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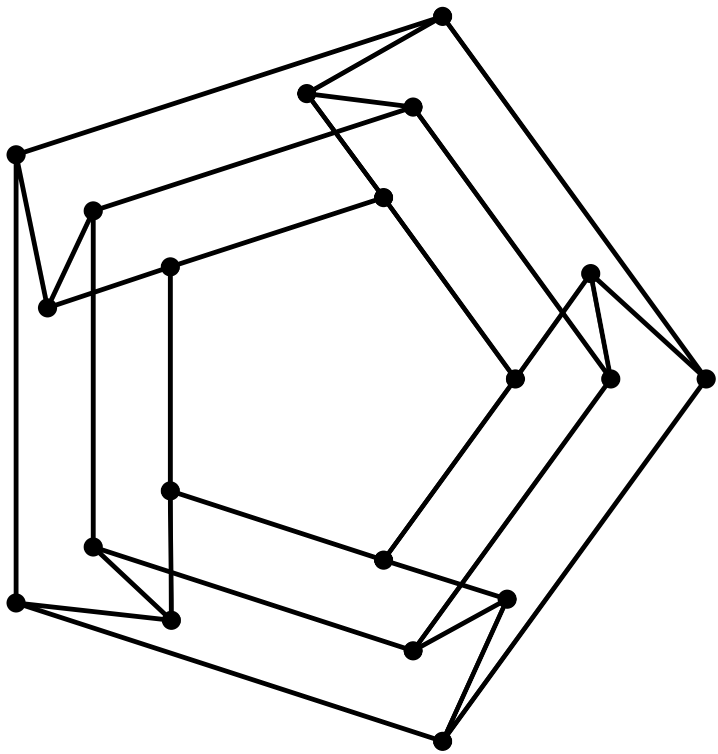
ISSN 1350-8539

M500 303



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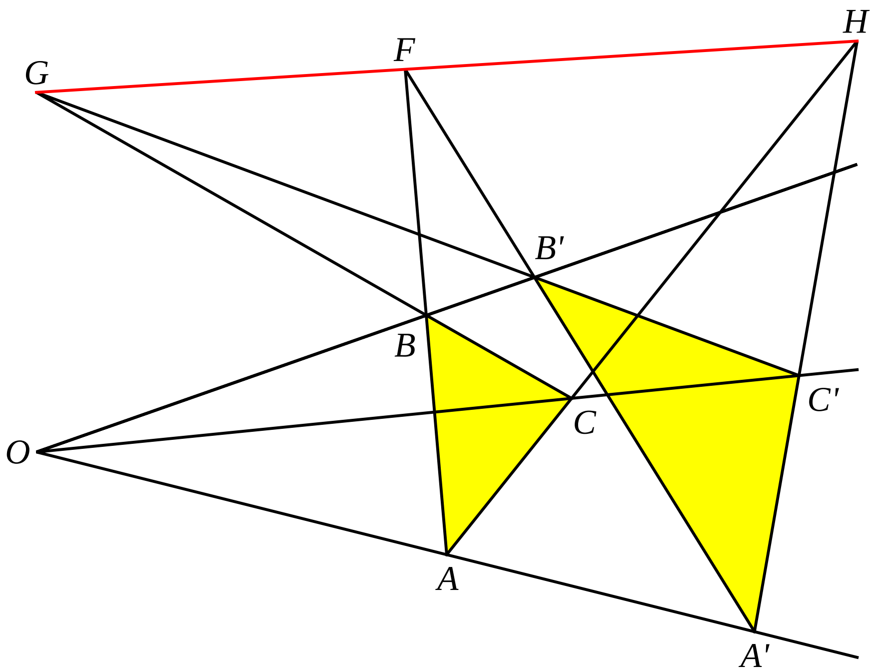
Desargues's theorem

Tony Forbes

As I expect you remember from your high-school geometry class, Desargues's theorem says that in the diagram F, G and H are collinear. There are (at least) two ways to prove it.



First, the clever method, which you will usually find in books on projective geometry such as, for example, Marshall Hall, *Projective Planes*, Amer. Math. Soc., 1943.

Imbed the diagram in a space of dimension 3. Pivoting OC' on O and whilst maintaining the straightness of all of the black lines, lift C' a little bit, ϵ say, into the third dimension so that the two yellow (grey) triangles are no longer coplanar. When this is done there will be a perspectivity situation. If you look at the triangles from position O, then $\triangle ABC$ will exactly obscure $\triangle A'B'C'$.

Clearly, A, A', B and B' are coplanar. Therefore lines AB and $A'B'$ have a point in common, F in the diagram. Unfortunately there will be a problem whenever AB is parallel to $A'B'$. There are several ways to get around this difficulty: (i) pretend it doesn't happen; (ii) work in a projective space, so that in this case F is a legitimate point but it is at infinity; (iii) leave the degenerate cases as exercises for the reader. For simplicity we shall adopt (i).

Similarly B, B', C and C' are coplanar; therefore lines BC and $B'C'$ have a point in common, G in the diagram. Similarly C, C', A and A' are coplanar; therefore lines CA and $C'A'$ have a point in common, H in the diagram.

Moreover, F is in the plane of triangle ABC as well as the plane of triangle $A'B'C'$. So are G and H. Furthermore, the intersection of two distinct planes is a line. This proves the theorem when the triangles are not coplanar. Finally, we let ϵ tend to zero and argue by continuity that the theorem also holds in two dimensions. $\square$

Now for the second method. Since I could not find this an any textbook I had to work it out for myself. It relies on computing the coordinates of F, G and H by brute force, a rather messy procedure. Nevertheless, the proof works entirely in the plane, which could be an advantage if a third dimension is not readily available.

Let $O = (0, 0)$. Since AA' goes through O, we can write

$$A \;=\; (t_A, a\, t_A), \qquad A' \;=\; (t'_A, a\, t'_A)$$

for some parameter a and variables t_A and t'_A. We are assuming that OA' is not vertical, a situation we can achieve by rotating the diagram if necessary. Similarly, let

$$B \;=\; (t_B, b\, t_B), \qquad B' = (t'_B, b\, t'_B), \qquad C \;=\; (t_C, c\, t_C) \qquad C' = (t'_C, c\, t'_C)$$

for parameters b, c and variables t_B, t'_B, t_C, t'_C. To compute the intersections F, G and H, we set up a system of six equations

$$A + (B - A)u_{AB} \;=\; A' + (B' - A')u'_{AB}, \tag{1}$$
$$B + (C - B)u_{BC} \;=\; B' + (C' - B')u'_{BC}, \tag{2}$$
$$C + (A - C)u_{CA} \;=\; C' + (A' - C')u'_{CA} \tag{3}$$

for the six variables

$$u_{AB}, \;\; u'_{AB}, \;\; u_{BC}, \;\; u'_{BC}, \;\; u_{CA}, \;\; u'_{CA}.$$

The equations are solved in the usual manner. We write down the unique solution and invite the reader to verify that it works.

$$\begin{aligned}
u_{AB} &= \frac{t_A t'_B - t'_A t'_B}{t_A t'_B - t_B t'_A}, & u_{BC} &= \frac{t_B t'_C - t'_B t'_C}{t_B t'_C - t_C t'_B}, & u_{CA} &= \frac{t_C t'_A - t'_A t'_C}{t_C t'_A - t_A t'_C}, \\
u'_{AB} &= \frac{t_A t_B - t_B t'_A}{t_A t'_B - t_B t'_A}, & u'_{BC} &= \frac{t_B t_C - t_C t'_B}{t_B t'_C - t_C t'_B}, & u'_{CA} &= \frac{t_A t_C - t_A t'_C}{t_C t'_A - t_A t'_C}.
\end{aligned}$$

Now we substitute u_{AB}, u'_{AB}, u_{BC}, u'_{BC}, u_{CA}, u'_{CA} into the left-hand sides of (1), (2) and (3) to obtain these three expressions for the intersection points in terms of a, b, c, t_A, t'_A, t_B, t'_B, t_C, t'_C:

$$\begin{aligned}
F &= \left(t_A - \frac{(t_A - t_B)(t_A - t'_A)t'_B}{t_A t'_B - t_B t'_A}, \; \frac{b\, t_B t'_B (t_A - t'_A) - a\, t_A t'_A (t_B - t'_B)}{t_A t'_B - t_B t'_A}\right), \\
G &= \left(t_B - \frac{(t_B - t_C)(t_B - t'_B)t'_C}{t_B t'_C - t_C t'_B}, \; \frac{c\, t_C t'_C (t_B - t'_B) - b\, t_B t'_B (t_C - t'_C)}{t_B t'_C - t_C t'_B}\right), \\
H &= \left(t_C - \frac{(t_C - t_A)(t_C - t'_C)t'_A}{t_C t'_A - t_A t'_C}, \; \frac{a\, t_A t'_A (t_C - t'_C) - c\, t_C t'_C (t_A - t'_A)}{t_C t'_A - t_A t'_C}\right).
\end{aligned}$$

Finally, to prove that F, G and H are collinear all we have to do (!) is show that

$$G = F + (H - F)\, v$$

has a unique solution, v. Indeed it has:

$$v = \frac{(t_B - t'_B)(t_C t'_A - t_A t'_C)}{(t_A - t'_A)(t_C t'_B - t_B t'_C)},$$

which you can verify. An amazingly simple expression, I think you will agree. Alternatively, you can compute the area of triangle FGH and confirm that it is exactly zero. $\square$

Of course, the most difficult part of any proof is remembering how to draw the Desargues configuration in such a manner that all of the intersection points F, G and H are on the page. Here is a simple strategy, which you might find useful if you ever find yourself lecturing about the subject. Draw the origin, O, and the three lines that pass through O roughly as shown. Choose A, A', G and H. Then draw lines and mark points in this order: AH, C, $A'H$, C', CG, B, $C'G$, B'. Now when AB and $A'B'$ are extended they meet at F, which will be on the page between G and H.

Solution 298.1 – Vectors

Given an integer $n \geq 2$, show how to construct a set of n mutually orthogonal linearly independent vectors of dimension n that includes the all-ones vector. Here's an example when $n = 4$:

$$\{(1,1,1,1),\ \ (1,1,-1,-1),\ \ (1,-1,1,-1),\ \ (-1,1,1,-1)\}.$$

Stuart Walmsley

It is recalled that for two linearly independent vectors $(a, b, c, \dots)$ and $(A, B, C, \dots)$ the orthogonality condition is

$$aA + bB + cC + \dots \ = \ 0;$$

that is, their scalar product is zero. A set of mutually orthogonal linearly independent vectors relative to $(1, 1, \dots)$ is required. A vector $(a, b, c, \dots)$ which is orthogonal to $(1, 1, \dots)$ satisfies the condition

$$a + b + c + \dots \ = \ 0.$$

One method of constructing such a set is as follows. The first vector is

$$(1, 1, \dots).$$

In the second vector, all components are zero except the first two, which must sum to zero. Hence the second vector is

$$(1, -1, 0, 0, \dots).$$

In the third, all components are zero except the first three. The first two take the value 1 to ensure orthogonality to the second vector and the third is -2 to give zero sum and hence orthogonality to the first vector.

The procedure is repeated to give the following solution.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & -1 & 0 & 0 & 0 & \dots \\ 1 & 1 & -2 & 0 & 0 & \dots \\ 1 & 1 & 1 & -3 & 0 & \dots \\ 1 & 1 & 1 & 1 & -4 & \dots \\ & & \dots & & & \end{bmatrix}.$$

It is noted that the condition for orthogonality of two vectors is independent of the magnitudes of the two vectors. In many applications it is required that all the vectors have the same length: the normalising factor is different for each of the vectors in this case. It is possible to get a solution to the problem in which the components of the vectors are 1 or -1, as in the example, if n is a power of 2. For $n = 2$,

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

If this array is taken to be a 2×2 matrix M, a solution for $n = 4$ is given by the matrix direct product $M \times M$:

$$\begin{bmatrix} M & M \\ M & -M \end{bmatrix};$$

that is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

If this new matrix is N, solution for $n = 8$ is $M \times N$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

Hence a solution is found when n is a power of two.

The numbers 1 and -1 are the two square roots of one. All higher roots are complex numbers. If the components of the vector are complex, the orthogonality condition is modified. For the vectors $(a, b, c, \dots)$ and $(A, B, C, \dots)$, the condition may be written

$$aA^* + bB^* + cC^* + \dots \;=\; 0$$

in which A^* is the complex conjugate of A. One of the vectors in the set continues to be $(1, 1, \dots)$ and the condition that any other vector in the set satisfies the condition

$$a + b + c + \dots \;=\; 0$$

still holds. For a given n, the n distinct nth roots of 1 are

$$\exp(2\pi ij/n), \quad j = 0, 1, \ldots n-1,$$

which will be contracted to e_j the value n being understood from the context.

The elements of the set $\{1, e_1, \ldots, e_{n-1}\}$ are the roots of the polynomial

$$z^n - 1 \;=\; 0$$

and hence their sum (being equal to minus the coefficient of z^{n-1} in the polynomial) is zero:

$$1 + e_1 + \cdots + e_{n-1} \;=\; 0.$$

Vectors of the required form may then be constructed as follows:

$$(e_0,\; e_j,\; e_j^2,\; \ldots,\; e_j^{n-1})$$

for $j = 0$ to $n - 1$. Here

$$e_j^k \;=\; e_{jk}, \qquad jk \pmod n.$$

As examples, when $n = 5$ we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & e_1 & e_2 & e_3 & e_4 \\ 1 & e_2 & e_4 & e_1 & e_3 \\ 1 & e_3 & e_1 & e_4 & e_2 \\ 1 & e_4 & e_3 & e_2 & e_1 \end{bmatrix},$$

and when $n = 6$,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e_1 & e_2 & e_3 & e_4 & e_5 \\ 1 & e_2 & e_4 & 1 & e_2 & e_4 \\ 1 & e_3 & 1 & e_3 & 1 & e_3 \\ 1 & e_4 & e_2 & 1 & e_4 & e_2 \\ 1 & e_5 & e_4 & e_3 & e_2 & e_1 \end{bmatrix}.$$

If j is coprime to n, the vector gives one cycle of length n. In general, if the highest common factor of n and j is f, there are f cycles of length n/f.

Mirrors

Robin Whitty

Søstrene Grene is a Danish chain of stores selling housewares, gifts and novelty items. At the time of writing it does not have much presence in the UK, but has branches all over France, including Toulouse, where I took this photograph.

Of course I spent the remainder of my time in the store (which does not otherwise offer much to hold the attention of a mathematician) tackling the implied exercise which, in case it is hard to decipher from the image, I summarise as follows.

> Three circular mirrors are priced according to diameter as follows: 20.8 euros for 0.5 m; 33.6 euros for 0.7 m; and 62.8 euros for 0.9 m. Which mirror offers the best value for money in terms of euros per square metre? What would you expect a 1m diameter mirror to cost? And how might you explain the distribution of prices suggested by the three mirrors?

Problem 303.1 – 16 sins

Show that

$$\sum_{i=1}^{3}\left(\frac{\sin 2c_i}{\sin(x-c_i)}\prod_{j=1,\ j\neq i}^{3}\frac{1}{\sin(c_i-c_j)}\right) \;=\; \frac{\sin 2x}{\prod_{i=1}^{3}\sin(x-c_i)}.$$

Solution 299.1 – Adjugate

The *adjugate*, adj M, of a square matrix M is the matrix defined by

$$[\operatorname{adj} M]_{i,j} \;=\; (-1)^{i+j} \det M_{\bar{j},\bar{i}},$$

where $M_{\bar{j},\bar{i}}$ is the matrix you get from M by deleting row j and column i. Show that if x is a variable, A is a matrix that is independent of x, and $M = xI - A$, then

$$\frac{d(\det M)}{dx} \;=\; \operatorname{trace}(\operatorname{adj} M).$$

More generally, show that

$$\frac{d(\det M)}{dx} \;=\; \operatorname{trace}\left((\operatorname{adj} M)\,\frac{dM}{dx}\right).$$

Tommy Moorhouse

The case $M = xI + A$

We will use component notation for matrices with the general layout M^a_b with a labeling rows and b labelling columns. We will use the δ notation for the unit matrix: $\delta^a_b = 1$ if $a = b$, being zero otherwise. A useful expression for the determinant of a matrix is

$$\epsilon_{a_1 a_2 \cdots a_n} \det M \;=\; \epsilon_{b_1 b_2 \cdots b_n} M^{b_1}_{a_1} M^{b_2}_{a_2} \cdots M^{b_n}_{a_n},$$

where ϵ is the n-index totally antisymmetric symbol with $\epsilon_{1\,2 \cdots n} = 1$, and repeated indices (one up, one down) are summed over. Setting $a_1 = 1, a_2 = 2$ and so on, and noting that in this case (i.e. $M = xI + A$) the derivative

$$\frac{dM^b_a}{dx} \;=\; \delta^b_a,$$

we have

$$\begin{aligned}
\frac{d}{dx}\det M \;&=\; \epsilon_{b_1 b_2 \cdots b_n}\Bigl(\Bigl(\frac{d}{dx}M^{b_1}_1\Bigr) M^{b_2}_2 \cdots M^{b_n}_n \\
&\qquad\qquad + M^{b_1}_1 \Bigl(\frac{d}{dx}M^{b_2}_2\Bigr) \cdots M^{b_n}_n + \cdots\Bigr) \\
&=\; \epsilon_{b_1 b_2 \cdots b_n}\Bigl(\delta^{b_1}_1 M^{b_2}_2 \cdots M^{b_n}_n + M^{b_1}_1 \delta^{b_2}_2 \cdots M^{b_n}_n + \cdots\Bigr).
\end{aligned}$$

Now we use the fact that $M_c^b \left(M^{-1}\right)_a^c = \delta_a^b$ to write

$$\begin{aligned}\frac{d}{dx}\det M &= \epsilon_{b_1 b_2 \cdots b_n}\Big(M_c^{b_1}\left(M^{-1}\right)_1^c M_2^{b_2}\cdots M_n^{b_n} \\ &\qquad + M_1^{b_1} M_c^{b_2}\left(M^{-1}\right)_2^c M_3^{B_3}\cdots M_n^{b_n} + \cdots\Big).\end{aligned}$$

The general term in the sum is

$$\epsilon_{b_1 b_2 \cdots b_n} M_1^{b_1} M_2^{b_2}\cdots M_{k-1}^{b_{k-1}} M_c^{b_k}\left(M^{-1}\right)_k^c M_{k+1}^{b_{k+1}}\cdots M_n^{b_n}.$$

The matrix product must be antisymmetric in the b_k so the index c can only take the value k (otherwise there would be a symmetric term $M_n^{b_k} M_n^{b_n}$ for some $n \neq k$ which would be killed by the antisymmetric ϵ), and the term becomes

$$\left(M^{-1}\right)_k^k \epsilon_{b_1 b_2 \cdots b_n} M_1^{b_1} M_2^{b_2}\cdots M_{k-1}^{b_{k-1}} M_k^{b_k} M_{k+1}^{b_{k+1}}\cdots M_n^{b_n}.$$

Summing over all the terms and using the expression for $\det M$ we find

$$\frac{d}{dx}\det M = \operatorname{trace} M^{-1}\det M.$$

Using $M^{-1} = \operatorname{adj} M/(\det M)$ we arrive at the result

$$\frac{d}{dx}\det M = \operatorname{trace}(\operatorname{adj} M).$$

General case

The general case is now in sight. We cannot simplify dM/dx further, but we can insert δ_b^as to assist:

$$\begin{aligned}\frac{d}{dx}\det M &= \epsilon_{b_1 b_2 \cdots b_n}\Big(\Big(\frac{d}{dx}M_1^{b_1}\Big)M_2^{b_2}\cdots M_n^{b_n} \\ &\qquad + M_1^{b_1}\Big(\frac{d}{dx}M_2^{b_2}\Big)\cdots M_n^{b_n} + \cdots\Big) \\ &= \epsilon_{b_1 b_2 \cdots b_n}\Big(\delta_c^{b_1}\Big(\frac{d}{dx}M_1^c\Big)M_2^{b_2}\cdots M_n^{b_n} \\ &\qquad + M_1^{b_1}\delta_c^{b_2}\Big(\frac{d}{dx}M_2^c\Big)\cdots M_n^{b_n} + \cdots\Big) \\ &= \epsilon_{b_1 b_2 \cdots b_n}\Big(M_d^{b_1}\left(M^{-1}\right)_c^d\Big(\frac{d}{dx}M_1^c\Big)M_2^{b_2}\cdots M_n^{b_n} \\ &\qquad + M_1^{b_1} M_d^{b_2}\left(M^{-1}\right)_c^d\Big(\frac{d}{dx}M_2^c\Big)\cdots M_n^{b_n} + \cdots\Big).\end{aligned}$$

The general term in the sum is

$$\epsilon_{b_1 b_2 \cdots b_n} M_1^{b_1} \cdots M_d^{b_k} \left(M^{-1}\right)_c^d \left(\frac{d}{dx} M_k^c\right) \cdots M_n^{b_n}$$

and again the antisymmetry of ϵ picks out the term $d = k$. But this is just

$$\left(M^{-1}\right)_c^k \left(\frac{d}{dx} M_k^c\right)$$

(no sum on k yet as we have simply isolated this term) which gives, when we perform the sum over k, trace$(M^{-1} dM/dx) \det M$. Again, using

$$M^{-1} \;=\; \frac{\mathrm{adj} M}{\det M}$$

we find

$$\frac{d}{dx} \det M \;=\; \mathrm{trace} \left(\frac{dM}{dx} \mathrm{adj}\; M\right).$$

Problem 303.2 – Regular graphs with girth 6

Tony Forbes

Given integer $n \geq 2$, show that an $(n+1)$-regular graph with $2(n^2+n+1)$ vertices and girth 6 must be the incidence graph of a projective plane of order n. Or find a counter-example.

Recall that in a projective plane of order n, there is a set P of n^2+n+1 points and a set L of n^2+n+1 lines such that:

(i) each line is incident with $n+1$ points,

(ii) each point is incident with $n+1$ lines,

(iii) for any two distinct points, there is a unique line incident with both points, and

(iv) for any two distinct lines, there is a unique point incident with both lines.

Its incidence graph has vertices $P \cup L$ and there is an edge $p \sim \ell$ whenever point p is incident with line ℓ. Clearly the graph is $(n+1)$-regular and has $2(n^2+n+1)$ vertices. Moreover, it is not too difficult to show that it has girth 6. Therefore it is sensible to pose the stated problem.

Problem 303.3 – Bin packing

Tony Forbes

There are infinitely many empty bins, each of capacity 100. At each tick of the clock you are presented with a random integer x in the range $[1, 100]$. You scan the partially filled bins that can accommodate an extra x. If there are none, you put x into an empty bin. Otherwise you put x into a bin that leaves the smallest unused capacity when x is added to it.

For example, you can verify that the sequence

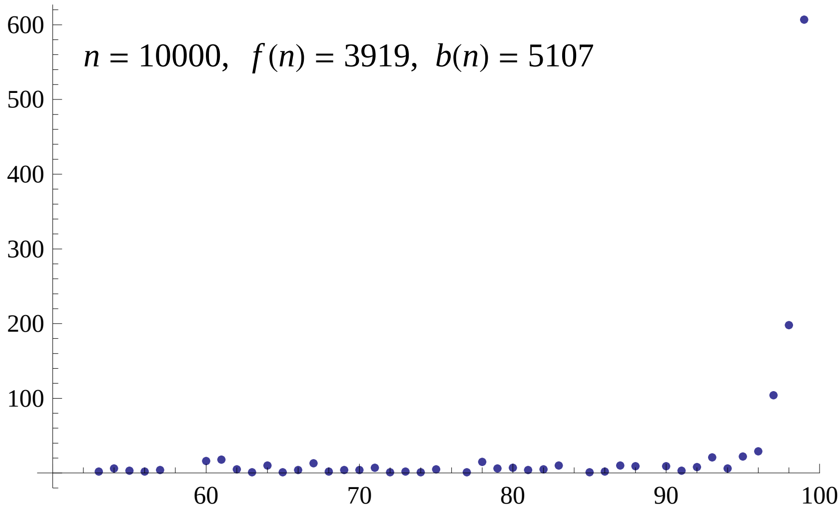
$$79, 21, 68, 90, 1, 1, 33, 78, 30, 65, 21, 10, 34, 96, 68, 59, 99, 24, 56, 42$$

requires 11 bins of which 2 are full and 9 are partly filled as follows:

$$98, 92, 98, 99, 96, 92, 59, 99, 42.$$

Notice that the last number, 42, is in a bin by itself because there is no non-empty bin that can accept it. If the next number is 5, you must put it into one of the 92% filled bins.

If n is the number of trials, let $b(n)$ be the number of bins required and $f(n)$ the number of full bins. What are the expected values of $b(n)/n$ and $f(n)/n$ as n tends to infinity? Here is an example.



Distribution of partly filled bins

Solution 299.2 – Integral

Show that

$$\int_0^{\pi/2} \left((\sin x)^{2/3} + (\cos x)^{2/3}\right)^3 \, dx \;=\; \frac{3\pi}{2}.$$

Tommy Moorhouse

To tackle this type of integral we can use the beta function defined, for p and q having positive real parts, by

$$B(p,q) \;=\; \int_0^1 y^{q-1}(1-y)^{p-1} dy.$$

Substituting $\cos^2 t = y$ so that $dy = -2\cos t \sin t \; dt$ we see that

$$\int_0^{\pi/2} (\cos t)^{2p-1}(\sin t)^{2q-1} dt \;=\; \frac{1}{2}\, B(p,q).$$

Now,

$$\left((\sin x)^{2/3} + (\cos x)^{2/3}\right)^3 \;=\; 1+3((\cos x)^{4/3}(\sin x)^{2/3}+(\cos x)^{2/3}(\sin x)^{4/3}).$$

Since $B(p,q) = B(q,p)$ the two trigonometric terms integrate to the same thing and the required integral is

$$\int_0^{\pi/2} \left(1 + 6\left((\cos x)^{4/3}(\sin x)^{2/3}\right)\right) dx \;=\; \frac{\pi}{2} + 3\, B\left(\frac{7}{6}, \frac{5}{6}\right).$$

We now use the identity

$$B(p,q) \;=\; \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

the property

$$m\Gamma(m) \;=\; \Gamma(m+1)$$

(so that $\Gamma(7/6) = \Gamma(1/6)/6$ and $\Gamma(5/6) = -\Gamma(-1/6)/6$), and the gamma function relation

$$\Gamma(z)\Gamma(-z) \;=\; \frac{-\pi}{z \sin \pi z}$$

to deduce that the integral is

$$\int_0^{\frac{\pi}{2}} \left((\sin x)^{2/3} + (\cos x)^{2/3}\right)^3 dx = \frac{\pi}{2} - \frac{3}{36}\frac{\Gamma(1/6)\Gamma(-1/6)}{\Gamma(2)}$$
$$= \frac{\pi}{2} + \frac{3}{36}\frac{\pi}{\left(\frac{1}{6}\sin\frac{\pi}{6}\right)} = \frac{3\pi}{2}.$$

The beta and gamma functions are discussed in many standard texts on complex analysis.

Problem 303.4 – Tetrahedron

Tony Forbes

The base of a tetrahedron is an equilateral triangle T with side d. Its other three sides, which meet at point P, have lengths a, b and c. Let h be the height of P above the plane in which T lies. Show that

$$3d^2h^2 = a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2 - a^4 - b^4 - c^4 - d^4.$$

Observe that this remarkable expression is symmetric in a, b, c, d, and you might find it interesting to compare it to equality (2) in Steve Moon's solution of Problem 266.3 – Equilateral triangle.

> There is an equilateral triangle. Point P is at distance a from one vertex and b from another vertex. What is the largest possible distance P can be from the third vertex?

See [M500, **300**, 10–12]. The situation in Problem 266.3 corresponds to the special case $h = 0$. When $a = b = c = d = 1$ we get $h = \sqrt{2/3}$, as expected.

I expect you remember from high-school geometry the formula for the height, h, above side c of a triangle with sides a, b and c:

$$4c^2h^2 = 2\left(a^2b^2 + a^2c^2 + b^2c^2\right) - a^4 - b^4 - c^4.$$

At once this leads to an interesting question. For what kind of n-dimensional object constructed from triangles with sides chosen from $\{a_1, a_2, \ldots, a_{n+1}\}$ is an expression of the form

$$\Delta_n \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} a_i^2 a_j^2 - \sum_{i=1}^{n+1} a_i^4$$

relevant?

Solution 298.4 – Sum

Show that $$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{5\cdot 6\cdot 7}+\frac{1}{9\cdot 10\cdot 11}+\ldots \;=\; \frac{\log 2}{4}.$$

Ted Gore

Let

$$x \;=\; \frac{1}{1\cdot 2\cdot 3}+\frac{1}{5\cdot 6\cdot 7}+\frac{1}{9\cdot 10\cdot 11}+\cdots \;=\; \sum \frac{1}{(2k-1)2k(2k+1)}$$

for odd integers $k>0$. Let

$$f \;=\; \sum_{n=1}^{\infty} \frac{1}{n(4n^2-1)} \;=\; \sum \frac{2}{(2m-1)2m(2m+1)}+\sum \frac{2}{(2k-1)2k(2k+1)}$$

for even integers $m>1$ and k as above. Let

$$g \;=\; \sum_{n=1}^{\infty} \frac{(-1)^n}{n(4n^2-1)} \;=\; \sum \frac{2}{(2m-1)2m(2m+1)}-\sum \frac{2}{(2k-1)2k(2k+1)}.$$

According to *Wikipedia*, (Natural logarithm of 2), $f = 2\cdot\log(2)-1$ and $g=\log(2)-1$. Now

$$f-g \;=\; 2\sum \frac{2}{(2k-1)2k(2k+1)} \;=\; 4x \;=\; \log 2,$$

so that $x=\log(2)/4$.

Peter Fletcher

We can write the given sum as

$$S \;=\; \sum_{n=0}^{\infty} \frac{1}{(4n+1)(4n+2)(4n+3)} \;=\; \sum_{n=0}^{\infty} \frac{(4n)!}{(4n+3)!}.$$

Recall that the beta function $B(\cdot\,,\cdot)$ is defined as

$$B(x,y) \;=\; \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \;=\; \frac{(x-1)!(y-1)!}{(x+y-1)!},$$

the expression involving factorials obviously only being valid for integer x, y.

This means that we can write the given sum as

$$S \;=\; \frac{1}{2!}\sum_{n=0}^{\infty} B(4n+1,3).$$

Now $B(x,y)$ may also be expressed in integral form,

$$B(x,y) \;=\; \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$$

per `https://en.wikipedia.org/wiki/Beta_function`. Therefore we can write the given sum as

$$S \;=\; \frac{1}{2}\int_0^1 \left(\sum_{n=0}^{\infty} t^{4n}\right)(1-t)^2\mathrm{d}t.$$

For the sum inside the integrand, we know that, for $|x| < 1$,

$$\sum_{n=0}^{N} x^n \;=\; \frac{1-x^{N+1}}{1-x}$$

so

$$\sum_{n=0}^{N} \left(t^4\right)^n \;=\; \frac{1-\left(t^4\right)^{N+1}}{1-t^4} \;=\; \frac{1-t^{4N+4}}{1-t^4}$$

and

$$\sum_{n=0}^{\infty} t^{4n} \;=\; \frac{1}{1-t^4}.$$

Therefore

$$\begin{aligned}
S \;&=\; \frac{1}{2}\int_0^1 \frac{(1-t)^2}{1-t^4}\mathrm{d}t \;=\; \frac{1}{2}\int_0^1 \frac{1-t}{(1+t)(1+t^2)}\mathrm{d}t \\
&=\; \frac{1}{2}\int_0^1 \left(\frac{1}{1+t} - \frac{t}{1+t^2}\right)\mathrm{d}t \\
&=\; \frac{1}{2}\left[\log(1+t) - \frac{1}{2}\log\bigl(1+t^2\bigr)\right]_0^1 \\
&=\; \frac{\log 2}{4}.
\end{aligned}$$

Problem 303.5 – Sum of two cube roots

Tony Forbes

For which positive integers a, b is $(a+\sqrt{b})^{1/3}+(a-\sqrt{b})^{1/3}$ an integer? Assume both cube roots are real. For example, the expression evaluates to 1 when $(a,b)=(2,5)$.

Solution 297.3 – Bridge

> Devise an arrangement of the cards and a bidding sequence where you could realistically and sensibly end up playing in 8♢. Assume such a bid is legitimate. Assume you and the other three players always bid and play intelligently. Whether bids at the eighth level should be allowed is a debatable point. The practice is currently banned by World Bridge Federation rules, but that has not always been the case.

Tony Forbes

I showed this to a few bridge players I know. Initially they weren't particularly impressed. They offered the simplistic explanation that you are prepared to go a few down in 8 diamonds to stop the other side from making a grand slam. Then I pointed out that surely the defenders have everything to gain and nothing to lose by doubling.

Now we appreciate the difficulty: explaining why the final contract is allowed to go undoubled. Here is a possible scenario.

N: ♠– ♡QJxxxxxxx ♢xxxx ♣–

W: ♠xx ♡xx ♢– ♣xxxxxxxxx   E: ♠AKxxxxxxxxx ♡– ♢AK ♣–

S: ♠– ♡AK ♢QJxxxxx ♣AKQJ

Dealer S, both sides vulnerable; bidding: S 2♢, W pass, N 4♡, E 7♠, S (Judging by the confident manner of E's bid the grand slam must be certain.) 8♢, W pass, N pass, E (If I double, N or S might transfer to 8♡, which could easily go just 1 down. On the other hand 8♢ goes at least 3 down.) **pass**.

Result: 7♠ makes for at least 1710 points, 8♢ goes 3 down for −300, 8♡$^{\times}$ goes 1 down for −200. Thus E's final pass gets a better result.

Problem 303.6 – Sixth powers

Tony Forbes

For which positive integers a and b is $(a(a+b)(a+2b)+1)^{1/6}$ an integer?

For instance, with $(a,b) = (28, 157)$ we have $28 \cdot 185 \cdot 342 + 1 = 11^6$. Curiously, $(a,b) = (45, 107)$ gets the same result. However, the most impressive example I have found so far is $(a,b) = (331, 1028) \to 32^6$, which leads to an even more interesting problem.

Problem 303.7 – Thirtieth powers

Either show that the only solution in positive integers a, b and c of

$$a(a+b)(a+2b)+1 \;=\; c^{30}$$

is $a = 331$, $b = 1028$, $c = 2$, or find another.

Problem 303.8 – Odd binomial coefficients

Show that $\binom{2n-1}{n}$ is odd iff n is a power of 2.

Polyhedra

Jeremy Humphries

Deduce the missing word in this poem.

Said the dodecahedron, "How cruel –
With the icosahedron, by rule
Of the fortunes of fate,
I can never be mate,
Though we're each one the other one's dual.

Does the earth perform actions tectonic?
Are there aural sensations symphonic?
Ah, would that it were so,
But the answer is 'No'.
Our relationship's purely _ _ _ _ _ _ _ _."

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04.01

Front cover A connected, 20-vertex, 3-regular graph that has girth 5.