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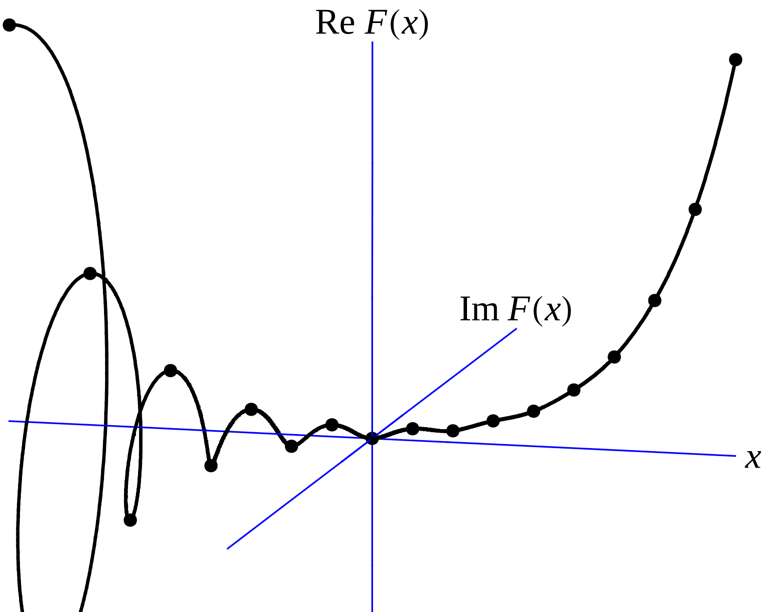
M500 312



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Complex Fibonacci

Martin Hansen

One of the highlights of an undergraduate mathematician's first year at university must surely be the revelation that many functions are amenable to being approximated by polynomials. In particular, there are the following three marvellous results, sometimes referred to as 'Taylor Polynomials' (about $x = 0$) or 'Maclaurin Series' or 'Power Series'.

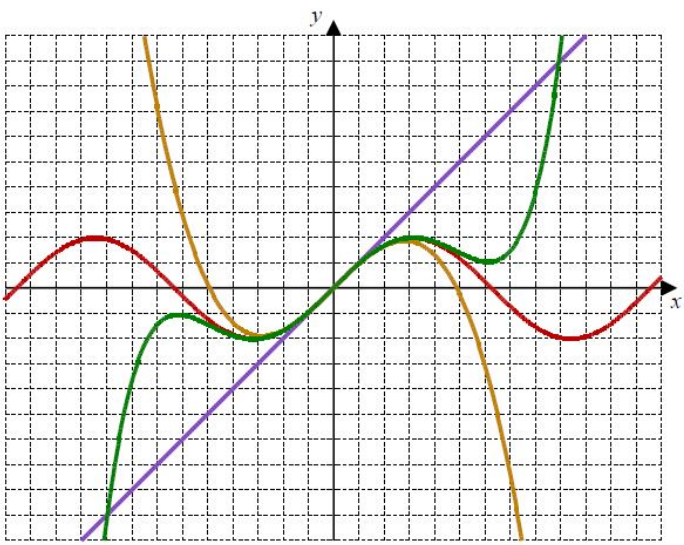
Power Series Expansions (centred on $x = 0$)

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^r}{r!} + \ldots \qquad \text{valid for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{valid for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{valid for all } x$$

Pleasingly, any one of these results can be visualised by taking successive partial sums of the series and plotting graphs. Below are shown approximations of the sine function with $y = x$, $y = x - x^3/3!$ and $y = x - x^3/3! + x^5/5!$. As the degree of the approximating polynomial is increased, it better follows the sine curve over a greater interval, centred on the origin.



A key fact about power series is that any two (with the same centre) can be added, multiplied or divided in the same way as polynomials. This suggests that the exponential series can be manipulated in the following adventurous manner when the index is a complex number, $z = a + ib$, $a, b \in \mathbb{R}$, $i^2 = -1$:

$$\begin{aligned}
e^z &= e^{a+ib} \\
&= e^a e^{ib} \\
&= e^a \left(1 + ib + \frac{(ib)^2}{2!} + \frac{(ib)^3}{3!} + \frac{(ib)^4}{4!} + \frac{(ib)^5}{5!} + \frac{(ib)^6}{6!} + \dots\right) \\
&= e^a \left(1 + ib - \frac{b^2}{2!} - \frac{ib^3}{3!} + \frac{b^4}{4!} + \frac{ib^5}{5!} - \frac{b^6}{6!} + \dots\right) \\
&= e^a \left(\left(1 - \frac{b^2}{2!} + \frac{b^4}{4!} - \frac{b^6}{6!} + \dots\right) + i\left(b - \frac{b^3}{3!} + \frac{b^5}{5!} - \dots\right)\right) \\
&= e^a(\cos b + i \sin b).
\end{aligned}$$

The real exponential function is thus extended into the world of complex numbers via the beautiful result known as Euler's Relation.

Euler's Relation

$$e^{ib} = \cos b + i \sin b.$$

When $b = \pi$ Euler's Relation yields $e^{\pi i} = -1$, usually written as

$$e^{\pi i} + 1 = 0$$

and often referred to as the most beautiful equation in all of mathematics. Of course, when $b = 0$ the complex exponential function is identical to the real exponential function. In this case, a real function has been extended into the complex realm by a simple multiplication of $\cos b + i \sin b$. It is natural to wonder if there are other functions extendable in this sort of way.

A drawback for beginners in trying to appreciate what has been achieved with the extension of the exponential from the real to the complex is that the complex exponential function is tricky to visualise. Its domain is two dimensional as is its codomain. Visualising four dimensions simultaneously does not come naturally to the average person. In this article I wanted to look at a similar extension to a well-known function but one which yields results more easily visualised.

The function I have in mind is that associated with the Fibonacci numbers. These are usually defined by means of the simple recursive formula,

$$F_{n+2} \;=\; F_{n+1} + F_n, \qquad n \in \mathbb{Z}, \quad n \geq 0,$$

along with two initial terms $F_0 = 0$ and $F_1 = 1$. It gives rise to the 'world famous' sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots .$$

This has so many astounding mathematical properties that a scholarly journal, *The Fibonacci Quarterly*, is devoted to ongoing research of this and related sequences such as those of Lucas, Jacobsthal, and Pell.

The domain can be extended readily enough to include the negative integers. The resulting 'extended leftward' sequence is

$$\ldots, -21, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots .$$

Part of the fascination of the Fibonacci sequence stems from the fact that it has a closed form formula for term n that, although we are working with the integers, contains fractions and square roots 'all over the place'. Yet it will yield an integer output no matter what integer input is assigned to n.

The Binet Formula for the Fibonacci Sequence (Version 1)

$$F_n \;=\; \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right), \qquad n \in \mathbb{Z}$$

Having extended the domain of the Fibonacci sequence to include the negative integers, the Binet formula provides an opportunity to go further and extend it to the reals. Of course, $1 - \sqrt{5}$ is negative, and so for some values of n its powers will be complex numbers. When I reached for my calculator (a Casio Classwiz fx-991EX in complex number mode) it gave, for $n = 0.5$,

$$\begin{aligned} F_{0.5} \;&=\; \frac{1}{\sqrt{5}}\left(\sqrt{\frac{1+\sqrt{5}}{2}} - \sqrt{\frac{1-\sqrt{5}}{2}}\right) \\ &=\; 0.569 - 0.352\,i. \end{aligned}$$

However, it could not cope with the same calculation presented in the form

$$F_{0.5} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{0.5} - \left(\frac{1-\sqrt{5}}{2}\right)^{0.5}\right).$$

In retrospect, this was a blessing; rather than reaching for more powerful software a deeper understanding was called for.

The key idea is to find a way of separating the real part of the calculation from the imaginary part and an ingenious way to do this is to be found in a paper from 1968 in *The Fibonacci Quarterly* by Alan Scott, [1].

To understand Scott's result we first need to look more carefully at $e^{i\pi} = -1$. Values of the function $f(b) = e^{ib}$ are best understood by visualising them as being on the unit circle in the complex plane. The variable b then has the interpretation of being the angle (in radians) of (anticlockwise) rotation, where the positive real axis corresponds to an angle of 0. The diagram on page 4 shows a few points plotted for b between $-\pi$ and π radians. The crucial point being made by this diagram is that the result can be equally well written as $e^{-i\pi} = -1$. In fact, given any point on the unit circle, a rotation of $2\pi k$ (about the centre) for any integer k gives the same point.

In general, we have that $e^{i(2k+1)\pi} = -1$, $k \in \mathbb{Z}$.

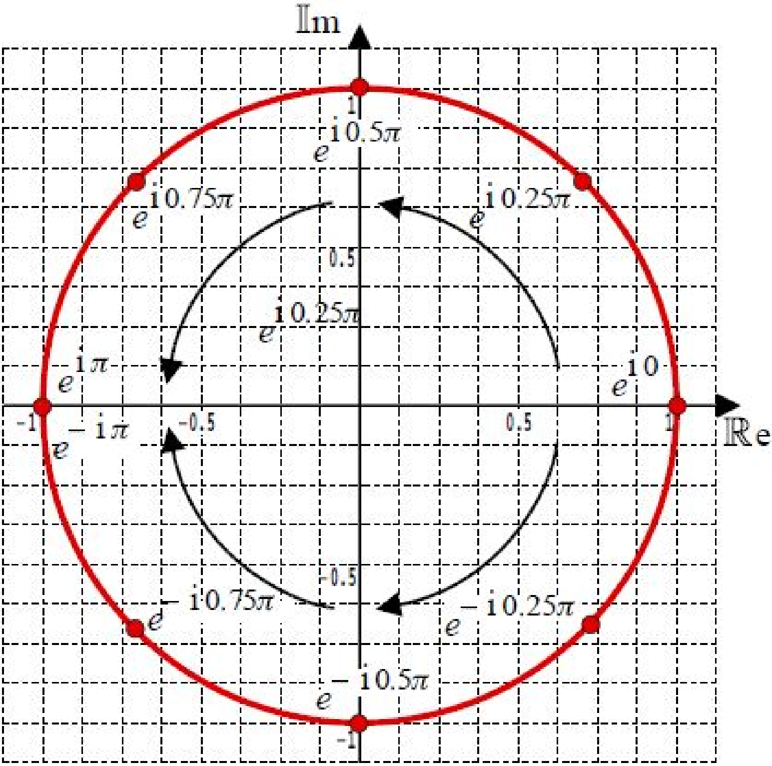
The digression over, we can pick up the main thread of the article. Observe that

$$\left(\frac{1-\sqrt{5}}{2}\right)^{n} = \left(\frac{2}{1-\sqrt{5}}\right)^{-n} = (-1)^{-n}\left(\frac{1+\sqrt{5}}{2}\right)^{-n}.$$

From this observation we obtain

The Binet Formula for the Fibonacci Sequence (Version 2)

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n} - (-1)^{-n}\left(\frac{1+\sqrt{5}}{2}\right)^{-n}\right)$$



Now, we could recall here that $e^{i\pi} = -1$ (so beautiful!) but, to match the result my calculator produced earlier, let's use $e^{-i\pi} = -1$ instead:

$$(-1)^{-n} \;=\; \left(e^{-\pi}\right)^{-n} \;=\; e^{i\pi n} \;=\; \cos(\pi n) + i\sin(\pi n).$$

Thus is obtained a third version of the Binet formula for the Fibonacci numbers.

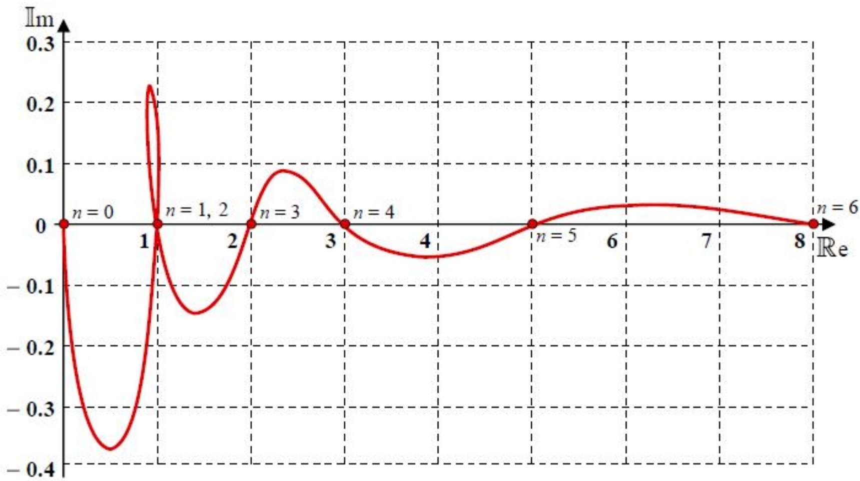
The Binet Formula for the Fibonacci Sequence (Version 3)

$$\operatorname{Re} F_n \;=\; \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \cos(\pi n)\left(\frac{1+\sqrt{5}}{2}\right)^{-n}\right)$$

$$\operatorname{Im} F_n \;=\; -\frac{1}{\sqrt{5}}\left(\sin(\pi n)\left(\frac{1+\sqrt{5}}{2}\right)^{-n}\right)$$

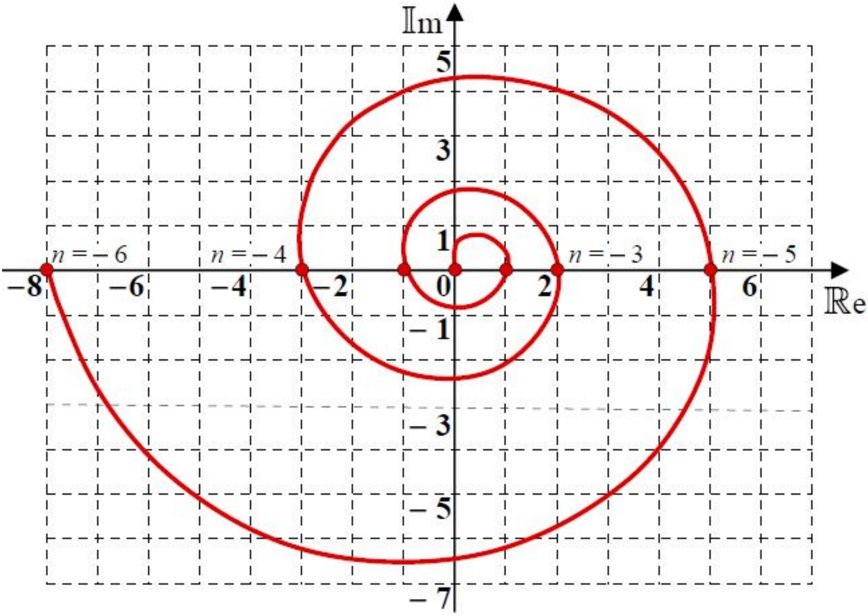
These can be interpreted as parametric equations where x is the real component and y the imaginary, and plotted as a smooth continuous curve where n is considered to be a variable over $\mathbb{R}$.

Such a plot for non-negative real n is presented below. The diagram is of the two dimensional output from the Fibonacci function with one dimensional real number domain and the ease with which it can be understood is aided considerably by knowing the integer sequence from which it was derived. Each time the curve crosses the real number axis as we move along the curve, the domain has integer value incremented by 1. So, three dimensions can be easily visualised in spite of having only the two dimensional codomain plot to study.



The loop from $n = 1$ to $n = 2$ is an attractive feature corresponding to $F_1 = F_2 = 1$ and, indeed, it can be viewed as a part of the interesting transition between the alternating sign of the integer outputs when n is a negative integer and the all positive integer outputs when n is a positive integer. The plot for the non-positive real n is given on the next page.

The earliest plots of this Fibonacci curve that I know of occurred in 1974 [2]. For any reader wishing to investigate the curve further, many clever mathematical results are to be found in a 1988 paper [3].



References (All from *FQ*, *The Fibonacci Quarterly*)

[1] A. M. Scott, Continuous Extensions of Fibonacci Identities, *FQ*, vol. 6(4), Oct 1968.

[2] F. J. Wunderlich, D. E. Shaw and M. J. Hones, Argand Diagrams of Extended Fibonacci and Lucas Numbers, *FQ*, vol. 12(3), Oct 1974.

[3] A. F. Horadam and A. G. Shannon, Fibonacci and Lucas Curves, *FQ*, vol. 26(1), Feb 1988.

Problem 312.1 – Product

Tony Forbes

Show that

$$P_N(x) \;=\; \prod_{i=1}^{N} \frac{4i+x}{4i+1}\,\frac{4i+3}{4i+2}, \qquad -1 < x < 1,$$

converges to something non-zero as $N \to \infty$ if and only if $x = 0$.

A trigonometric series

David Sixsmith

1 The problem

In M500 Problem 310.1 we were are asked to show that

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} \;=\; \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2} \;=\; \frac{\pi - 1}{2}.$$

By taking a quite general approach, we will succeed in proving both these equalities. We will also prove many other equalities previously calculated by MATHEMATICA, as well as a number of other well-known series results.

2 A solution

To solve a set of slightly more general problems, define, for $p \in \mathbb{N}$,

$$s_p(x) \;=\; \sum_{n=1}^{\infty} \frac{\sin^p nx}{n^p}.$$

So we are asked in the problem to calculate both $s_1(1)$ and $s_2(1)$. To do this we will develop a formula for s_1. It is useful to note first that

$$\log(1 - z) \;=\; -\sum_{n=1}^{\infty} \frac{z}{n}, \quad \text{for } \{z \in \mathbb{C} : |z| \le 1 \text{ and } z \ne 1\}. \tag{1}$$

It is also helpful to recall that

$$\arctan(\tan x) \;=\; x - \pi \left\lfloor \frac{x}{\pi} + \frac{1}{2} \right\rfloor, \tag{2}$$

where $\lfloor x \rfloor$ denotes the floor function.

Now, by de Moivre's theorem, by (1), by (2), and using the fact that

$$\lfloor y \rfloor \;=\; -\lfloor 1 - y \rfloor,$$

we obtain

$$
\begin{aligned}
s_1(x) &= \operatorname{Im}\left(\sum_{n=1}^{\infty} \frac{(\exp ix)^n}{n}\right) \\
&= -\operatorname{Im}(\log(1-\exp(ix))) \\
&= -\arg(1-\exp(ix)) \\
&= \arctan\left(\frac{\sin x}{1-\cos x}\right) \\
&= \arctan\left(\cot\left(\frac{x}{2}\right)\right) \\
&= \arctan\left(\tan\left(\frac{\pi-x}{2}\right)\right) \\
&= \frac{\pi-x}{2} + \pi\left\lfloor\frac{x}{2\pi}\right\rfloor
\end{aligned}
$$

whenever x is not an integer multiple of 2π. In particular,

$$s_1(1) = \frac{\pi-1}{2},$$

as required. We also get a proof of the well-known result that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n-1)} = s_1(\pi/2) = \frac{\pi}{4}.$$

We also have that

$$\frac{ds_2(x)}{dx} = \sum_{n=1}^{\infty}\frac{\sin 2nx}{n} = s_1(2x) = \frac{\pi}{2} - x.$$

Integration, and the fact that $s_2(0) = 0$, gives that

$$s_2(x) = \frac{x(\pi-x)}{2} \quad \text{when } x \in (0,\pi).$$

Hence

$$s_2(1) = \frac{\pi-1}{2},$$

as required. It is also interesting to note the well known result that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = s_2(\pi/2) = \frac{\pi^2}{8}.$$

3 The third to sixth series

We can in fact continue this approach. Using the fact that

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4},$$

we obtain by differentiating twice that

$$\frac{d^2 s_3}{dx^2} = \frac{9s_1(3x) - 3s_1(x)}{4} = \frac{3\pi}{4} - 3x.$$

Since s_3 and its derivative are both zero at $x = 0$, we obtain

$$s_3(x) = \frac{3\pi}{8}x^2 - \frac{x^3}{2},$$

and so

$$s_3(1) = \frac{3\pi - 4}{8}.$$

Using this technique we find that

$$s_4(x) = \frac{\pi}{3}x^3 - \frac{x^4}{2}, \quad \text{and so} \quad s_4(1) = \frac{\pi}{3} - \frac{1}{2},$$

$$s_5(x) = \frac{115\pi}{384}x^4 - \frac{x^5}{2}, \quad \text{and so} \quad s_5(1) = \frac{115\pi}{384} - \frac{1}{2},$$

and finally (and computer calculations are a boon here)

$$s_6(x) = \frac{11\pi}{40}x^5 - \frac{x^6}{2}, \quad \text{and so} \quad s_6(1) = \frac{11\pi}{40} - \frac{1}{2}.$$

4 The seventh term and onwards

Note that all the terms $s_1(1), \ldots, s_6(1)$ are of the form $p\pi - 1/2$, where p is rational. However, as was noted in M500[1], we have

$$s_7(1) = \frac{1}{46080}\big(-23040 + 129423\pi - 201684\pi^2 + 144060\pi^3 - 54880\pi^4 + 11760\pi^5 - 1344\pi^6 + 64\pi^7\big).$$

This change in the fundamental nature of the terms when we transition from 6 to 7 was (rightly) described as 'mystifying'.

[1] Tony Forbes, A trigonometric series, M500 **310**, 12–13

In fact we can see that this change follows from our formula for $s_1(x)$, which is easiest to integrate when $x \in (0, 2\pi)$. Since $6 < 2\pi < 7$, this implies that all calculations up to $s_6(1)$ use this simplest form of s_1, but from s_7 onwards we need to integrate a more complicated expression.

The calculations are complicated, and best left to MATHEMATICA. I will only outline the start of the calculations. First we write

$$\sin^7 x = \frac{1}{64}(35 \sin x - 21 \sin 3x + 7 \sin 5x - \sin 7x).$$

Differentiating six times gives

$$\frac{d^6(\sin^7 x)}{dx^6} = -\frac{7}{64}(5 \sin x - 2187 \sin 3x + 15635 \sin 5x - 16807 \sin 7x),$$

and so

$$\frac{d^6 s_7}{dx^6} = -\frac{7}{64}(5 s_1(x) - 2187 s_1(3x) + 15635 s_1(5x) - 16807 s_1(7x)).$$

The difficulty comes in integrating the final term. For values of x close to 7 (in particular when $2\pi < x < 4\pi$, we have that

$$\int_0^x s_1(t)\ dt = \int_{2\pi}^x \frac{3\pi - t}{2}\ dt = \frac{6\pi x - x^2 - 8\pi^2}{4}.$$

It should now be clear where the powers of π come from – the six necessary repeated integrations of s_1.

Note that between 6 and 7, $x = 7$ is the only point where this transition happens. For example, when calculating the integral of $s_1(x)$ for values near 13 (note that $12 < 4\pi < 13$) we continue to get a polynomial in π of degree 2. It follows (with a little thought!) that $s_k(1)$ is a polynomial in π of degree 1 when $1 \le k \le 6$, and of degree k otherwise.

5 Problem

Prove that the constant term (i.e. the term independent of π) in $s_k(1)$ is always $-1/2$.

6 The series with cosine

An obvious extension is to consider the related series, for $p \in \mathbb{N}$,

$$c_p(x) = \sum_{n=1}^{\infty} \frac{\cos^p nx}{n^p}.$$

We will find a formula for c_1. By de Moivre's theorem again, we obtain

$$\begin{aligned} c_1(x) &= \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{(\exp ix)^n}{n}\right) \\ &= -\operatorname{Re}(\log(1-\exp(ix))) \\ &= -\log|1-\exp(ix)| \\ &= -\log\sqrt{(1-\cos x)^2+\sin^2 x} \\ &= -\log|2\sin(x/2)| \end{aligned}$$

whenever x is not an integer multiple of 2π. In particular,

$$\sum_{n=1}^{\infty} \frac{\cos n}{n} = c_1(1) = -\log|2\sin(1/2)|.$$

We also have that

$$\frac{dc_2(x)}{dx} = -\sum_{n=1}^{\infty} \frac{\sin 2nx}{n} = s_1(2x) = -\frac{\pi}{2} + x.$$

Integration, and the fact that

$$c_2(0) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

gives that

$$c_2(x) = \frac{x(x-\pi)}{2} + \frac{\pi^2}{6}, \qquad \text{for } 0 \le x < \pi.$$

Thus

$$\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2} = c_2(1) = \frac{1-\pi}{2} + \frac{\pi^2}{6}.$$

This is to be expected in fact, as

$$c_2(x) + s_2(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

7 A new problem

It seems natural to ask if series such as

$$t_p = \sum_{n=1}^{\infty} \frac{\tan n}{n^p}$$

can converge, at least for large enough values of p. MATHEMATICA (via Wolfram Alpha) thinks t_1 and even t_{10} are divergent. It also thinks

$$\sum_{n=1}^{\infty} \frac{\tan n}{e^n}$$

is divergent, even though it gives the sum of the first $100,000$ terms as just over 0.26255. It claims that

$$\sum_{n=1}^{\infty} \frac{\tan n}{e^{e^n}}$$

is convergent.

PROBLEM: Prove convergence (or otherwise) of any of these series.

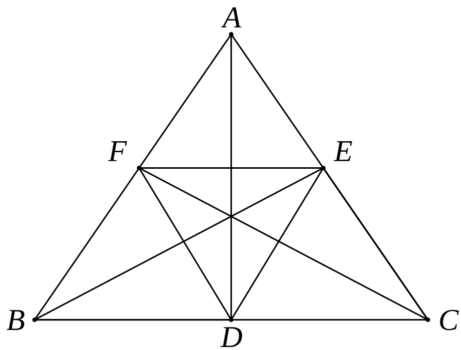
8 Acknowledgement

I am very grateful to Tony Forbes for helpful comments, suggestions, and use of MATHEMATICA to check and suggest formulae.

Problem 312.2 – Isosceles triangles

Take a triangle with vertices ABC, and denote by D, E, F the intersections on the opposite sides of the bisectors of the angles A, B, C, respectively.

If $\triangle ABC$ is isosceles, then clearly so is $\triangle DEF$. Either show that the converse is true, or find a counter-example.



Folding a square

Tommy Moorhouse

This investigation was inspired by a puzzle concerning the folding of a strip of paper, although I'm not sure who thought it up. Take a strip of paper, lay it on a table and mark the left edge. Take the other end of the strip and fold in half, right over left, so that the marked edge is lowermost. When you unfold the strip there is a single downward-pointing fold. Fold in half as before, then take the folded edge and fold the strip in half again. Unfolding again we get the following pattern of folds (Figure 1):

Figure 1: Paper strip after second turn.

or, symbolically,

$$\downarrow\downarrow\uparrow.$$

The idea is to fold repeatedly, right over left, and see if you can find a rule for the pattern of folds after n turns. You could try this if you've not seen it before.

I wondered if it would make sense to do a similar thing with a square sheet of paper (or some easily folded material). Take the sheet and mark the top left corner. Fold in half, bottom over top, then fold the rectangle in half, right over left.

Call this 'Folding 1'. When the paper is unfolded (see Figure 2) the pattern of folds, going clockwise from the marked corner, is

$$\downarrow\downarrow\uparrow\downarrow.$$

Re-fold the square and repeat the folding (top/bottom then right/left) with this smaller square. This is 'Folding 2'. Now there are more folds and in order to get a one-dimensional sequence of folds we have to decide on a path over the paper that crosses (or at least visits) every fold. This is where graphs come in. We will associate a graph ϕ_k with Folding k.

Starting from the marked corner draw a vertex for each fold. Connect together those vertices that represent folds on the edge of the same square.

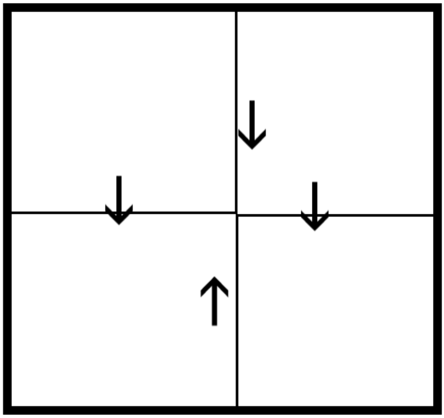


Figure 2: Paper after Folding 1

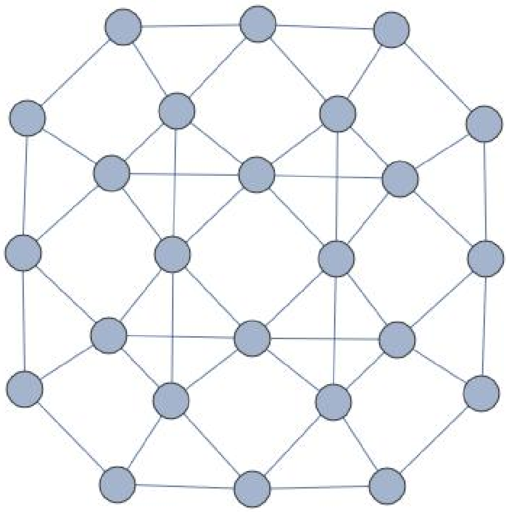


Figure 3: The graph ϕ_2.

The case $k = 2$ is shown in Figure 3.

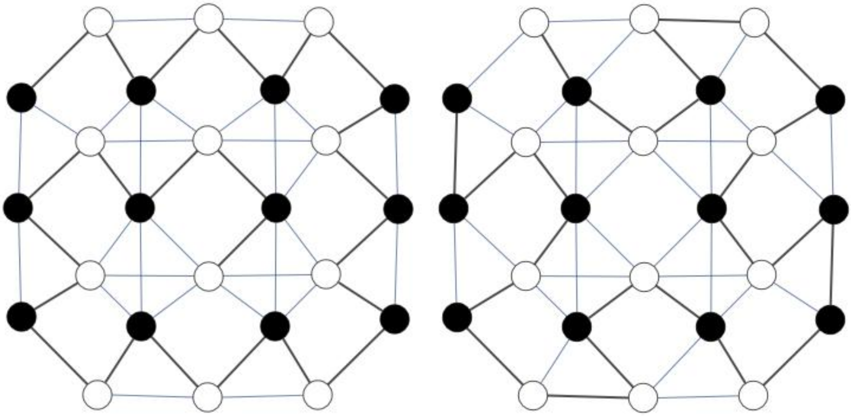


Figure 4: Two Hamilton walks in ϕ_2.

Investigation What is the structure of the graph ϕ_k? For example, how many edges and vertices are there and how are they connected? In particular, is there a Hamilton walk, i.e. a path starting from any vertex and following edges through every other vertex exactly once, for every ϕ_k? An open walk (starting and ending at different vertices) is what we need. Figure 4 shows two Hamilton walks in ϕ_2. The colouring of the vertices is simply intended to indicate some of the structure of the graph. Both walks are perfectly good, but I think that the right-hand walk can more easily be extended to higher ϕ_k. I could be wrong!

If there is a Hamilton walk in every ϕ_k then it may make sense to study the one-dimensional sequence of folds given by Folding k, i.e. the 2-d extension of the 1-d folded strip problem. I believe (but have not proved) that the choice of Hamilton walk in each graph can be made consistently, and I have drawn a walk in ϕ_3 illustrating a walk w_3 that may hint at a general approach. Can you prove this, or perhaps find a counterexample?

One line of investigation is whether it is possible to find a rule for the pattern of folds for Folding k defined above, given a well defined sequence of Hamilton walks $(w_1 \in \phi_1, w_2 \in \phi_2, \cdots, w_k \in \phi_k)$. Each w_n must contain all the vertices of w_{n-1} – in some sense 'in the same order' – with additional vertices inserted. I have in mind an inclusion map from the set of vertices of ϕ_k to a subset of those of ϕ_{k+1}, but the details need to be worked out.

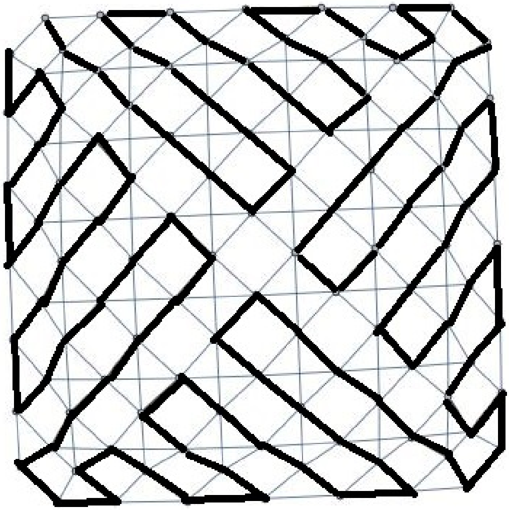


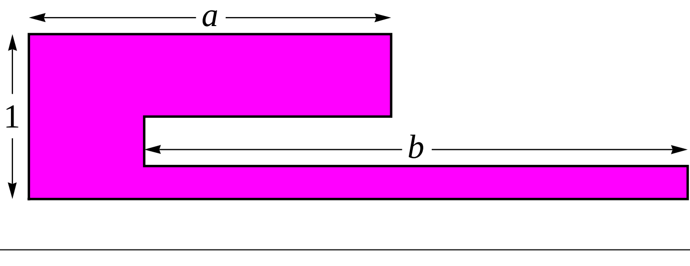
Figure 5: Hamiltonian walk in ϕ_3.

Looking at the graphs above it seems that the ϕ_k are included in a sequence of graphs that can be drawn with n vertices on the first, third, fifth etc. rows, with $n + 1$ vertices on the second, fourth etc. rows and connected 'in the same way' as ϕ_2. Is this in fact the case, and do the more general graphs arise from foldings of a square?

Problem 312.3 – Area and perimeter

Jeremy Humphries

What's the area of the shape? What about the perimeter?



Solution 306.6 – Pistachio nuts

> Pistachio nuts can be an important component of a well-balanced diet and have featured significantly in past issues of this magazine as Problems 275.6, 278.1, 290.5 and 283.6 as well as in some cases their solutions. Just when I [TF] thought the subject was dead I discovered an interesting simple variation that I had overlooked. There is a bowl containing n pistachio nuts. How may times would you expect to perform the following procedure in order to consume all of the edible material in the bowl?

(i) You select uniformly at random one object from the bowl. It might be a whole pistachio nut in its shell, or just a pistachio nut kernel without its shell, or just half of a pistachio nut shell.

(ii) If it is a half-shell, you return it to the bowl.

(iii) If it is a naked pistachio nut kernel, you consume it.

(iv) If it is a pistachio nut in its shell, you split it into its three components, kernel and two half-shells, which you return to the bowl.

Ted Gore

Let n be the number of whole nuts at the start of the process. A computer simulation was run for some chosen values of n and the average number of picks is shown in the table on page 21.

I approached this problem by considering that the process for a given n is made up of a number of batches. A batch consists of a number of picks that result in a half-shell being selected until it is terminated by selection of either a whole nut or a kernel. There will always be $2n$ batches when there are n nuts to start with; one batch for each whole nut and one for each kernel.

The simulation keeps a count of the total number of picks for each n. This appears in the table as $P(\text{sim})$. It also counts the number of kernels and half-shells in the bowl at the end of each batch.

For batch i, let $W(i)$ be the number of whole nuts in the bowl at the end of the batch; let $K(i)$ be the number of kernels and let $H(i)$ be the number of half-shells. At the start of batch i the probability of picking a whole nut is

$$\frac{W(i-1)}{W(i-1)+H(i-1)+K(i-1)}$$

and the probability of a kernel is

$$\frac{K(i-1)}{W(i-1)+H(i-1)+K(i-1)}.$$

The probability of picking either a nut or a kernel is

$$\frac{W(i-1)+K(i-1)}{W(i-1)+H(i-1)+K(i-1)}$$

and the expected number of picks required to get that result is

$$\begin{aligned}&\frac{W(i-1)+H(i-1)+K(i-1)}{W(i-1)+K(i-1)}\\ =&\ 1+\frac{H(i-1)}{W(i-1)+K(i-1)}\\ =&\ 1+\frac{H(i-1)}{n-H(i-1)/2+K(i-1)}\end{aligned}$$

since $W(i-1) = n - H(i-1)/2$. If $P(n)$ is the total number of picks required for a given n, then

$$P(n) = \sum_{i=1}^{2n} 1 + \frac{H(i-1)}{n-H(i-1)/2+K(i-1)}.$$

The functions H and K can be mapped to canonical functions h and k with domain $[0,1]$ and image $[0,1]$ by dividing all i, $K(i)$ and $H(i)$ by $2n$.

The number of kernels in the bowl increases from 0 to a maximum and then reduces to 0 when the last one is eaten. Let $i_{\max}$ be the batch that has the maximum number of kernels and let

$$x_{\max} = \frac{i_{\max}}{2n}.$$

The values of h and k at $x_{\max}$ are $h_{\max}$ and $k_{\max}$.

Let $x = \dfrac{i-1}{2n}$, let $H(i-1) = 2nh(x)$ and let $K(i-1) = 2nk(x)$. This will give us

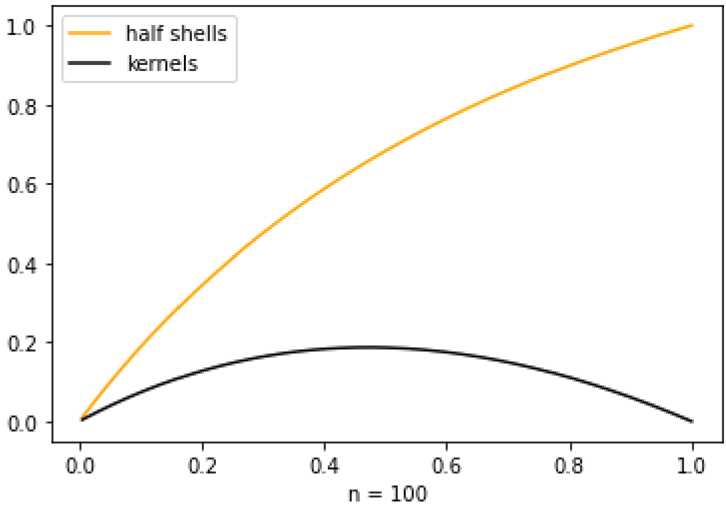
$$P(n) = \sum_{i=1}^{2n} 1 + \frac{2nh(x)}{n-2nh(x)/2+2nk(x)}$$

$$= \sum_{i=1}^{2n} 1 + \frac{h(x)}{0.5 - h(x)/2 + k(x)},$$

so that

$$\frac{P(n)}{2n} = \sum_{i=1}^{2n} \frac{1}{2n} + \frac{h(x)}{n - nh(x) + 2nk(x)}.$$

It would be useful to have functions that tell us the values of h and k for any batch for any n. The graphs of h and k are similar for $n \geq 10$. The one shown is for $n = 100$.



By experimentation it seems that reasonable functions are

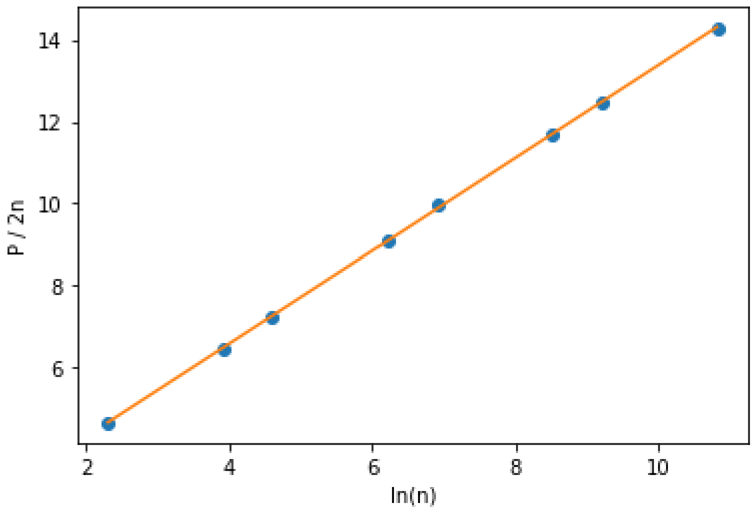
$$h(x) = \frac{a}{1 + be^x} + c,$$
$$k(x) = px + qx^2 + rx^3.$$

The values in the table suggest that it also seems reasonable to work with average values for $x_{\max}$, $k_{\max}$ and $h_{\max}$.

We have three simultaneous equations for each of h and k. Solving the two sets of equations gives the values of a, b, c, p, q, r. A description of the values obtained is detailed in an appendix below the table.

The results for $P/(2n)$ using these values appear in the table as $\frac{P}{2n}$(calc).

This leads to a further result. Using PYTHON's curve fitting software to plot $P(n)/(2n)$(calc) against $\ln(n)$ I obtained the graph below. It suggests that $P(n)/(2n) \approx 2.067 + 1.1312\ln(n)$ and these results appear in the column marked $P/(2n)$(final).



n	P(sim)	$\frac{P}{2n}$(sim)	$x_{\max}$	$k_{\max}$	$h_{\max}$	$\frac{P}{2n}$(calc)	$\frac{P}{2n}$(final)
10	92.112	4.6506	0.45	0.2011	0.6511	4.6609	5.0135
50	652.793	6.5279	0.46	0.1866	0.6466	6.4787	6.4924
100	1468.995	7.3450	0.45	0.1866	0.6366	7.2622	7.2765
500	9091.452	9.0915	0.438	0.1839	0.6220	9.0817	9.0971
1000	19,703.631	9.8518	0.4415	0.1841	0.6256	9.9865	9.8812
5000	115,717.121	11.5717	0.4521	0.1841	0.6362	11.6852	11.7018
10000	250,242.633	12.5121	0.4499	0.1838	0.6337	12.4690	12.4859
50000	1,459,062.264	14.5906	0.4492	0.1839	0.6331	14.2888	14.3065
mean			0.4488	0.1867	0.6356		

Appendix: Solving the equations for h and k

Let Θ, Φ, Ψ be the averages of $x_{\max}$, $h_{\max}$, $k_{\max}$. For h, let $L = e^1$, $M = e^\Theta$, $S = e^{1/(2n)}$. We have three equations,

$$\begin{aligned} h(1/(2n)) &= 1/n = \frac{a}{1+bS} + c, \\ h(\Theta) &= \Phi = \frac{a}{1+bM} + c, \end{aligned}$$

$$h(1) \;=\; 1 \;=\; \frac{a}{1+bL}.$$

Solving these we get

$$\begin{aligned} b &= \frac{(n-1)(M-L) - n(1-\Phi)(S-L)}{n(1-\Phi)(S-L)M - (n-1)(M-L)S}, \\ a &= \frac{(n-1)(1+bS)(1+bL)}{nb(S-L)}, \\ c &= \Phi - \frac{a}{1+bM}. \end{aligned}$$

For $n = 100$, $(a, b, c) = (2.8826, -3.0697, 1.3925)$.

For k, let $L = 1$, $M = \Theta$, $S = 1/(2n)$. We have three equations

$$\begin{aligned} k(1/(2n)) \;=\; 1/(2n) &\;=\; pS + qS^2 + rS^3, \\ k(\Theta) \;=\; \Psi &\;=\; pM + qM^2 + rM^3, \\ k(1) \;=\; 0 &\;=\; pH + qH^2 + rH^3. \end{aligned}$$

Solving these we get

$$\begin{aligned} q &= \frac{(2n)^2}{1-(2n)^2}, \\ p &= \frac{\Psi L^2}{M(L^2-M^2)} - \frac{qLM}{(M+L)}, \\ r &= \frac{\Psi}{M(M^2-L^2)} - \frac{q}{(M+L)}. \end{aligned}$$

For $n = 100$, $(p, q, r) = (0.8307, -1.000, 0.1693)$.

Problem 312.4 – Counting solutions

Tony Forbes

(i) Let q be a prime. Show that the number of solutions (x, y), $0 \le x, y < q$ of $x^2 + y^2 + 1 \equiv 0 \pmod q$ plus the number of solutions y, $0 \le y < q$ of $1 + y^2 \equiv 0 \pmod q$ is $q + 1$.

(ii) Let q be a composite prime power, and assume in what follows that addition and multiplication are done in the finite field $\mathrm{GF}(q)$. Show that the number of solutions (x, y), $x, y \in \mathrm{GF}(q)$ of $x^2 + y^2 + 1 = 0$ plus the number of solutions y, $y \in \mathrm{GF}(q)$ of $1 + y^2 = 0$ is $q + 1$.

Solution 310.1 – A trigonometric series

Show that

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} \;=\; \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2} \;=\; \frac{\pi-1}{2}.$$

Henry Ricardo

Using Euler's formula, we write $(e^{ix})^n = \cos nx + i\sin nx$ and $(e^{-ix})^n = \cos nx - i\sin nx$, so that $\sin nx = \left((e^{ix})^n - (e^{-ix})^n\right)/(2i)$. Then we have

$$\begin{aligned}
S \;&=\; \sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{1}{2i}\left(\sum_{n=1}^{\infty} \frac{(e^{ix})^n}{n} \;-\; \sum_{n=1}^{\infty} \frac{(e^{-ix})^n}{n}\right) \\
&= \frac{1}{2i}\log\frac{1-\,e^{-ix}}{1-e^{ix}} \qquad \left(\text{using } -\log(1-z) = \sum_{n=1}^{\infty} z^n/n,\ |z|\le 1,\ z\ne 1\right) \\
&= \frac{1}{2i}\log\frac{1-\cos x + i\sin x}{1-\cos x - i\sin x} \\
&= \frac{1}{2i}\log\frac{1+iu}{1-iu},\ \text{ where } u = \frac{\sin x}{1-\cos x}.
\end{aligned}$$

Since

$$\log\frac{1+iu}{1-iu} \;=\; 2\,i\,\tan^{-1} u,$$

we conclude that

$$S \;=\; \tan^{-1}\frac{\sin x}{1-\cos x} \;=\; \frac{\pi-x}{2},\ \ 0<x<2\pi. \qquad (1)$$

Setting $x=1$ completes the proof of this part of the problem.

Now integrating both sides of formula (1) with respect to x from 0 to t, we get

$$\sum_{n=1}^{\infty} \frac{1-\cos nt}{n^2} \;=\; \frac{\pi t}{2} \;-\; \frac{t^2}{4} \text{ for } t\in(0,2\pi).$$

Then

$$\sum_{n=1}^{\infty} \frac{\sin^2 nt}{n^2} \;=\; \frac{1}{2}\sum_{n=1}^{\infty} \frac{1-\cos 2nt}{n^2} \;=\; \frac{\pi t}{2} \;-\; \frac{t^2}{2}.$$

Setting $t=x=1$ yields the desired result.

Solution 306.8 – Floor, ceiling and square root

If n is a positive integer that is not an integer square, show that

$$\left\lceil \frac{n}{\lfloor\sqrt{n}\rfloor} \right\rceil - \lfloor\sqrt{n}\rfloor \;=\; \frac{3}{2} - \frac{1}{2}\big(\operatorname{sign}\sin(2\pi\sqrt{n})\big).$$

Ted Gore

Let $k^2 < n < (k+1)^2$ where k is an integer. There are $2k$ possible values for n. For all these, $\lfloor\sqrt{n}\rfloor = k$. The table shows results for $k = 3$.

r (row)	n	$\sqrt{n} = k + \varepsilon$	n/k	$\lceil n/k \rceil$	$\lceil n/k \rceil - k$	$\sin(2\pi\varepsilon)$
1	10	3.1623	10/3	4	1	0.8520
2	11	3.3166	11/3	4	1	0.9137
3	12	3.4641	4	4	1	0.2237
4	13	3.6056	13/3	5	2	−0.6159
5	14	3.7417	14/3	5	2	−0.9986
6	15	3.8730	5	5	2	−0.7159

Now, $n/k = (k^2 + r)/k$ and this will only be an integer when r is a multiple of k.

For row k, $n/k = k+1$ and this is the ceiling of n/k for the first k rows. Likewise, the value of n/k in row $2k$ is the ceiling of n/k for the last k rows and this will always be $k + 2$. The value of $\lceil n/k \rceil - k$ is therefore 1 for the first k rows and 2 for the last k rows.

Let $\sqrt{n} = k+\varepsilon$. When $\lceil n/k \rceil - k = 1$, ε is less than 0.5. For $\lceil n/k \rceil - k = 2$, ε is greater than 0.5.

For the proof, we only need to consider rows k and $k + 1$. For row k, we have $n = k^2 + k = (k + 0.5 + w)^2$, where $|w| \in (0, 0.5)$. Hence

$$w \;=\; \sqrt{k^2 + k} - (k + 0.5) \;=\; \sqrt{k^2 + k} - \sqrt{k^2 + k + 0.25}$$

so that $w < 0$ and $\varepsilon < 0.5$.

For $k = 3$, we have $w = -0.0359$ and $\varepsilon = 0.4641$ (in agreement with the table). For row $k + 1$, we have $n = k^2 + k + 1 = (k + 0.5 + x)^2$, where $|x| \in (0, 0.5)$. Thus

$$x \;=\; \sqrt{k^2 + k + 1} - (k + 0.5) \;=\; \sqrt{k^2 + k + 1} - \sqrt{k^2 + k + 0.25}$$

so that $x > 0$ and $\varepsilon > 0.5$.

For $k = 3$, we have $x = 0.1056$ and $\varepsilon = 0.6056$ (in agreement with the table). Now

$$\sin(2\pi(k+0.5)) \;=\; \sin(2k\pi+\pi) \;=\; \sin(\pi) \;=\; 0.$$

Similarly,

$$\sin(2\pi\sqrt{n}) \;=\; \sin(2\pi(k+\varepsilon)) \;=\; \sin(2\pi\varepsilon).$$

For $\varepsilon < 0.5$, this is positive since $2\pi\varepsilon$ is less than π. For $\varepsilon > 0.5$ it is negative.

Taking all this together we have

$$\left\lceil \frac{n}{\lfloor\sqrt{n}\rfloor} \right\rceil - \lfloor\sqrt{n}\rfloor \;=\; \frac{3}{2} - \frac{1}{2}\big(\operatorname{sign}\sin(2\pi\sqrt{n})\big).$$

Problem 312.5 – Gas cloud

A spherical cloud of hydrogen has radius r, temperature T and uniform density ρ atoms per cubic metre. How big must r be for the cloud to begin collapsing under gravity?

Problem 312.6 – 53 bricks

You cannot fit 54 $1 \times 1 \times 4$ bricks into a $6 \times 6 \times 6$ box. If you can devise a simple proof, we would like to see it. What about 53 bricks?

Problem 312.7 – Service

If you are a professional tennis player, most likely you employ two types of service, F and S, say, where F has greater probabilities than S of (i) getting declared a fault by the line judges or the umpire, and (ii) winning the point whenever (i) does not happen.

(1) Explain why it is never a good idea to do S as your first service and F as your second.

(2) Completely solve the tennis game initiation problem. Which should you do, FF, FS, SF or SS? (I [TF] suspect the problem has already been done to death in the high-school mathematical and how-to-play-tennis literature. However, there is no harm airing it in M500 for the benefit of wider readership. Contrary to what you might see on television in July, the answer is not necessarily FS.)

Contents

04.00

Problem 312.8 – Irrational eigenvalues

Suppose a and b are positive integers with $b \neq 2a$. Show that

$$M = \begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}$$

has irrational eigenvalues unless $b < a$, or find a counter-example.

Front cover The complex Fibonacci function; see page 1.