) and (3) we have

$$\overline{z}_1 \;=\; \int_0^{\pi/2}\int_0^1 \frac{4r}{\pi}\left(r\cos\theta+\sqrt{1-r^2\sin^2\theta}\right)dr\,d\theta \;=\; \frac{4}{\pi},$$

which is greater than 1, as expected. For Case I, (2) and (3) give

$$\overline{z}_2 \;=\; \int_0^{\pi/2}\int_0^1 \frac{4r}{\pi}\left(-r\cos\theta+\sqrt{1-r^2\sin^2\theta}\right)dr\,d\theta \;=\; \frac{4}{3\pi},$$

less than 1, as expected.

Now we are in a position to calculate the expectation value for the distance from the random point to the unit circle. We have

$$\overline{z} \;=\; \frac{\overline{z}_1 + \overline{z}_2}{2} \;=\; \frac{1}{2}\left(\frac{4}{\pi} + \frac{4}{3\pi}\right) \;=\; \frac{8}{3\pi}$$

because each of $\overline{z}_1$ and $\overline{z}_1$ represents a range of $\pi/2$ in the angle distribution $0 \le \theta \le 0$, which by symmetry with the $x$-axis represents the result for the whole disc.

This result appears to be sensible. If a random point lies at the centre of the disc, $z = 1$ in all directions. For any other position, $z < 1$ for a range of directions greater than $\pi$ and $z > 1$ for a range of directions less than $\pi$, and the sum of $z(\theta)$ and $z(\theta + \pi)$, i.e. collinear, opposite directions, is less than 2. So the average $(z(\theta) + z(\theta + \pi))/2$ is less than 1.

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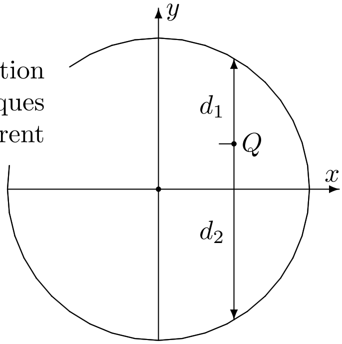
## Tony Forbes




Whist playing around with Steve Moon's solution it occurred to me to apply the same techniques to attack a similarly worded but totally different problem.

> Choose a point at random in the unit disc. Choose a *vertical* direction at random, up or down. What is the expected distance from the point to the unit circle in the chosen direction?

I was then surprised to discover that (i) the solution seems to be very much easier, and (ii) the answer is the same!

The probability of selecting the random point $Q$ between the vertical chords with $x$ coordinates $r$ and $r + \delta r$ is $2\sqrt{1 - r^2}\,\delta r/\pi$, the ratio of the area sandwiched between the chords to the area of the whole disc. But there are two vertical distances from $Q$ to the unit circle, $d_1$ and $d_2$, say, and one of them is chosen with probability $\frac{1}{2}$. Moreover, $d_1 + d_2 = 2\sqrt{1 - r^2}$. So the expected distance we want is given by

$$\int_{-1}^{1} \left(\frac{d_1}{2} + \frac{d_2}{2}\right) \frac{2\sqrt{1 - r^2}}{\pi}\, dr \;=\; \int_{-1}^{1} \frac{2(1 - r^2)}{\pi}\, dr \;=\; \frac{8}{3\pi}.$$

Is there any connection between the two problems?

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Front cover: Squares dissected into similar pieces whose (areas, lengths) are in (arithmetic, harmonic, geometric) progression. See page 5.

## M500 Society Committee – call for applications

The M500 committee invites applications from members to join the Committee; in particular we are seeking someone to help with publicity. Please apply to the Secretary by 1st October 2013.