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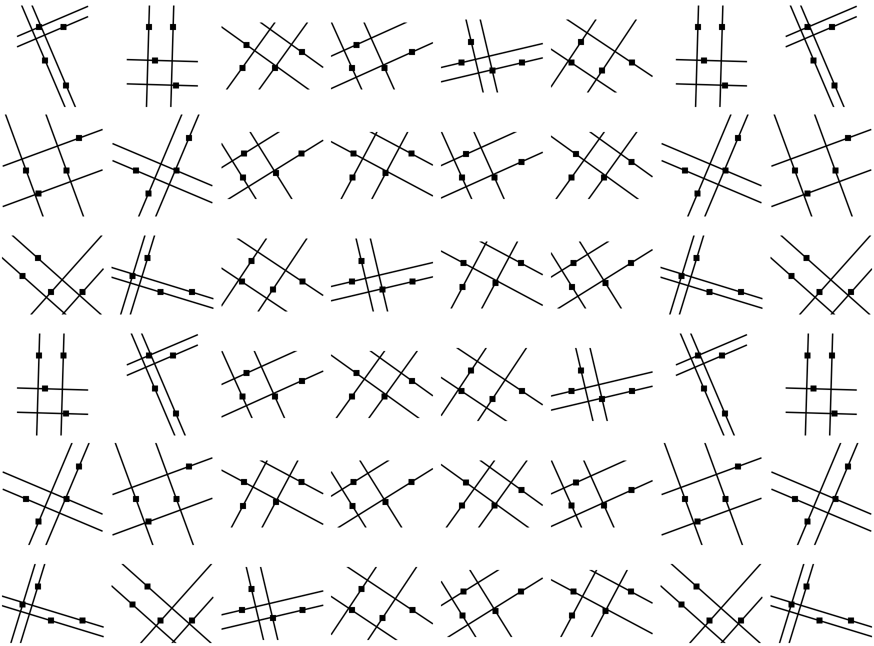
M500 254



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Integer partitions

Identities involving partitions of an integer – generalization of a standard result

Tommy Moorhouse

Introduction A partition of a non-negative integer n is an expression for n as a sum of positive integers, where the terms in the sum are known as the 'parts' of the partition. In this article we will set up the notation to express a well-known identity relating partitions of different types, and prove it. The first step will be to establish some tools and notation. Finally we prove an interesting generalization of the identity using the tools we have developed.

Notation In previous articles we have considered certain generating ('partition') functions. Given any integer-valued function ξ we define the logarithm L_ξ through the action

$$L_\xi\left(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}\right) \;=\; k_1\xi(p_1) + k_2\xi(p_2) + \cdots + k_r\xi(p_r).$$

Clearly this is a partition of the integer $L_\xi(n)$ into parts of the form $\xi(p_j)$. Then

$$Z_\xi(s) \;=\; \sum_{n=1}^{\infty} e^{-sL_\xi(n)} \;=\; \sum_{n=0}^{\infty} P_\xi(n) e^{-sn}$$

is the generating function for $P_\xi(m)$, the number of partitions of m having parts given by $\xi(p_j)$. This follows by looking at the coefficient of e^{-sk}, which comes from those $e^{-sL_\xi(n)}$ for which $L_\xi(n) = k$.

More general functions than $Z(s)$ can be written down. These functions can be used to define a wider class of partitions, as we will see. Given any function f we can write

$$F(s) \;=\; \sum_{n=1}^{\infty} f(n) e^{-sL_\xi(n)}.$$

Gathering together terms as before we find

$$F(s) \;=\; \sum_{n=0}^{\infty} E_f^\xi(n) e^{-sn},$$

where

$$E_f^\xi(n) \;=\; \sum_{L_\xi(m) \,=\, n} f(m).$$

Since L_ξ is a logarithm $L_\xi(1) = 0$ and, assuming that ξ is non-trivial, $E_f^\xi(0) = \sum_{L_\xi(m)=0} f(m) = f(1)$. In this notation $P_\xi = E_u^\xi$, where $u(n) = 1$ for every n.

It is straightforward to show that

$$E_{f*g}^\xi = E_f^\xi \circ E_g^\xi, \quad \text{where} \quad A \circ B(n) = \sum_{j+k=n} A(j)B(k)$$

and j and k are non-negative integers.

Partitions into distinct parts We now order the prime numbers by size, so that $2 = p_1, 3 = p_2$ and so on. We then specify ξ by setting $\xi(p_k) = k$. A partition of n into distinct parts is an expression of the form

$$n \;=\; j_1 + j_2 + \cdots + j_m \;=\; \xi(p_{j_1}) + \cdots + \xi(p_{j_k})$$

, where each j is different. Clearly, $n = L_\xi(m)$ if m is a product of distinct prime factors (that is, $m = p_{j_1} p_{j_2} \cdots p_{j_k}$), and only these m give rise to partitions into distinct parts. It is possible to show, and the reader is encouraged to try this, that in fact the number of such partitions is exactly

$$\sum_{L_\xi(m)=n} \sigma(m) \;=\; E_\sigma^\xi(n),$$

where $\sigma(n)$ vanishes if n has any square factors greater than 1, and is 1 otherwise [Hint: $2sL_\xi(n) = sL_\xi(n^2)$].

There is an interesting interplay between the infinite product and infinite sum representations of $Z_f(s)$ which we will exploit in the sequel.

Partitions into odd parts Defining the function $\omega(p_i) = 2i - 1$ we find that the number of partitions of an integer into odd parts is generated by $Z_\omega(s)$ and will be denoted P_ω.

We wish to show that

$$P_\omega \;=\; E_\sigma^\xi.$$

First we note that

$$Z_\xi(s) \;=\; Z_\xi(2s) Z_\omega(s) \tag{1}$$

as can be seen using the product representation

$$Z_f(s) \;=\; \prod_p \frac{1}{1 - e^{-sL_f(p)}}$$

and the fact that the right hand side of the expression (1) for $Z_\xi(s)$ is just the product of the odd and even contributions to the product on the left.

This tells us that

$$Z_\omega(s) \;=\; \frac{Z_\xi(s)}{Z_\xi(2s)}.$$

The right-hand side can be expanded in the series representation using

$$\frac{1}{Z_\xi(2s)} \;=\; \sum_{n=1}^{\infty} \mu(\sqrt{n})\eta(n)e^{-sL_\xi(n)},$$

where $\eta(n) = 1$ if n is a square and is zero otherwise. What we have done here is put the series into the form $F(s)$ for $f(n) = \mu(\sqrt{n})\eta(n)$ so that we can use the methods developed so far.

The product becomes, by the rules established above,

$$\begin{aligned}
\sum_{n=0}^{\infty} P_\omega(n)e^{-sn} \;&=\; \sum_{n=1}^{\infty} \{\mu(\surd)\cdot\eta) * u\}(n)e^{-sL_\xi(n)} \\
&=\; \sum_{n=1}^{\infty} \Big\{ \sum_{m^2|n} \mu(m) \Big\} e^{-sL_\xi(n)} \\
&=\; \sum_{n=1}^{\infty} \sigma(n)e^{-sL_\xi(n)} \;=\; \sum_{k=0}^{\infty} E_\sigma^\xi(k)e^{-sk},
\end{aligned}$$

where we have used the notation $\mu(\surd)$ as shorthand for the function sending k to $\mu(\sqrt{k})$, and used the following.

Lemma

$$\sum_{m^2|n} \mu(m) \;=\; \sigma(n),$$

which means that the sum of $\mu(d)$ over square divisors d of n vanishes unless n is prime-square-free, in which case it is 1.

Proof Suppose n is prime-square-free. Then the only square dividing n is 1, and $\mu(1) = 1$. If, on the other hand, $n = ad^2$ where a is square-free and $d > 1$ then

$$\sum_{m^2|n} \mu(m) \;=\; \sum_{k|d} \mu(k) \;=\; \mu * u(d) \;=\; I(d) \;=\; 0,$$

where $I(d) = 1$ for $d = 1$ and is zero otherwise (see Apostol, for example). Here $I(d)$ vanishes because of our stipulation that $d > 1$.

The identity Comparing coefficients of e^{-sn} above we see that

$$P_\omega \;=\; E_\sigma^\xi.$$

This says that the number of partitions of an integer into odd parts is equal to the number of partitions of that integer into distinct parts.

Although this seems like a circuitous route to a straightforward result, we have established that the partition function approach can accommodate a range of ideas not immediately apparent at first sight. The notation and reasoning are robust and may be extended, as we will see.

A generalization Using the ideas above we can show the following.

Theorem The number of partitions of an integer n in which no part appears more than $p-1$ times is equal to the number of partitions of n in which none of the parts are congruent to 0 (mod p). (The result above represents the case $p = 2$.)

Proof Write

$$Z_\xi(s) \;=\; Z_\xi(ps) Z_\theta(s),$$

where $\theta(p_i) = \tilde{i}$ and $\tilde{i}$ is the ith member of the sequence $1, 2, \ldots, p-1, p+1, \ldots$. In this sequence every integer divisible by p has been removed. $Z_\theta(s)$ is the generating function for partitions into parts none of which is congruent to 0 (mod p) (i.e. divisible by p). We have

$$Z_\theta(s) \;=\; \frac{Z_\xi(s)}{Z_\xi(ps)}.$$

All we have to do to establish the result is to show that the right-hand side is equal to $\sum E_{\sigma_p}^\xi(n)$, where $\sigma_p(n)$ is 1 if n has no factor that is prime to a power greater than $p-1$. The reasoning follows that for the case $p = 2$ very closely.

$$\begin{aligned}
\sum_{n=0}^{\infty} E_u^\theta(n) e^{-sn} \;&=\; \sum_{n=1}^{\infty} \{\mu(\sqrt[p]{\;}) \cdot \eta_p) * u\}(n) e^{-sL_\xi(n)} \\
&=\; \sum_{n=1}^{\infty} \Big\{ \sum_{m^p | n} \mu(m) \Big\} e^{-sL_\xi(n)} \\
&=\; \sum_{n=1}^{\infty} \sigma_p(n) e^{-sL_\xi(n)} \qquad\qquad (2) \\
&=\; \sum_{k=0}^{\infty} E_{\sigma_p}^\xi(k) e^{-sk},
\end{aligned}$$

where $\eta_p(n)$ is 1 if n is a pth power and vanishes otherwise. Equation (2) follows by similar reasoning to that in the case of $p = 2$. This establishes the Theorem, since $E^{\xi}_{\sigma_p}(k)$ is the number of partitions of k into parts, no part occurring more than $p-1$ times, and $E^{\theta}_{u}(k)$ is the number of partitions of k into parts none of which is congruent to 0 (mod p).

Further reading Much more interesting material on partitions can be found in *The Theory of Partitions* by G. E. Andrews (Cambridge, 1984). This book uses methods different from those presented here. I found a good starting point for basic number theory to be *Elementary Number Theory* by D. Burton (McGraw–Hill, 1995 (3rd Ed.)) or the more demanding *Introduction to Analytic Number Theory* by T. Apostol (Springer).

Problem 254.1 – Four bottles

Tony Forbes

Find a two-variable function that provides a convincing model for the shape of the stretched plastic sheeting in this typical example of four one-litre bottles of fizzy stuff that I bought from my local supermarket for £1.65 (50 pence each if purchased separately).

Observe (for which I have no explanation) that the cross-section through the centre in the NW–SE direction, a parabola-like curve, differs from the NE–SW cross-section, which looks as if it could be a quartic with two local maxima and a local minimum at the central saddle point.

Problem 254.2 – Interesting integral

Show that

$$\int_0^{\pi/2} \cos(\tan x)dx = \frac{\pi}{2e}$$

and hence that $\int_0^a \cos(\tan x)dx = a/e$ if a is an integer multiple of $\pi/2$.

Solution 253.1 – A Diophantine equation

Given that $P = 2$, $Q = 1$, and $R = 7$ is a solution to the Diophantine equation

$$P^4 + 8P^2Q^2 + Q^4 \;=\; R^2,$$

use this to find further solutions.

Vincent Lynch

We start by reducing the number of variables by the substitutions $p = P/Q$, $r = R/Q^2$. This gives the equation in rational numbers $p^4 + 8p^2 + 1 = r^2$. Completing the square gives $(p^2 + 4)^2 - r^2 = 15$, which factorizes as

$$(p^2 + 4 + r)(p^2 + 4 - r) \;=\; 15.$$

We may now put $p^2 + 4 + r = 15x$ and $p^2 + 4 - r = 1/x$. Eliminating r gives $2(p^2 + 4) = 15x + 1/x$.

The next step is the most difficult to think of. Multiply through by $2x^2$ and substitute $y = 2px$ to give

$$y^2 \;=\; 30x^3 - 16x^2 + 2x. \tag{1}$$

This is the equation of an elliptic curve. Knowing the coordinates of a rational point on such a curve means we can find the equation of the tangent at the point and solve it with the curve to find another rational point. The given solution leads to $p = 2$, $x = 1$, $y = 4$.

The gradient of the tangent at this point is $15/2$ and its equation is $y = (15x - 7)/2$. Solving with (1) gives

$$120x^3 - 289x^2 + 218x - 49 \;=\; 0.$$

But we know $x = 1$ is a double root. So we can divide by $x^2 - 2x + 1$ to give $120x - 49 = 0$. So $x = 49/120$ and $y = 7/16$. Further substitution gives $p = 15/28$ and $r = 1441/784$. So we have a solution to the original equation: $P = 15$, $Q = 28$, $R = 1441$.

This process can be repeated. The final chapter of the sixth edition of Hardy and Wright's famous book on number theory is on elliptic curves. The addition of two points on such a curve is obtained by finding where the chord joining the points meets the curve again and reflecting in the x-axis. To duplicate a point, you find where the tangent at the point meets the curve again and reflect in the x-axis. We didn't need this last step here. In the chapter there are formulae for these, but the equation needs to be in standard form: $y^2 = x^3 + Ax + B$.

We can put our equation in this form using the substitutions $30y = u$, $30x = w + 16/3$ to give

$$u^2 \;=\; w^3 - \frac{76w}{3} + \frac{448}{27};$$

so $A = -76/3$, $B = 448/27$. We can now use the duplication formula given in the chapter. Let x_{2P} denote the x coordinate of the duplication of the point whose x coordinate is x_P. Then

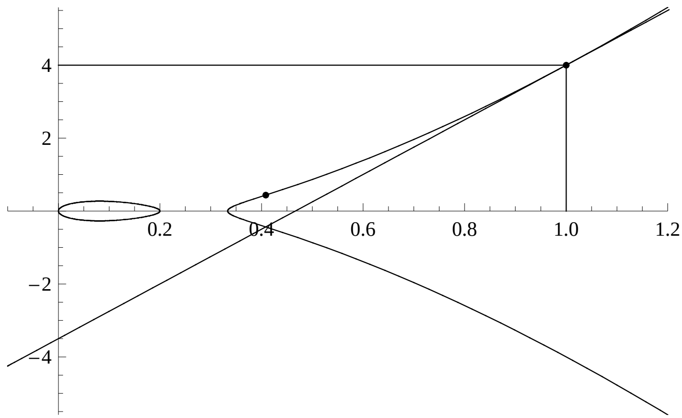
$$x_{2P} \;=\; \frac{x_P^4 - 2Ax_P^2 - 8Bx_P + A^2}{4(x_P^3 + Ax_P + B)}.$$

Using the solution $x = 49/120$ of (1) and applying the duplication formula with $x_P = 30x - 16/3 = 83/12$ gives the next solution to the original equation: $P = 564031$, $Q = 1210440$, $R = 2444755743361$.

Dick Boardman

Clearly $P = 2n$, $Q = n$, $R = 7n^2$ is a solution for all integer n. Further, $p^2 + 8pq + q^2$ forms a multiplicative domain, so that the techniques I mentioned in my other submission [Solution 253.2 – Quadratic, to appear] also apply. Having found all of these solutions, I am afraid I didn't search further.

Tony Forbes In case you were wondering what they look like, here is the graph of $y^2 = 30x^3 - 16x^2 + 2x$, equation (1), together with the line $y = (15x - 7)/2$. The other dot is at $(49/120, 7/16)$.



Rotations of the sphere

Rob Evans

This article will be concerned with the following interesting problem.

> Let S^2 denote an arbitrary sphere. In turn, let ρ_1 and ρ_2 denote arbitrary non-trivial rotations of S^2 about different diameters of S^2. In turn, let θ_{12}, θ_{21} denote respectively the angles of turn of the composite rotations $\rho_1\rho_2$ and $\rho_2\rho_1$. Prove that $\theta_{12} = \theta_{21}$.

This problem (with a less formal wording) appeared in M500 **216** under the title 'Rotations'. It is a remarkable fact that our solution to this problem has no need to resort to far-from-obvious results from the geometry of the sphere. Indeed, it highlights some of the features that are common to the geometry of the sphere and the geometry of the plane.

In order to solve the above problem it is natural to modify it in a way that distinguishes between 'clockwise' and 'anticlockwise' rotations. (How exactly we do this shall soon become clear.) In order to state the modified problem as succinctly as possible, we introduce the following notation.

Euclidean 3-space is denoted by $\mathbb{E}^3$. As in the original problem, S^2 denotes an arbitrary sphere in $\mathbb{E}^3$. In turn, O denotes the centre of S^2.

Let $P, Q \in \mathbb{E}^3$ such that $\#\{P, Q\} = 2$. Then $l(P, Q)$ denotes the line that passes through P and Q.

Finally, let $P, Q \in \mathbb{E}^3$ such that $\#\{P, Q\} = 2$ and $\theta \in \mathbb{R}$. Then $\rho(P, Q; \theta)$ denotes the rotation of $\mathbb{E}^3$ through θ radians about $l(P, Q)$, where the angle of turn, as seen from P towards Q, is measured clockwise. Using this notation, we state the modified problem as follows.

> Let $\rho_1 = \rho(O, P_1; \theta_1)$ and $\rho_2 = \rho(O, P_2; \theta_2)$ for some $P_1, P_2 \in S^2$ such that O, P_1, P_2 are non-collinear, and for some $\theta_1, \theta_2 \in (0, \pi]$. Prove that $\rho_1\rho_2 = \rho(O, P^+; \theta)$ and $\rho_2\rho_1 = \rho(O, P^-; \theta)$ for some $P^+, P^- \in S^2$ and $\theta \in (0, 2\pi)$ where, moreover, $\rho_1(P^-) = P^+$ and $\rho_2(P_+) = P^-$. (See figure. This figure is intended only as an aid to visualizing $P^\pm$. The detailed construction of those points will be made clear in the course of the solution.)

(N.B. For each $P \in S^2$ and $\theta \in (0, 2\pi)$ we have that $\rho(O, P; \theta) = \rho(O, P'; \theta - \pi)$, where P' denotes the point that is antipodal to P on S^2 (i.e. $P' \in S^2$ such that $P' \neq P$ and O, P, P' are collinear). In other words, the condition that $\theta_1, \theta_2 \in (0, \pi]$ implies no less generality than the condition that $\theta_1, \theta_2 \in (0, 2\pi)$. The reason for the choice of the former over the latter condition will become clear in the remarks following the solution.)

In order to present our solution of the modified problem as succinctly as possible, we introduce the following additional notation.

Let $P, Q \in \mathbb{E}^3$ such that $\#\{P, Q\} = 2$, and $\theta \in [0, 2\pi)$. In turn, let $\rho = \rho(P, Q; \theta)$ and $x \in \mathbb{R}$. Then, ρ^x denotes $\rho(P, Q; x\theta)$. (N.B. The condition that $\theta \in [0, 2\pi)$ ensures that ρ^x is well defined.)

Let $P, Q, R \in \mathbb{E}^3$ such that P, Q, R are non-collinear. Then, $\Pi(P, Q, R)$ denotes the plane that passes through P, Q, R.

As in the hypothesis, let $\rho_1 = \rho(O, P_1; \theta_1)$ and $\rho_2 = \rho(O, P_2; \theta_2)$ for some $P_1, P_2 \in S^2$ such that O, P_1, P_2 are non-collinear, and for some $\theta_1, \theta_2 \in (0, 2\pi)$. Then, eq denotes the circle $S^2 \cap P(O, P_1, P_2)$.

In turn, $H^{\pm}$ denotes the hemispherical region of S^2 that has eq as its boundary and that contains $\rho_1^{\pm 1/2}(P_2)$ and $\rho_2^{\mp 1/2}(P_1)$.

In turn, $s_1^{\pm}$ and $s_2^{\pm}$ denote the semicircles $\Pi[O, P_1, r_1^{\pm 1/2}(P_2)] \cap H^{\pm}$ and $\Pi[O, P_2, r_2^{\mp 1/2}(P_1)] \cap H^{\pm}$ respectively.

Finally, q denotes reflection of $\mathbb{E}^3$ in $P(O, P_1, P_2)$.

Using this notation and our previous notation, we proceed to solve the modified problem as follows.

Firstly, since O, P_1, P_2 are non-collinear we can already conclude that neither $\rho_1\rho_2$ nor $\rho_2\rho_1$ is the identity transformation. In other words, we can already conclude that each of $\rho_1\rho_2$ and $\rho_2\rho_1$ is a non-trivial rotation of S^2 about a diameter of S^2. In order to show that $\rho_1\rho_2$ and $\rho_2\rho_1$ have the form stated in the conclusion we proceed as follows.

Next, we demonstrate that $q(P^{\pm}) = P^{\mp}$ where $P^{\pm} = s_1^{\pm} \cap s_2^{\pm}$. (See figure.) From the relevant definitions it is obvious that we have:

$$\begin{aligned} q(H^{\pm}) &= H^{\mp}; \\ q(\Pi[O, P_1, \rho_1^{\pm 1/2}(P_2)]) &= \Pi[O, P_1, \rho_1^{\mp 1/2}(P_2)]; \\ q(\Pi[O, P_2, \rho_2^{\pm 1/2}(P_1)]) &= \Pi[O, P_2, \rho_2^{\mp 1/2}(P_1)]. \end{aligned}$$

Moreover, since q is 1–1 on $\mathbb{E}^3$ we have $q(A \cap B) = q(A) \cap q(B)$ for $A, B \in \mathbb{E}^3$. Consequently, in turn, we have:

$$\begin{aligned} q(s_1^{\pm}) &= q(\Pi[O, P_1, \rho_1^{\pm 1/2}(P_2)] \cap H^{\pm}) = q(\Pi[O, P_1, \rho_1^{\pm 1/2}(P_2)]) \cap q(H^{\pm}) \\ &= \Pi[O, P_1, \rho_1^{\mp 1/2}(P_2)] \cap H^{\mp} = s_1^{\mp}; \\ q(s_2^{\pm}) &= q(\Pi[O, P_2, \rho_2^{\mp 1/2}(P_1)] \cap H^{\pm}) = q(\Pi[O, P_2, \rho_2^{\mp 1/2}(P_1)]) \cap q(H^{\pm}) \\ &= \Pi[O, P_2, \rho_2^{\pm 1/2}(P_1)] \cap H^{\mp} = s_2^{\mp}. \end{aligned}$$

Consequently, in turn, we have

$$q(P^{\pm}) \;=\; q(s_1^{\pm} \cap s_2^{\pm}) \;=\; q(s_1^{\pm}) \cap q(s_2^{\pm}) \;=\; s_1^{\mp} \cap s_2^{\mp} \;=\; P^{\mp}. \qquad \text{Q.E.D.}$$

Next, we demonstrate that $\rho_1(P^-) = P^+$ and $\rho_2(P^+) = P^-$.

From the relevant definitions it is obvious that we have: $P^{\pm} \in s_1^{\pm}$ where, in turn, $s_1^+ = \rho_1(s_1^-)$, and $P^{\pm} \in s_2^{\pm}$ where, in turn, $s_2^- = \rho_2(s_2^+)$. However, since $q(P^{\pm}) = P^{\mp}$ it is also obvious that we have

$$|P_1P^+| \;=\; |P_1P^-| \quad \text{and} \quad |P_2P^+| \;=\; |P_2P^-|.$$

From the last two statements we deduce that $\rho_1(P^-) = P^+$ and $\rho_2(P^-) = P^-$. Q.E.D.

At the beginning of this solution we established that each of $\rho_1\rho_2$ and $\rho_2\rho_1$ is a non-trivial rotation of S^2 about a diameter of S^2. We now demonstrate that P^+ and P^- are respectively invariant under $\rho_1\rho_2$ and $\rho_2\rho_1$. Since $\rho_1(P^-) = P^+$ and $\rho_2(P^+) = P^-$ we have:

$$\begin{aligned}
(\rho_1\rho_2)(P^+) \;&=\; \rho_1[\rho_2(P^+)] \;=\; \rho_1(P^-) \;=\; P^+ \quad \text{and} \\
(\rho_2\rho_1)(P^-) \;&=\; \rho_2[\rho_1(P^-)] \;=\; \rho_2(P^+) \;=\; P^-. \quad \text{Q.E.D.}
\end{aligned}$$

We now know that $\rho_1\rho_2 = \rho(O, P^+; \theta^+)$ and $\rho_2\rho_1 = \rho(O, P^-; \theta^-)$ for some $\theta^+, \theta^- \in (0, 2\pi)$ where, moreover, $\rho_1(P^-) = P^+$ and $\rho_2(P^+) = P^-$.

Finally, we demonstrate that $\theta^+ = \theta^-$.

Firstly, it is obvious that to show that $\theta^+ = \theta^-$ it is sufficient to show that there exist an isometry $i : \mathbb{E}^3 \to \mathbb{E}^3$ and points Q, Q' such that we have

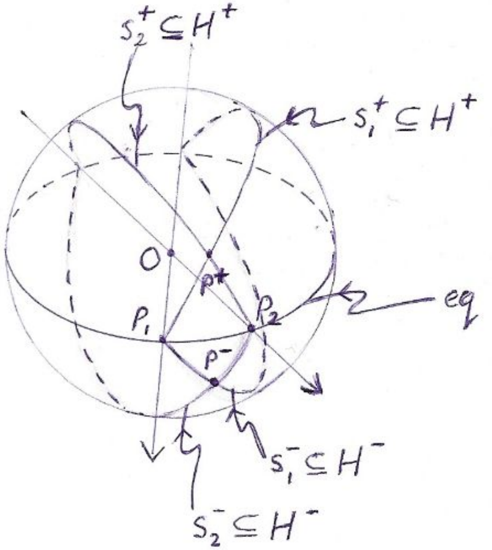
$$i(P^+) = P^-; \quad (\rho_1\rho_2)(Q) = Q'; \quad (\rho_2\rho_1)[i(Q)] = i(Q').$$

However, we already know that $\rho_2(P^+) = P^-$ where, moreover, it is obvious that ρ_2 is an isometry. Also (as readers can confirm), straightforward calculations show that the last two of the above equations hold for $(i, Q, Q') = (\rho_2, \rho_2^{-1}(P_1), P_1)$. (N.B. In these calculations, one needs to assume that $\rho_1(P_1) = P_1$. However, one is justified in doing so since, by definition, ρ_1 is a rotation about the line $l(O, P_1)$.)

From the above line of argument, we deduce that $\theta^+ = \theta^-$. Q.E.D. This completes the solution.

Readers of the above solution might suspect that an analogous one could be constructed within the context of the geometry of the plane. Except for the cases whereby $\theta_1 + \theta_2 = 0 \pmod{2\pi}$, this is indeed so. In these exceptional cases both $\rho_1\rho_2$ and $\rho_2\rho_1$ would be translations! These exceptional cases aside, it would be (as readers can confirm) a straightforward matter to extend the analogous solution to obtain the equation $\theta = \theta_1 + \theta_2 \pmod{2\pi}$ where $\theta = \theta^+ = \theta^- \pmod{2\pi}$. (N.B. For the purposes of this paragraph only we do not need to put conditions on θ_1, θ_2 and θ.)

In light of the last sentence of the last paragraph, it is natural to ask what the corresponding extension to our solution of the original problem might be. It turns out that one can extend the above solution to obtain $\theta < \theta_1 + \theta_2$ where $\theta = \theta^+ = \theta^-$. (Recall that here we had that $\theta_1, \theta_2 \in (0, \pi]$ and $\theta \in (0, 2\pi)$.) Moreover, using some results from the trigonometry of the sphere one can obtain an explicit equation for θ in terms of θ_1, θ_2 and the angular separation between $l(O, P_1)$ and $l(O, P_2)$.



Problem 254.3 – Three integers

Given three positive integers a, b, c, with a and b co-prime, show that the number of solutions in non-negative integer pairs (x, y) of the equation $xa + yb = c$ is given by

$$\frac{c}{ab} - \left\{ \frac{(a^{-1} \bmod b)\, c}{b} \right\} - \left\{ \frac{(b^{-1} \bmod a)\, c}{a} \right\} + 1,$$

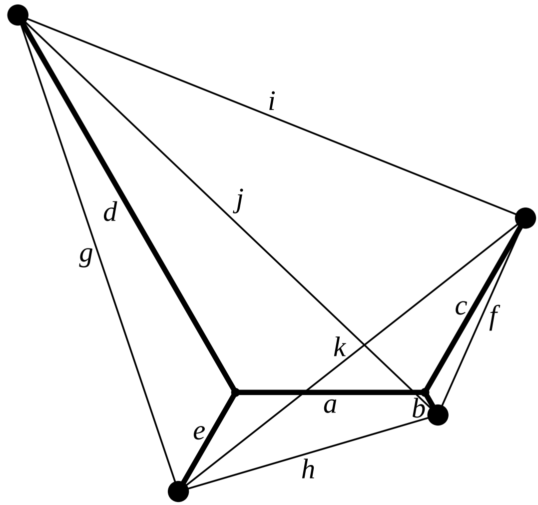
where $\{u\}$ denotes the fractional part of u (that is, $\{u\} = u - \lfloor u \rfloor$), and $(u^{-1} \bmod v)$ is the smallest (or in fact any) positive integer t that satisfies $tu \equiv 1 \pmod{v}$. For example, with $a = 2$, $b = 5$ and $c = 99$ the formula gives

$$\frac{99}{10} - \left\{ \frac{3 \cdot 99}{5} \right\} - \left\{ \frac{99}{2} \right\} + 1 \;=\; \frac{99}{10} - \frac{2}{5} - \frac{1}{2} + 1 \;=\; 10,$$

using $(2^{-1} \bmod 5) = 3$ and $(5^{-1} \bmod 2) = 1$. On the other hand, one can verify by counting that the number of solutions really is 10.

Solution 251.4 – Four more towns

Four towns are to be serviced by a road network of minimum length. This can often be achieved by the creation of two junctions, called *Steiner points*, where three roads meet at 120°. A typical layout is shown on the right.



Find a solution where the lengths of the five road segments, a, b, c, d, e, and the six distances between the towns, f, g, h, i, j, k, are distinct integers.

Tony Forbes

This came about as a result of my misunderstanding of Dick Boardman's Problem 251.3 – Four towns. (Four towns lie at the corners of a quadrilateral with integer sides and integer diagonals, no two the same. They are each connected to a single point such that the sum of the four distances is minimum. Find solutions where all of the individual lengths are integers.) Somehow I chose to ignore the bit about a single point and instead found myself attacking a completely different problem.

I do not have a solution Dick's original 'Four towns' problem. However, I have managed to find one solution of 251.4 and moreover it was actually presented in the picture that accompanied the original statement of the problem in M500 **251**. The diagram is repeated above with letters a–k added for reference. If you carefully measure the 11 lengths and multiply by a suitable number, you should obtain 11 distinct integers.

Remembering that a, b and c as well as a, d and e meet at 120 degrees, the cosine rule gives

$$f^2 \;=\; b^2 + c^2 + bc \qquad \text{and} \qquad g^2 = d^2 + e^2 + de.$$

So it makes sense (at least to me) to create a collection of positive integer triples $(x, y, \sqrt{x^2 + y^2 + xy})$ where $x^2 + y^2 + xy$ is a square. Then we can try combining two distinct triples from the collection together with an integer a to make a road network where each segment has integer length. And as a bonus we get two integer inter-town distances, f and g. If we do this often enough we can maybe hope to find a combination where the four numbers

$$\begin{aligned}4h^2 &= (2a+b+e)^2+3(b-e)^2,\\ 4i^2 &= (2a+c+d)^2+3(c-d)^2,\\ 4j^2 &= (2a+b+d)^2+3(b+d)^2,\\ 4k^2 &= (2a+c+e)^2+3(c+e)^2\end{aligned}$$

are even squares. The method actually works and produces

$$a=240, \quad b=33, \quad c=255, \quad d=552, \quad e=145,$$

$$f=273, \quad g=637, \quad h=343, \quad i=693, \quad j=735, \quad k=560,$$

as given by the diagram.

With a little effort one can prove that the road network really does have the smallest length. The only other possibility is the dual network, where the Steiner points occur at the intersections of c and d and of b and e, in which case the section joining them (corresponding to our a) is approximately north–south orientated rather than east–west. But it turns out that the dual network is slightly longer, at approximately 1272.09, compared with $a+b+c+d+e=1225$.

Incidentally, 1225 is a square, 35^2. I leave it for someone else to explain why this is significant. I also leave it for others to obtain a general solution.

Problem 254.4 – Gaussian binomial coefficients

For positive integer n, define

$$[n]_q \;=\; 1+q+\cdots+q^{n-1}, \qquad [n]_q! \;=\; [1]_q\,[2]_q\,\ldots\,[n]_q$$

and by analogy with the usual binomial coefficient $\displaystyle\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$, define the *Gaussian binomial coefficient* by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \;=\; \frac{[n]_q!}{[k]_q!\,[n-k]_q!}.$$

(For example, $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1+q+2q^2+q^3+q^4$.) Show that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q with integer coefficients.

An amazing construction

Dick Boardman

Problem Given four points, construct a square such that each side of the square, extended if necessary, passes through one of the points.

I am sure that this problem has an elegant solution using Euclidean methods. However, my solution uses methods of coordinate geometry.

Let the points be $\{P_1, P_2, P_3, P_4\}$. Choose P_1 to be the origin and choose the line joining it to P_3 to be the x-axis so that the coordinates of this second point are $(x_s, 0)$. Let the other two points be $P_4 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. From this there are two solutions.

Firstly, from the origin, along the y-axis measure a distance $x_2 - x_1$ and, perpendicular to it, a distance $x_s + y_1 - y_2$. The line joining this new point to the origin is one side of the square, the line through $(x_s, 0)$ parallel to it is another, and the two lines through (x_1, y_1) and (x_2, y_2), perpendicular to them complete the square.

Secondly, from the origin, along the y-axis measure a distance $x_1 - x_2$ and, perpendicular to it, a distance $x_s - y_1 + y_2$. The line joining this new point to the origin is one side of the square, the line through $(x_s, 0)$ parallel to it is another. As before, the two lines through (x_1, y_1) and (x_2, y_2), perpendicular to them complete the square.

If we choose a different pair of points to define the x-axis we arrive at another pair of solutions. It turns out that for a general configuration, there are six distinct squares that can be drawn through the four points. The front cover of this magazine illustrates the 48 solutions obtained by performing the two constructions on the 24 permutations of a given set of four points. Readers can verify that each square appears eight times.

Proof A point can be defined by its distances from two perpendicular axes (x, y) and the equation of a line is often given as $y = mx + c$, where m is called the gradient. Thus I can define a line by two numbers $[m, c]$. If a line has gradient m then a line perpendicular to it has gradient $-1/m$.

These two definitions are linked. A line through two points (x, y) and (u, v) becomes

$$\left[\frac{y - v}{x - u},\ y - x\frac{y - v}{x - u}\right].$$

The intersection of two lines $[m_1, c_1]$ and $[m_2, c_2]$ becomes

$$\left(\frac{c_2 - c_1}{m_1 - m_2},\ m_1\frac{c_2 - c_1}{m_1 - m_2} + c_1\right).$$

A line through (x, y) with gradient m becomes $[m,\ y-mx]$, and the distance between (x, y) and (u, v), squared, is $(x-u)^2 + (y-v)^2$.

Let

$S_1 = [m,\ 0]$ be the line through the origin (P_1) with gradient m,

$S_3 = [m,\ -mx_s]$ be the line through P_3 with gradient m,

$S_2 = [-1/m,\ x_2/m + y_2]$ be the line through P_2 with gradient $-1/m$,

$S_4 = [-1/m,\ x_1/m + y_1]$ be the line through P_4 with gradient $-1/m$.

Denoting the intersection of S_i and S_j by $S_{i,j}$, let

$$\begin{aligned} V_1 &= S_{1,2} = \left(\frac{x_2 + my_2}{1+m^2},\ \frac{m(x_2 + my_2)}{1+m^2}\right), \\ V_2 &= S_{2,3} = \left(\frac{x_2 + m(mx_s + y_2)}{1+m^2},\ \frac{m(x_2 - x_s + my_2)}{1+m^2}\right), \\ V_3 &= S_{3,4} = \left(\frac{x_1 + m(mx_s + y_1)}{1+m^2},\ \frac{m(x_1 - x_s + my_1)}{1+m^2}\right), \\ V_4 &= S_{1,4} = \left(\frac{x_1 + my_1}{1+m^2},\ \frac{m(x_1 + my_1)}{1+m^2}\right). \end{aligned}$$

Then these four points form the vertices of a rectangle. We choose m such that the sides of the rectangle are equal. It is sufficient that the squares of the sides are equal; so we equate the distances squared between V_1 and V_2 and between V_1 and V_4,

$$\begin{aligned} &\big(x_2 + m(mx_s + y_2) - (x_2 + my_2)\big)^2 + \big(m(x_2 - x_s + my_2) - m(x_2 + my_2)\big)^2 \\ &\quad = \big(x_1 + my_1 - (x_2 + my_2)\big)^2 + \big(m(x_1 + my_1) - m(x_2 + my_2)\big)^2, \end{aligned}$$

to obtain these two solutions,

$$m = \frac{-x_1 + x_2}{x_s + y_1 - y_2} \quad \text{and} \quad m = \frac{x_1 - x_2}{x_s - y_1 + y_2},$$

from which we can derive the two alternative constructed points:

$$P_{5,1} = (x_s + y_1 - y_2,\ x_2 - x_1) \quad \text{and} \quad P_{5,2} = (x_s - y_1 + y_2,\ x_1 - x_2).$$

As an example, consider the set of four points defined by

$$x_1 = 2, \quad y_1 = 2, \quad x_2 = 1.3, \quad y_2 = 1.7, \quad x_s = 1.5.$$

These give

$$P_1 = (0,\ 0), \quad P_2 = (1.3,\ 1.7), \quad P_3 = (1.5,\ 0), \quad P_4 = (2,\ 2),$$

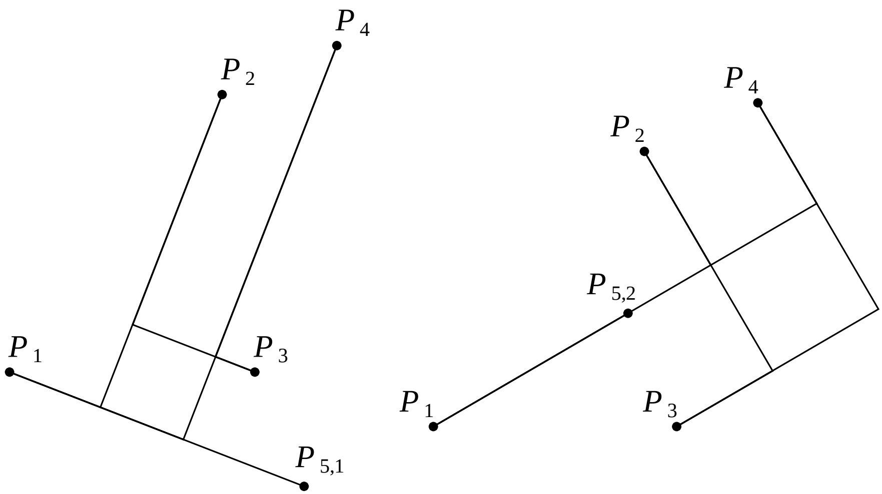
from which we derive

$$m = -0.388889, \quad P_{5,1} = (1.8,\ -0.7),$$

$$V_1 = (0.55496,\ -0.215818), \quad V_2 = (0.752011,\ 0.290885),$$
$$V_3 = (1.25871,\ 0.0938338), \quad V_4 = (1.06166,\ -0.412869),$$

and

$$m = 0.583333, \quad P_{5,2} = (1.2,\ 0.7),$$
$$V_1 = (1.70984,\ 0.997409), \quad V_2 = (2.09067,\ 0.34456),$$
$$V_3 = (2.74352,\ 0.725389), \quad V_4 = (2.36269,\ 1.37824).$$



Problem 254.5 – Descending integers

Find a nice formula as a function of n for the big number you get by writing down all the n-digit integers in *descending* order, as in, for example, 999998997…101100 when $n = 3$.

Editor's email address change In case you haven't noticed on page 0, the Editor's email address has changed. If you have sent a contribution to tony@m500.org.uk without receiving an acknowledgement, would you please resend it to editor@m500.org.uk.

Solution 251.1 – Increasing digits

> How many positive integers have the property that their digits increase when read from left to right? For example, 3, 26, 1357, but not 10, 43, 778, 34592.

Reinhardt Messerschmidt

In base b, the number of positive integers with increasing digits is

$$2^{b-1} - 1.$$

For example, in the decimal system there are $2^9 - 1 = 511$ such integers, and in base 4 there are $2^3 - 1 = 7$, namely

$$1_4, \quad 2_4, \quad 3_4, \quad 12_4, \quad 13_4, \quad 23_4, \quad 123_4.$$

Proof. A positive integer has increasing digits in base b if and only if its digits can be found by first writing out the $b-1$ nonzero digits in increasing order,

$$1, \quad 2, \quad \ldots, \quad b-1,$$

and then removing either 0 or 1 or ... or $b-2$ of them. The number of ways this can be done is

$$\sum_{k=0}^{b-2} \binom{b-1}{k} \;=\; \sum_{k=0}^{b-1} \binom{b-1}{k} - 1 \;=\; 2^{b-1} - 1. \qquad \square$$

Vincent Lynch

This is a good example of selections any number at a time. Each of the nine digits can either be in the selection or not be in it. That is two possibilities for each digit, making 2^9 selections. Once a selection has been made, there is only one way to arrange them, numerical order. The one selection where no digits are selected could correspond to the number zero, but this is excluded by the positive requirement. This leaves $2^9 - 1 = 511$.

TF — I did seriously consider including zero to make the total exactly 2^9 in the interests of mathematical nicety. Then I got distracted by a lengthy discussion involving Eddie Kent, Jeremy Humphries and myself concerning the meaningfulness of the phrase 'their digits increase when read from left to right' when applied to numbers less than 10. So it never happened.

Solution 251.2 – Thirteen boxes

> How big must h be in order to pack thirteen $17 \times 6 \times 6$ cuboids into a $33 \times 23 \times h$ box?

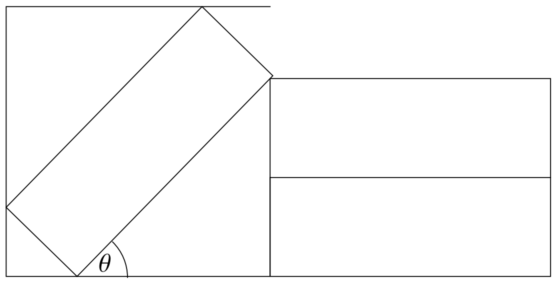
Vincent Lynch

I've tried quite a number of possibilities. This is the best I have found. There are 12 cuboids arranged in two layers to leave a hole of length 16 and width $23 - 17 = 6$. We now fit the 13th box in the hole.

Let this box make an angle of θ to the horizontal. It is clear from the diagram that $6 \sin\theta + 12 \cot\theta = 16$. Rearranging, we have $\tan\theta = 6/(8 - 3\sin\theta)$, which we can solve using the iteration

$$\theta_{n+1} \;=\; \tan^{-1} \frac{6}{8 - 3\sin\theta_n}.$$

Starting with $\theta_1 = 30°$ we end with $\theta = 45.7133°$ to 4 decimal places. Then the height needed is $h = 6\cos\theta + 17\sin\theta = 16.359$ to 3 decimal places.



Birthday cake

Colin Davies

I had a birthday in March, and my son decorated the cake with the symbols herewith. How old was I?

$$\sum_{r=\varphi^2-\varphi}^{(\varphi^2-1/\varphi)^{(\varphi^2-1/\varphi)^{(\varphi^2-1/\varphi)}}} \varphi(r)$$

Solution 250.9 – Product

Show that

$$\prod_{n=2}^{\infty} \frac{n^3+1}{n^3-1} \;=\; \frac{3}{2}.$$

Reinhardt Messerschmidt

Note that

$$\frac{n^3+1}{n^3-1} \;=\; \frac{a_n b_n}{c_n d_n},$$

where

$$a_n = n+1, \qquad b_n = n^2-n+1, \qquad c_n = n-1, \qquad d_n = n^2+n+1.$$

For every $n \geq 2$, $a_n = n+1 = (n+2)-1 = c_{n+2}$, and $b_{n+1} = (n+1)^2 - (n+1)+1 = n^2+n+1 = d_n$. It follows that for every $N \geq 3$,

$$\prod_{n=2}^{N} \frac{n^3+1}{n^3-1} \;=\; \frac{b_2}{c_2 c_3} \cdot \frac{a_{N-1} a_N}{d_N} \;=\; \frac{3}{2} \cdot \frac{N^2+N}{N^2+N+1},$$

therefore

$$\prod_{n=2}^{N} \frac{n^3+1}{n^3-1} \longrightarrow \frac{3}{2} \;\text{ as }\; N \longrightarrow \infty. \qquad \square$$

Solution 252.2 – Can

A tin can has radius r, height h and surface area 1. Choose r and h to maximize its volume.

Vincent Lynch

This is another maximization problem which can be solved without calculus. We have that, with the usual notation, $2\pi r^2 + 2\pi rh = 1$, and we seek to maximize $V = \pi r^2 h$. Consider the quantities $2\pi r^2$, πrh, πrh. Their sum is 1, so their mean 1/3 is fixed. Their product is $2\pi^3 r^4 h^2$. But this is $2\pi V^2$. The product of numbers with fixed mean or sum is greatest when the numbers are equal. So $2\pi r^2 = \pi rh$; hence $h = 2r$. Substituting, $2\pi r^2 + 4\pi r^2 = 1$. Therefore

$$r \;=\; \sqrt{\frac{1}{6\pi}}, \quad h \;=\; 2\sqrt{\frac{1}{6\pi}} \quad \text{and} \quad V_{\max} \;=\; \frac{1}{3\sqrt{6\pi}}.$$

Solution 253.2 – Quadratic

This is like finding Pythagorean triples but slightly different. Solve the quadratic $3x^2 + y^2 = z^2$ for positive integers x, y, z.

Tommy Moorhouse

We consider only solutions in which x, y and z have no factors in common (they are relatively prime in pairs). Rewriting the quadratic as

$$3x^2 = (z - y)(z + y)$$

we see that any prime factor of x divides $z + y$ or $z - y$ but not both, since then x, y and z would have a common factor. Similarly 3 divides, say, $z + y$ (the relative signs of z and y are not important). Set $3w = z + y$. Then if s is an odd prime factor of x we must have $s|z + y$ or $s|z - y$. Thus if x factorizes into relatively prime factors $x = st$ we have $w = s^2, z - y = t^2$. This leads to the solution set

$$(x, y, z) \;=\; \left(st,\; \frac{1}{2}(3s^2 + t^2),\; \frac{1}{2}(3s^2 - t^2)\right).$$

The complete set of solutions is obtained by switching the signs of x, y and z. Note that if x is even then both y and z are odd. But then $z - y$ and $z + y$ are even and we can cancel factors of 4 until we have x odd. Thus we need only consider x odd.

Solutions with small x include $(1, 1, 2)$ (with $s = t = 1$), and $(77, 13, 134)$ $(s = 7, t = 11)$.

Problem 254.6 – Two octics

Solve

$$x^8 + 4x^5 + 8 \;=\; 0.$$

This occurred in an entry submitted by Noam Elkies to the internet forum *NMBRTHRY* in which he asserts amongst other things that its Galois group is solvable. Therefore it must have a derivable exact solution. And when you have succeeded in finding all eight roots, try another equation from the same source, also with a solvable Galois group:

$$x^8 + 16x^3 + 32 \;=\; 0.$$

(Just in case you were wondering: no, neither equation has a root near π.)

Twenty-five years ago (M500 109)

Shakespeare by numbers, Stewart Cresswell

I am sure you have heard of Shakespeare's attempt to code his plays ($2B \vee \overline{2B} = ?$), but were you aware of his influence on the translation of the Bible?

Despite his literary ability, in common with many people then (and now), his spelling was not consistent, and in particular he is known to have spelt his name in many different ways (you may like to try listing the possible variants), for example, Shaksper, Shakespere, Shakspeare, etc.

Being a budding numerologist and wanting to ensure that his contribution was recorded for posterity, William first noticed that Shakspeare has two syllables which are words in themselves of four and six letters, so 46 would be a suitable key. Then he has a word with King James.

Initially James (who was, after all, 1st and 6th) had trouble with numerical codes, but then he had a brainwave. Why not use a psalm, obviously the 46th? After all, they have a special place and rota in services. William, being the wordmonger, noticed that Psalm 46 appeared to be talking about tidal waves and cessation of wars with the destruction of military weapons, and suggested that 'shake' and 'spear' (our spellings) could be incorporated. As 'shake' was the first part of his name, it should be 46 (of course) words from the beginning, and as 'spear' was the last part of his name, it should be 46 words from the end.

As a result, we have these magnificent passages.

'Though the waters thereof roar and be troubled, though the mountains shake with the swelling thereof.'

'He maketh wars to cease unto the end of the earth; he breaketh the bow, and cutteth the spear in sunder; he burneth the chariot in the fire.'

Problem 108.1 – Darts

On a standard dartboard, what is the lowest total you can't score with one, two, three, ..., n darts?

TF writes (in 2013) — Whilst reading in M500 **109** the extensive solutions offered by Colin Lindsay and Graham Hawes, who show that the answer is $60n - 17$, I cannot help wondering if there is an easy way to get this result. That is why I have recycled the problem. To make things clear, on the dartboard in question a single dart can score any of 0, 1, ..., 22, 24, 25, 26, 27, 28, 30, 32, 33, 34, 36, 38, 39, 40, 42, 45, 48, 50, 51, 54, 57, 60.

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Front cover: Four points and a square; 48 solutions; see page 14.

M500 Winter Weekend 2014

The **thirty-third M500 Society Winter Weekend** will be held at

Florence Boot Hall, **Nottingham University**

Friday 3rd – Sunday 5th January 2014.

Cost: £205 to M500 members, £210 to non-members. Get a booking form from the M500 web site: http://www.m500.org.uk/winter/booking.pdf.

The Weekend provides you with an opportunity to do some non course based, recreational mathematics with a friendly group of like-minded people. The relaxed and social approach delivers maths for fun. In addition, on Friday we will be running a pub quiz with Valuable Prizes. We hope to see you there.