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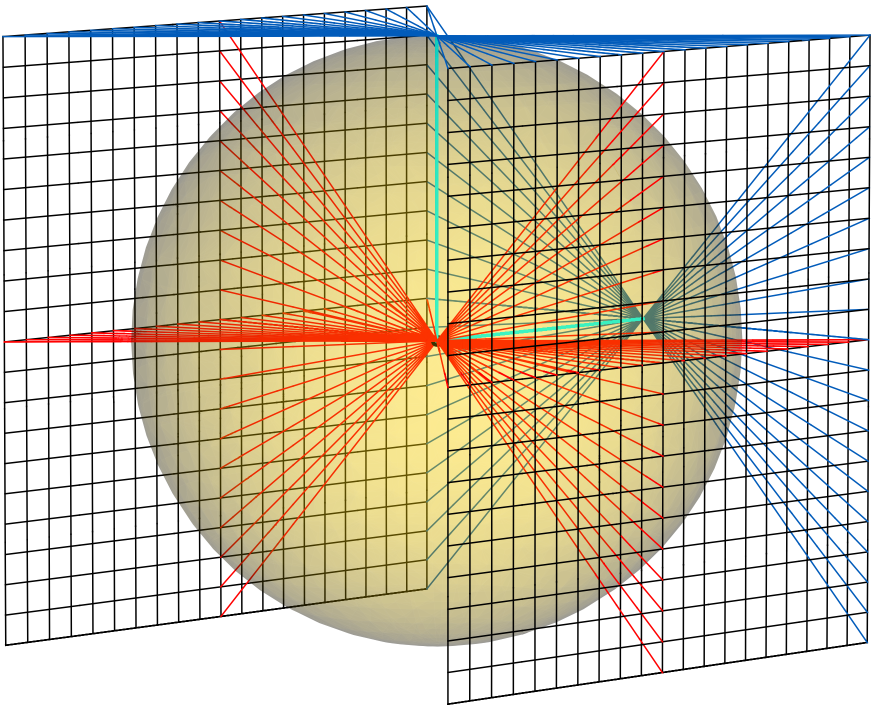
M500 313



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: `m500.org.uk`.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The Revision Weekend is a residential Friday to Sunday event providing revision and examination preparation for both undergraduate and postgraduate students. For details, please go to the Society's website.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, please go to the Society's website.

Editor – *Tony Forbes*

Editorial Board – *Jeremy Humphries*

Edward Arthur Philpott–Kent

We are very sorry to have to tell you that Eddie Kent, one of the founding members of the M500 Society, died on 19 June 2023 after a long battle with cancer. He was 90 years old.

Eddie was a member of the M500 Society Committee from its beginning. He edited M500 from issue 25 (July 1975) to issue 84 (September 1983), and in December 1988 was elected to Chairman, retiring from that post in October 2012. He remained on the Committee as Editorial Board member. Eddie is of course well known to readers of M500. He was a frequent contributor to the magazine, especially the early issues.

We offer our sympathy to his family and his friends.

Solution 310.3 – Unit sum

Show that

$$\sum_{n=1}^{\infty} \frac{3+10n+10n^2-6n^4}{(n^2+n)^4} \;=\; 1.$$

Henry Ricardo

The partial fraction decomposition of the nth term of the series is

$$\frac{3+10n+10n^2-6n^4}{(n^2+n)^4} \;=\; \frac{2}{(n+1)^3} \;-\; \frac{2}{n^3} \;-\; \frac{3}{(n+1)^4} \;+\; \frac{3}{n^4},$$

so that we have the telescoping series

$$\begin{aligned}\sum_{n=1}^{N} \frac{3+10n+10n^2-6n^4}{(n^2+n)^4} \;&=\; -2\sum_{n=1}^{N}\left(\frac{1}{n^3}-\frac{1}{(n+1)^3}\right)\\ &\quad+\;3\sum_{n=1}^{N}\left(\frac{1}{n^4}-\frac{1}{(n+1)^4}\right)\\ &=\; -2\left(1-\frac{1}{(N+1)^3}\right) \;+\; 3\left(1-\frac{1}{(N+1)^4}\right),\end{aligned}$$

which tends to $-2(1)+3(1)=1$ as $N\to\infty$.

Alternatively, considering Riemann's zeta function,

$$\zeta(k) \;=\; \sum_{n=1}^{\infty} 1/n^k,$$

we can write

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{3+10n+10n^2-6n^4}{(n^2+n)^4} \;&=\; \sum_{n=1}^{\infty}\frac{2}{(n+1)^3} - \sum_{n=1}^{\infty}\frac{2}{n^3} - \sum_{n=1}^{\infty}\frac{3}{(n+1)^4} + \sum_{n=1}^{\infty}\frac{3}{n^4}\\ &=\; 2\,(\zeta(3)-1) \;-\; 2\,\zeta(3) \;-\; 3\,(\zeta(4)-1) \;+\; 3\,\zeta(4)\\ &=\; -2 \;+\; 3 \;=\; 1.\end{aligned}$$

Lines in $\mathbb{R}^3$ and vector fields on S^2

Tommy Moorhouse

Introduction and recap The correspondence between sets of lines in $\mathbb{R}^3$ and vector fields on the 2-sphere S^2 was discussed in a previous investigation (M500 **301**). In essence, if a base point (x_0, y_0, z_0) is chosen in $\mathbb{R}^3$ and a set of lines is given, then the pair of vectors given by a unit vector $\vec{u}$ along a line and the shortest vector $\vec{v}$ from the base point to the line defines a vector on S^2, since these vectors are orthogonal and $\vec{u}$ defines a point of S^2. The aim of this follow-up is to outline the solution to the problem previously stated and find some explicit coordinate descriptions. In the process we'll find amusing analogies with electrostatics on the sphere. The calculations can appear messy so I've put them at the end and hopefully the underlying idea will be clear.

Vector fields and allowed sets of lines It is a well-known theorem that any smooth vector field on the sphere vanishes at at least one point. Thus any allowed set must have a line passing through any point of $\mathbb{R}^3$. If the point in question is chosen to be the base point for constructing the field then $|\vec{v}| = 0$ at this point.

Electric field The first example is the set of lines passing through the origin with any other fixed point taken to be the base point. The construction above leads to a vector field on S^2 with two zeros and the vector field pointing from one to the other (see Figure 1). This is analogous to the field between two point charges. An explicit construction is given in the appendix. This type of field actually arises as a special case of the construction for magnetic fields below, but the details are beyond the scope of this discussion.

Magnetic field The second example uses what at first may seem an arbitrary choice of lines. We start from a set of lines passing through the unit disk in the (x, y) plane of $\mathbb{R}^3$ (see Figure 2). On each circle of radius $r < 1$ there are two sets of lines, pointing upwards and downwards respectively: at the origin the lines point along the z axis, and on the circle $r = 1$ the lines are the positively oriented tangents to the circle. Inside the disk the tangents to the lines project to anticlockwise (positively oriented) tangent vectors. We do not regard the coincident lines at $r = 1$ to be double lines. One explicit description of this set of lines in $\mathbb{R}^3$ is the field of directed unit

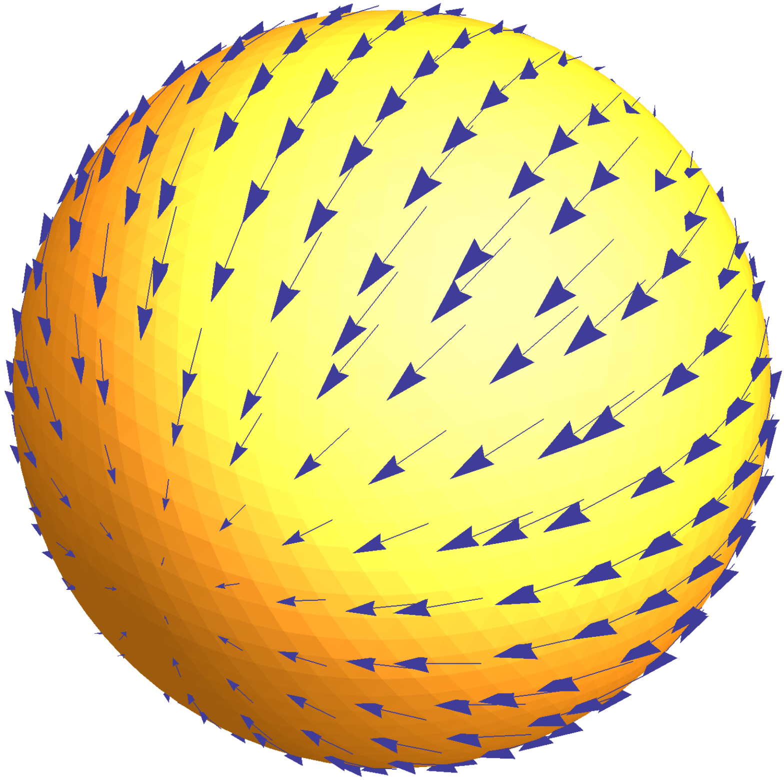


Figure 1: Electric field on the sphere.

vectors

$$\vec{u}_{\pm} \; = \; (y, \;\; -x, \;\; \pm\sqrt{1-(x^2+y^2)})$$

on the unit disk.

Choosing the base point to be the origin we can show that the resulting vector field on S^2 has zeros at the poles with the vector field flowing from east to west, encircling the poles (see Figure 3). By analogy with the field around a long current-carrying wire we call this a magnetic field.

The magnetic field is just one of the fields arising from this set of lines. Choosing a different base point on the unit disk gives rise to distorted magnetic fields if the point lies inside the unit disk in the (x, y)-plane, but electric fields if the base point is in the plane outside the disk. Choosing a

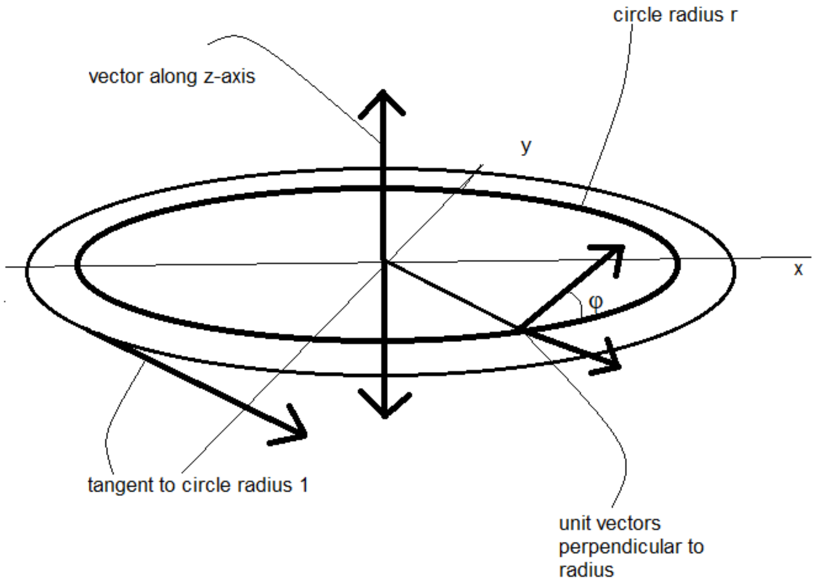


Figure 2: Line construction.

base point on the z-axis (e.g. $(0, 0, 1)$) gives the field shown in Figure 4. This represents a 'four-screw', as described by Roger Penrose in his description of Lorenz transformations. I am curious whether the electric and magnetic fields arising from base points related by inversion in the unit disk on the (x, y)-plane are in some sense dual.

An interesting link This isn't essential to the main theme, but the seemingly arbitrary set of lines producing the magnetic field actually has a fascinating link to the Hopf bundle (also known as the Hopf fibration, discussed in M500 **227**, p.1ff.). The total space of the bundle is S^3 and projection from the 'north pole' of S^3 gives a picture of the bundle in $\mathbb{R}^3$. The projected fibres are linked circles wrapping around tori. The allowed lines are then the projected tangent lines to the fibres on the tori, as they pass through the unit disk. There is a construction in twistor theory describing the 'Robinson congruence' that gives rise to the same picture and this could be an interesting topic for investigation in its own right. However, the main thing here is to enjoy the simplicity of the connection between lines and vector fields!

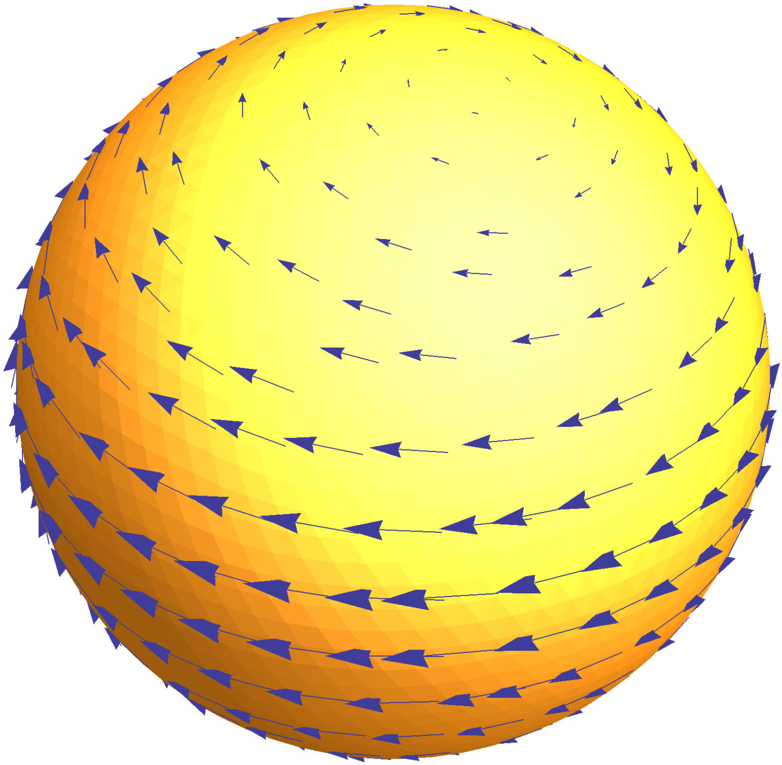


Figure 3: Magnetic field on the sphere.

Appendix

The key to the calculations is finding the vector $\vec{v}$ linking the base point $\vec{x_0}$ and a line. The line is defined by any point $\vec{p}$ it passes through and the unit vector $\vec{u}$ along the line. We can realise this as the curve $t \mapsto \vec{p} + t\vec{u}$. The distance from the point to the line is $|\vec{p} - \vec{x_0} + t\vec{u}|$. It's easier to work with the squared distance, and we use elementary calculus to minimise with respect to t, finding

$$\vec{v} \;=\; (\vec{p} - \vec{x_0}) - (\vec{u} \cdot (\vec{p} - \vec{x_0}))\vec{u}.$$

The electric field can be constructed from the lines through the origin and a base point $(0, 0, z_0)$ on the z-axis. The unit vectors are given by (x, y, z) with $x^2 + y^2 + z^2 = 1$ and $\vec{v} = z_0(-xz, -yz, 1 - z^2)$ in this case.

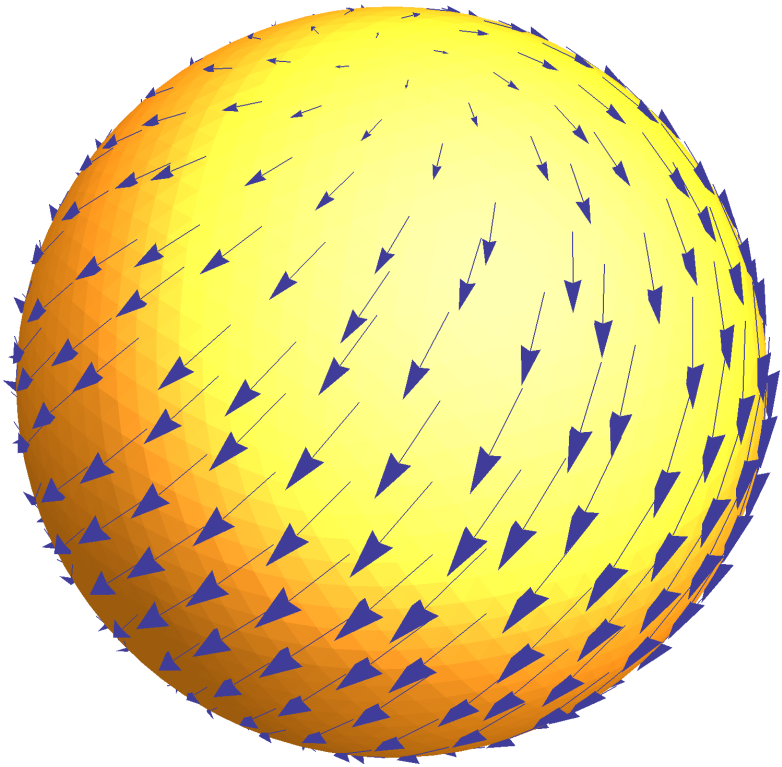


Figure 4: Four-screw field.

The magnetic field is a little trickier but working through the same process gives

$$\vec{v}(x, y, z) = \left(-\frac{y}{1+|z|} - x_0 + x\Delta, \frac{x}{1+|z|} - y_0 + y\Delta, -z_0 + z\Delta\right),$$

where

$$\Delta = xx_0 + yy_0 + zz_0.$$

To get to this result we start from the field of unit vectors on the unit disk, which at the point $(x, y, 0)$ is given by

$$\vec{u}_{\pm} = \left(\frac{2y}{1+x^2+y^2}, \frac{-2x}{1+x^2+y^2}, \pm\frac{1-x^2-y^2}{1+x^2+y^2}\right)$$

as stated above. This unit vector field is calculated from the projection of the tangents to the fibres of the Hopf bundle, but we will not elaborate here. Finding the shortest distance from (x_0, y_0, z_0) to the line $\vec{p} + t\vec{u}$ proceeds as above, giving $\vec{v}$ as a function of x, y and z. We actually need to express $\vec{v}$ in terms of u_1, u_2 and u_3 using

$$(u_1, u_2, u_3) \;=\; \vec{u}_{\pm} \;=\; \left(\frac{2y}{1+x^2+y^2},\ \frac{-2x}{1+x^2+y^2},\ \pm\frac{1-x^2-y^2}{1+x^2+y^2}\right).$$

The modulus sign on $|z|$ comes from the need to cover S^2 by two patches, S^2_+, where $z \geq 0$ and S^2_- with $z \leq 0$. This actually comes out of the calculation quite naturally.

It is possible to visualise the vector fields directly from sketches of the set of lines and base points, although it does take a bit of getting used to. For those who have access to a mathematics program and prefer computer graphics it is fairly straightforward to use the above methods to find explicit expressions that can be used in, for example, MATHEMATICA. I used the function `SlicedVectorPlot3D` to produce the figures above: you could try the code below to reproduce the four-screw field. The options `Axes→False`, `Boxed→False`, inserted after the main options, remove the axes and box around the displayed sphere:

$$\texttt{SliceVectorPlot3D}[v, \texttt{"CenterSphere"}, \{x,-1,1\}, \{y,-1,1\}, \{z,-1,1\}]$$

with v written as a list such as

$$\begin{aligned}\{-y - x_0 + x(xx_0 + yy_0 + zz_0),\ x - y_0 + y(xx_0 + yy_0 + zz_0),\\ - z_0 + z(xx_0 + yy_0 + zz_0)\}\end{aligned}$$

for some choice of x_0, y_0 and z_0.

Problem 313.1 – Floored and summed

Show that if a and b are positive integers such that $\gcd(a, b) = 1$, then

$$\left\lfloor\frac{a}{b}\right\rfloor + \left\lfloor\frac{2a}{b}\right\rfloor + \cdots + \left\lfloor\frac{(b-1)a}{b}\right\rfloor \;=\; \frac{(a-1)(b-1)}{2}.$$

Observe that this is $(b-1)/2$ less than what you would get without the floor brackets.

Pythagoras' theorem revisited

Tony Forbes

Students Calcea Johnson and Ne'Kiya Jackson of St Mary's Academy in New Orleans have generated a certain amount of excitement amongst the mathematical community by proving Pythagoras' theorem in a new and interesting manner. See Leila Sloman's article of 10 April 2023 in *Scientific American*, which is available at

`https://www.scientificamerican.com/article/2-high-school-students-prove-pythagorean-theorem-heres-what-that-means/`.

Unfortunately no details of the proof were given.

At a London South Bank University Maths Study Group meeting in April 2023 we were shown a video claiming to explain how the two students might have proved the theorem. However, I found the long-winded presentation rather tedious and the temptation to get some sleep became almost irresistible. Nevertheless I managed to salvage the main features of the lecturer's diagram from which a proof is quite easy to concoct.

Here is a proof based on the first picture on the next page. I do not know if this is actually how the students did it. All the relevant triangles are similar, and it helps if $b > a$. We have

$$\begin{aligned} |B_1B_2| &= \frac{2ca}{b}, \quad & |D_1B_2| &= \frac{2a^2}{b}, \quad & |D_1D_2| &= \frac{2ca^2}{b^2}, \\ |B_2B_3| &= \frac{2ca^3}{b^3}, \quad & |D_2B_3| &= \frac{2a^4}{b^3}, \quad & |D_2D_3| &= \frac{2ca^4}{b^4}, \\ & \ldots . \end{aligned}$$

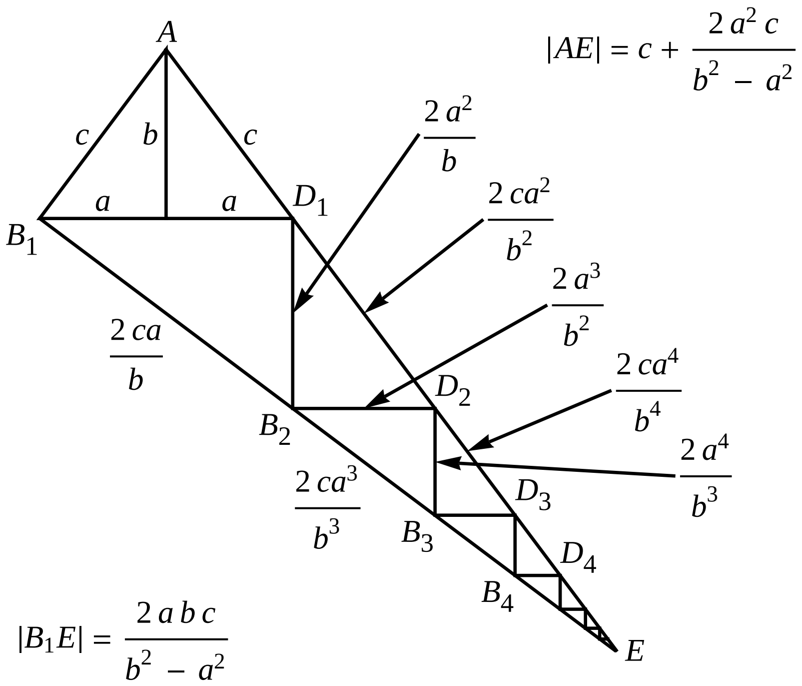
Hence

$$\begin{aligned} |B_1E| &= \frac{2ca}{b}\sum_{n=0}^{\infty}\left(\frac{a^2}{b^2}\right)^n = \frac{2abc}{b^2-a^2}, \\ |AE| &= c + 2c\sum_{n=1}^{\infty}\left(\frac{a^2}{b^2}\right)^n = c + \frac{2a^2c}{b^2-a^2}. \end{aligned}$$

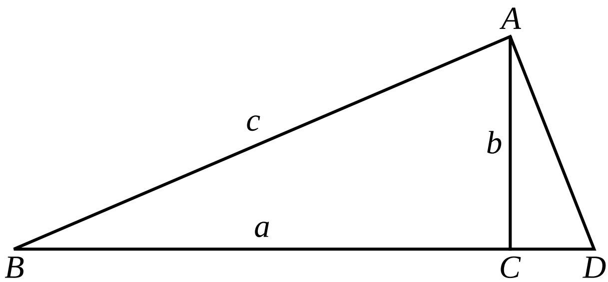
Observe that AB_1E is an arbitrary right-angled triangle. Its hypotenuse is AE, and

$$|AE|^2 - |B_1E|^2 = \left(c + \frac{2a^2c}{b^2-a^2}\right)^2 - \left(\frac{2abc}{b^2-a^2}\right)^2 = c^2 = |AB_1|^2.$$

QED



A simple proof, you will agree. However, **Graham Lovegrove** reminded the LSBU Maths Study Group of an even simpler proof, also based on triangles and the similarity thereof. Look at the next diagram …



… in which angles ACB and BAD are 90°. Clearly,

$$|AD| = \frac{cb}{a}, \quad |CD| = \frac{b^2}{a}, \quad |BD| = \frac{c}{b}\,|AD| = \frac{c^2}{a},$$

and hence $c^2/a = a + b^2/a$.

Inspired by Sloman's article and the debates regarding the use or otherwise of trigonometry in Johnson & Jackson's argument, I thought I would have a go at proving Pythagoras' theorem in its most fundamental form. No diagram, no triangles, no trigonometry, and no funny stuff involving $\sqrt{-1}$.

Theorem 1 (Pythagoras) *We have*

$$\left(\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}\right)^2 + \left(\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\right)^2 \;=\; 1.$$

Proof It is not too difficult to show that for positive integer n,

$$\left(\sum_{k=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!}\right)^2 + \left(\sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\right)^2 - 1.$$

is a polynomial in x of the form

$$P_n(x) \;=\; a_{2n+2}x^{2n+2} + a_{2n+4}x^{2n+4} + \cdots + a_{4n+2}x^{4n+2}, \tag{1}$$

where

$$|a_k| \;\le\; \frac{2^k}{k!}, \quad k = 2n+2,\ 2n+4,\ \ldots,\ 4n+2.$$

Hence for any x, $P_n(x) \to 0$ as $n \to \infty$. The theorem follows.

For completeness, we now address in detail the 'not too difficult' part of the proof. Let

$$\begin{aligned} C_n \;&=\; \sum_{k=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!}, \\ S_n \;&=\; \sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}, \end{aligned}$$

where n is a positive integer. Then

$$C_n^2 + S_n^2 \;=\; \sum_{j=0}^{n}\sum_{k=0}^{n}(-1)^{j+k}\left(\frac{x^{2j+2k}}{(2j)!(2k)!} + \frac{x^{2j+2k+2}}{(2j+1)!(2k+1)!}\right).$$

Put $r = k + j$:

$$C_n^2 + S_n^2 \;=\; \sum_{j=0}^{n}\sum_{r=j}^{n+j}(-1)^{r}\left(\frac{x^{2r}}{(2j)!(2r-2j)!} + \frac{x^{2r+2}}{(2j+1)!(2r-2j+1)!}\right).$$

Reverse the order of summation. Then r goes from 0 to $2n$, and we split its range into $[0, n]$ and $[n+1, 2n]$. Observe that j goes from 0 to r when $r \le n$ and from $r-n$ to n when $r \ge n+1$. Thus we have

$$C_n^2 + S_n^2 \;=\; X_1 + X_2,$$

where

$$\begin{aligned} X_1 &= \sum_{r=0}^{n} (-1)^r \left(\frac{x^{2r}}{(2r)!} \sum_{j=0}^{r} \binom{2r}{2j} + \frac{x^{2r+2}}{(2r+2)!} \sum_{j=0}^{r} \binom{2r+2}{2j+1} \right), \\ X_2 &= \sum_{r=n+1}^{2n} (-1)^r \left(\frac{x^{2r}}{(2r)!} \sum_{j=r-n}^{n} \binom{2r}{2j} + \frac{x^{2r+2}}{(2r+2)!} \sum_{j=r-n}^{n} \binom{2r+2}{2j+1} \right). \end{aligned}$$

Consider X_1. The two binomial sums can be evaluated:

$$\begin{aligned} \sum_{j=0}^{r} \binom{2r}{2j} &= \begin{cases} 1 & \text{when } r = 0, \\ 2^{2r-1} & \text{otherwise,} \end{cases} \\ \sum_{j=0}^{r} \binom{2r+2}{2j+1} &= 2^{2r+1}. \end{aligned}$$

Therefore

$$\begin{aligned} X_1 &= 1 + \sum_{r=1}^{n} (-1)^r \frac{x^{2r} 2^{2r-1}}{(2r)!} + \sum_{r=0}^{n} (-1)^r \frac{x^{2r+2} 2^{2r+1}}{(2r+2)!} \\ &= 1 + x^2 + \sum_{r=1}^{n} \frac{(-1)^r}{2} \left(\frac{(2x)^{2r}}{(2r)!} + \frac{(2x)^{2r+2}}{(2r+2)!} \right). \end{aligned}$$

In this beautiful formula the last sum collapses into $-x^2$ (which cancels the x^2) and a term involving x^{2n+2}:

$$X_1 \;=\; 1 + \frac{(-1)^n}{2} \frac{(2x)^{2n+2}}{(2n+2)!}.$$

When we add X_2 to X_1 we do indeed get a polynomial of the form (1):

$$\begin{aligned} C_n^2 + S_n^2 - 1 &= \frac{(-1)^n}{2} \frac{(2x)^{2n+2}}{(2n+2)!} \\ &\quad + \sum_{r=n+1}^{2n} (-1)^r \left(\frac{x^{2r}}{(2r)!} \sum_{j=r-n}^{n} \binom{2r}{2j} + \frac{x^{2r+2}}{(2r+2)!} \sum_{j=r-n}^{n} \binom{2r+2}{2j+1} \right). \end{aligned}$$

On gathering expressions involving the same power of x, we have

$$\begin{aligned} C_n^2 + S_n^2 - 1 \;&=\; (-1)^n\,\frac{x^{2n+2}}{(2n+2)!}\left(\frac{2^{2n+2}}{2} - \sum_{j=1}^{n}\binom{2n+2}{2j}\right) \\ &\quad + \frac{x^{4n+2}}{(4n+2)!}\binom{4n+2}{2n+1} \\ &\quad + \sum_{r=n+2}^{2n}\frac{(-1)^r x^{2r}}{(2r)!}\left(\sum_{j=r-n}^{n}\binom{2r}{2j} - \sum_{j=r-1-n}^{n}\binom{2r}{2j+1}\right). \end{aligned}$$

Next, observe that

$$\sum_{j=1}^{n}\binom{2n+2}{2j} \;=\; 2^{2n+1} - 2.$$

Therefore

$$\begin{aligned} C_n^2 + S_n^2 - 1 \;&=\; (-1)^n\,\frac{2\,x^{2n+2}}{(2n+2)!} + \frac{x^{4n+2}}{(4n+2)!}\binom{4n+2}{2n+1} \\ &\quad + \sum_{r=n+2}^{2n}\frac{(-1)^r x^{2r}}{(2r)!}\left(\sum_{j=r-n}^{n}\binom{2r}{2j} - \sum_{j=r-1-n}^{n}\binom{2r}{2j+1}\right). \end{aligned}$$

But $\binom{4n+2}{2n+1}$ is less than or equal to 2^{4n+2}. Moreover, the two inner sums involve distinct binomial coefficients of the form $\binom{2r}{s}$ and so when taken together they sum to something bounded by $\pm 2^{2r}$. Hence the coefficient of x^k in $C_n^2 + S_n^2 - 1$ is zero unless $k \in \{2n+2,\ 2n+4,\ \ldots,\ 4n+2\}$ in which case it is bounded by $\pm 2^k/k!$. $\square$

If we want to avoid all those rather complicated and somewhat messy power series manipulations, there is an interesting alternative and much simpler argument. However, we have to assume some familiarity with high-school calculus, which might be considered an excessive demand for what should be a 'pure' proof of the Pythagorean theorem.

To emphasize the dependence of the sums on a variable, let us write

$$C(x) \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}, \qquad S(x) \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}.$$

Lemma 1 *We have*

$$\frac{d}{dx}\Big(C^2(x) + S^2(x)\Big) \;=\; 0.$$

Proof On differentiating term by term we obtain

$$\begin{aligned}\frac{dC(x)}{dx} &= \sum_{k=1}^{\infty}(-1)^k \frac{2k\,x^{2k-1}}{(2k)!} \;=\; \sum_{k=1}^{\infty}(-1)^k \frac{x^{2k-1}}{(2k-1)!} \\ &= \sum_{j=0}^{\infty}(-1)^{j+1} \frac{x^{2j+1}}{(2j+1)!} \;=\; -\,S(x)\end{aligned}$$

and

$$\frac{dS(x)}{dx} \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{(2k+1)x^{2k}}{(2k+1)!} \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} \;=\; C(x).$$

Hence $\displaystyle\frac{d}{dx}\Big(C^2(x) + S^2(x)\Big) \;=\; -\,2C(x)S(x) + 2S(x)C(x) \;=\; 0.$ □

It is clear that $C^2(0) + S^2(0) = 1$. Therefore Theorem 1 follows from Lemma 1.

Problem 313.2 – Power ratio

Tony Forbes

Given integers a and b such that $2 \le b < a < b^2$, show that there are only a finite number of positive integers m such that

$$\left\lfloor \frac{a^m}{b^m} \right\rfloor \;=\; \left\lfloor \frac{a^m - 1}{b^m - 1} \right\rfloor \qquad (1)$$

is false. Or find a and b for which (1) is false for infinitely many positive integers m.

For example, when $a = 5$ and $b = 3$ you can verify that (1) is false for $m \in \{1, 2\}$, and if there are further values of m, they must be rather large.

The special case $a = 3$, $b = 2$ was presented as Problem 925 in P. J. Cameron and D. A. West, Research problems from the 20th British Combinatorial Conference, *Discrete Math.* **308** (2008), 621–630, where it was conjectured that (1) holds for all $m \ge 2$.

Solution 308.2 – Minifigures

> There are 12 distinct items in a set, and you can buy them in packs of six. The packs consist of six different items, apparently randomly selected from the 12 available. If we assume that is true, how many six-packs would you expect to buy in order to get a complete set of all 12 items?

Reinhardt Messerschmidt

For every positive integer n, let X_n be the random variable for the number of different Minifigures that we have after buying n packs, and let

$$T \;=\; \min\{n : X_n = 12\}.$$

We want to find the expected value of T.

Step 1: Transition probabilities. If $n \geq 2$ and we have $j \in \{6, \ldots, 12\}$ different Minifigures after buying $n-1$ packs, then we will have $k \in \{j, \ldots, 12\}$ of them after buying the next pack if and only if the pack contains $k - j$ of the $12 - j$ Minifigures that we don't have and $6 - (k - j)$ of the j ones that we do have. It follows that

$$\mathbb{P}[X_n = k \,|\, X_{n-1} = j] \;=\; \frac{\dbinom{12-j}{k-j}\dbinom{j}{6-k+j}}{\dbinom{12}{6}},$$

which does not depend on n. Let Q be the matrix with columns indexed by $j \in \{6, \ldots, 11\}$, rows indexed by $k \in \{6, \ldots, 11\}$, and entries

$$\mathbb{P}[X_n = k \,|\, X_{n-1} = j].$$

With rounding to four decimal places,

$$Q \;=\; \begin{bmatrix} 0.0011 & 0 & 0 & 0 & 0 & 0 \\ 0.0390 & 0.0076 & 0 & 0 & 0 & 0 \\ 0.2435 & 0.1136 & 0.0303 & 0 & 0 & 0 \\ 0.4329 & 0.3788 & 0.2424 & 0.0909 & 0 & 0 \\ 0.2435 & 0.3788 & 0.4545 & 0.4091 & 0.2273 & 0 \\ 0.0390 & 0.1136 & 0.2424 & 0.4091 & 0.5455 & 0.5000 \end{bmatrix}.$$

Step 2: Probability distribution of X_n. Let $\boldsymbol{x}_n$ be the column vector with rows indexed by $k \in \{6, \ldots, 11\}$ and entries $\mathbb{P}[X_n = k]$; therefore

$$\boldsymbol{x}_1 \;=\; [1, 0, 0, 0, 0, 0]^{\mathrm{T}}, \qquad \mathbf{1}^{\mathrm{T}}\boldsymbol{x}_n \;=\; \mathbb{P}[X_n \in \{6, \ldots, 11\}].$$

For every $n \geq 2$ and $k \in \{6, \ldots, 11\}$,

$$\mathbb{P}[X_n = k] \;=\; \sum_{j=6}^{k} \mathbb{P}[X_n = k \mid X_{n-1} = j]\,\mathbb{P}[X_{n-1} = j];$$

therefore $\boldsymbol{x}_n = Q\boldsymbol{x}_{n-1}$; therefore $\boldsymbol{x}_n = Q^{n-1}\boldsymbol{x}_1$.

Step 3: Expected value of T. We have

$$\begin{aligned}
\mathbb{E}[T] &= \sum_{m=1}^{\infty} m\mathbb{P}[T = m] \;=\; \sum_{m=1}^{\infty}\sum_{n=0}^{m-1} \mathbb{P}[T = m] \\
&= \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} \mathbb{P}[T = m] \;=\; \sum_{n=0}^{\infty} \mathbb{P}[T > n] \\
&= 1 + \sum_{n=1}^{\infty} \mathbb{P}[T > n];
\end{aligned}$$

and

$$\mathbb{P}[T > n] \;=\; \mathbb{P}[X_n \in \{6, \ldots, 11\}] \;=\; \mathbf{1}^{\mathrm{T}}\boldsymbol{x}_n \;=\; \mathbf{1}^{\mathrm{T}}Q^{n-1}\boldsymbol{x}_1;$$

therefore

$$\mathbb{E}[T] \;=\; 1 + \sum_{n=1}^{\infty} \mathbf{1}^{\mathrm{T}}Q^{n-1}\boldsymbol{x}_1 \;=\; 1 + \mathbf{1}^{\mathrm{T}}(I - Q)^{-1}\boldsymbol{x}_1.$$

Since

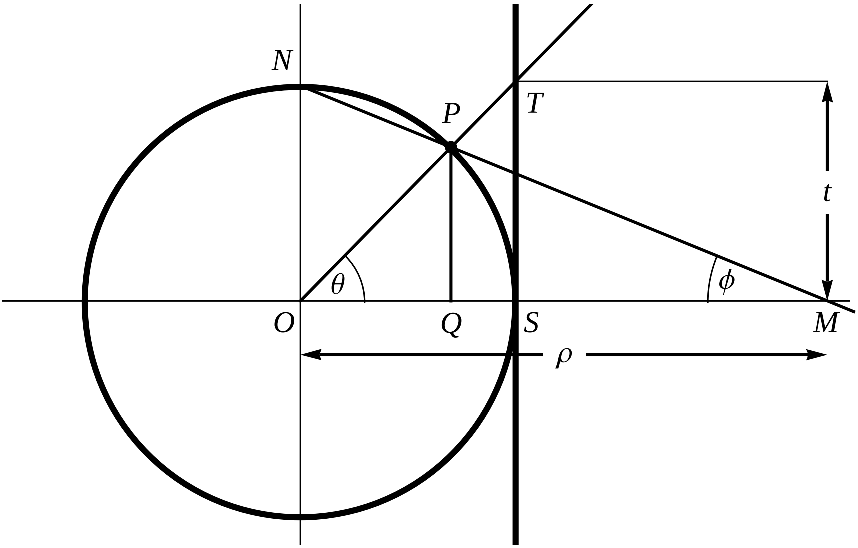
$$(I - Q)^{-1} \;=\; \begin{bmatrix} 1.0011 & 0 & 0 & 0 & 0 & 0 \\ 0.0393 & 1.0076 & 0 & 0 & 0 & 0 \\ 0.2560 & 0.1181 & 1.0313 & 0 & 0 & 0 \\ 0.5613 & 0.4513 & 0.2750 & 1.1000 & 0 & 0 \\ 0.7825 & 0.8023 & 0.7522 & 0.5824 & 1.2941 & 0 \\ 1.5240 & 1.5308 & 1.5456 & 1.5353 & 1.4118 & 2.0000 \end{bmatrix},$$

it follows that

$$\mathbb{E}[T] \;=\; 1 + 1.0011 + 0.0393 + 0.2560 + 0.5613 + 0.7825 + 1.5240 \;=\; 5.1642.$$

Solution 311.3 – Circle construction

Take a circle of radius 1 centred on the origin in $\mathbb{R}^2$, and a vertical line L parallel to the y-axis, passing through the point $(1, 0)$. Given a point P on the right hand half of the circle we define the height t to be given by the intersection of the production of $\overrightarrow{OP}$ with L. Let ρ be the distance between the origin and the intersection of the line $\overrightarrow{NP}$ with the x-axis. Show that $\rho^2 - 2t\rho - 1 = 0$.



Dave Clark

Let M be the intersection of $\overrightarrow{NP}$ with the x-axis. Drop a perpendicular from P to the x-axis at Q. Denote by θ the angle $\angle MOP$, so $-\pi < \theta < \pi$.

The radius of the circle is 1, so $t = \tan\theta$, $\overrightarrow{OQ} = \cos\theta$, and $\overrightarrow{QP} = \sin\theta$. If $\theta = 0$ then $M = P$, so $\rho = 1$ and $t = 0$, giving $\rho^2 - 2t\rho - 1 = 0$. If $\theta \neq 0$, so P is not on the x-axis, then $\triangle MON$ and $\triangle MQP$ are similar, so

$$\frac{1}{\rho} = \frac{\sin\theta}{\rho - \cos\theta},$$

giving

$$\rho = \frac{\cos\theta}{1 - \sin\theta} \quad \text{and} \quad t\rho = \frac{\sin\theta}{1 - \sin\theta},$$

so that

$$\begin{aligned}\rho^2 - 2t\rho - 1 &= \frac{\cos^2\theta}{(1-\sin\theta)^2} - \frac{2\sin\theta}{(1-\sin\theta)} - 1\\ &= \frac{(1-\sin^2\theta) - 2\sin\theta(1-\sin\theta) - (1-\sin\theta)^2}{(1-\sin\theta)^2}\\ &= \frac{1-\sin^2\theta - 2\sin\theta + 2\sin^2\theta - 1 + 2\sin\theta - \sin^2\theta}{(1-\sin\theta)^2}\\ &= 0.\end{aligned}$$

Ted Gore

I have added a vertical line from P to Q on the x-axis as well as labels T for the point $(1, t)$, M for the point $(\rho, 0)$ and S for the point $(1, 0)$. Let angle POS be θ and angle PMQ be ϕ.

Since P is on the unit circle we know that

$$PQ = \sin\theta \quad \text{and} \quad OQ = \cos\theta. \tag{1}$$

Also we have

$$\tan\phi = \frac{ON}{OM} = \frac{1}{\rho} = \frac{QP}{QM} = \frac{\sin\theta}{\rho - \cos\theta}. \tag{2}$$

From (1) we have

$$\frac{\sin\theta}{\cos\theta} = t \quad \text{so that} \quad \sin\theta = t\cos\theta.$$

From (2) we have

$$\frac{1}{\rho} = \frac{\sin\theta}{\rho - \cos\theta}$$

so that

$$\rho - \cos\theta = \rho t\cos\theta \quad \text{and} \quad \cos\theta = \frac{\rho}{\rho t + 1}.$$

Now

$$\sin^2\theta + \cos^2\theta = 1 = \frac{t^2\rho^2}{(\rho t+1)^2} + \frac{\rho^2}{(\rho t+1)^2}$$

so that $\rho^2 t^2 + 2\rho t + 1 = \rho^2 t^2 + \rho^2$ and $\rho^2 - 2\rho t - 1 = 0$.

Solution 308.5 – Integral

Let a be a positive integer. Show that

$$\int_0^1 \frac{\sqrt{1-x^a}}{\sqrt{1+x^a}}\,dx \;=\; \sqrt{\pi}\left(\frac{\Gamma\left(\dfrac{2a+1}{2a}\right)}{\Gamma\left(\dfrac{a+1}{2a}\right)} - \frac{1}{a+1}\,\frac{\Gamma\left(\dfrac{3a+1}{2a}\right)}{\Gamma\left(\dfrac{2a+1}{2a}\right)}\right).$$

Tommy Moorhouse

The integral itself is readily transformed into a familiar one which can be written in terms of gamma functions. The final step is to re-express the resulting gamma functions. First we note that

$$\int_0^{\pi/2} (\cos\theta)^{m-1} d\theta \;=\; \frac{\Gamma\left(\dfrac{1}{2}\right)\Gamma\left(\dfrac{m}{2}\right)}{2\,\Gamma\left(\dfrac{1+m}{2}\right)}.$$

The proof can be found in standard texts. Next we set out two useful relations satisfied by the gamma function.

$$\begin{aligned} z\Gamma(z) &= \Gamma(1+z), \\ \Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi}. \end{aligned}$$

Now we rework the integral by multiplying top and bottom by $\sqrt{1-x^a}$ and making the substitution $x^a = \cos\theta$:

$$\begin{aligned} I &= \int_0^1 \frac{\sqrt{1-x^a}}{\sqrt{1+x^a}} dx \\ &= \int_0^1 \frac{1-x^a}{\sqrt{1-x^{2a}}} dx \\ &= \frac{1}{a}\int_0^{\pi/2} \{(\cos\theta)^{(1-a)/a} - (\cos\theta)^{1/a}\} d\theta. \end{aligned}$$

Referring to our identities to integrate the two terms we see that this is just

$$I \;=\; \frac{\sqrt{\pi}}{2a}\left(\frac{\Gamma\left(\dfrac{1}{2a}\right)}{\Gamma\left(\dfrac{1+a}{2a}\right)} - \frac{\Gamma\left(\dfrac{1}{2a}+\dfrac{1}{2}\right)}{\Gamma\left(\dfrac{2a+1}{2a}\right)}\right).$$

The first term can be treated as follows, 'absorbing' the factor of $1/(2a)$:

$$\frac{1}{2a}\Gamma\left(\frac{1}{2a}\right) \;=\; \Gamma\left(\frac{2a+1}{2a}\right).$$

The second term can be rewritten

$$\begin{aligned}
-\frac{1}{2a}\frac{\Gamma\left(\frac{1}{2a}+\frac{1}{2}\right)}{\Gamma\left(\frac{2a+1}{2a}\right)} \;&=\; -\frac{1}{2a}\left(\frac{\frac{1}{2a}+\frac{1}{2}}{\frac{1}{2a}+\frac{1}{2}}\right)\frac{\Gamma\left(\frac{1}{2a}+\frac{1}{2}\right)}{\Gamma\left(\frac{2a+1}{2a}\right)} \\
&=\; -\frac{1}{a+1}\frac{\Gamma\left(\frac{1}{2a}+\frac{3}{2}\right)}{\Gamma\left(\frac{2a+1}{2a}\right)}.
\end{aligned}$$

Adding the two terms gives

$$I \;=\; \sqrt{\pi}\left(\frac{\Gamma\left(\frac{2a+1}{2a}\right)}{\Gamma\left(\frac{a+1}{2a}\right)} - \frac{1}{a+1}\frac{\Gamma\left(\frac{3a+1}{2a}\right)}{\Gamma\left(\frac{2a+1}{2a}\right)}\right).$$

Problem 313.3 – Tetrahedron

A tetrahedron has vertices A, B, C, D, and

$$\angle BAD \;=\; \angle BAC \;=\; \angle CAD \;=\; 90^\circ.$$

Show that the face areas $\triangle BCD$, $\triangle BAC$, $\triangle CAD$ and $\triangle DAB$ satisfy

$$(\triangle BCD)^2 \;=\; (\triangle BAD)^2 + (\triangle BAC)^2 + (\triangle CAD)^2.$$

Problem 313.4 – Power series

Tony Forbes

Show that

$$\sum_{k=0}^{\infty}\frac{\pi^{4k+3}}{(4k+3)!} \;=\; \sum_{k=0}^{\infty}\frac{\pi^{4k+1}}{(4k+1)!} \quad\text{and}\quad \sum_{k=0}^{\infty}\frac{\pi^{4k+2}}{(4k+2)!} \;=\; \sum_{k=0}^{\infty}\frac{\pi^{4k}}{(4k)!} + 1.$$

Solution 310.5 – 7NT

Here is a bridge deal where it is possible (with some cooperation from defence) for South to make 7NT.

North: ♠ 3 2 ♡ 9 8 7 6 5 4 3 2 ♢ 9 7 ♣ 2

West: ♠ A K Q J 10 8 6 4 ♡ A K ♢ – ♣ A K Q

East: ♠ – ♡ Q J 10 ♢ A K Q J 10 8 6 ♣ J 10 9

South: ♠ 9 7 5 ♡ – ♢ 5 4 3 2 ♣ 8 7 6 5 4 3

The play: West ♠8, ♠2, ♡Q, ♠9; South ♠7, ♠6, ♠3, ♡J; South ♠5, ♠4, ♣2, ♡10; South ♢3, ♡A, ♢9, ♢8; North ♢7, ♢6, ♢2, ♡K; North ♡9 and claims the rest.

Either (i) make the North–South hands weaker, or (ii) prove that (i) cannot be done. Alternatively, find a deal with the strongest North–South hands that will fail to make 7NT under every possible defence.

Colin Aldridge

Obviously the problems are different.

1. Defence plays as badly as possible to allow 7NT to make.
2. However badly or well defence plays 7NT never makes.

Clearly in your example E–W can always make 7NT if they decide to take the first trick. Your example is indeed the lowest example not involving a 10, i.e. two yarboroughs.

You need five cards lower in two suits to enable one hand to lose the first five tricks the other hands discarding the top five hearts having voids in the two suits in question. You have given N–S the lowest cards that allow this. If we allow North the 10 of hearts, then we only need to win four tricks and we can give South 8, 7, 5 of spades. This swaps two 9s for an 8 and a 10, the same pip count but with a 10, which is not a yarborough and hence a superior hand I guess. So you have the two worst hands in 310.5.

The following is I think the best pair of hands that always fails, and involves two super yarboroughs with no card above the 8!

Explanation: whatever is led is won by the west hand, who has 12 top tricks. South must follow and the lowest spade he is left with if he follows from the top down is the 5 of spades which beats West's 4 of spades.

North: ♠ 3 2 ♡ 8 7 6 5 ♢ 8 7 6 5 ♣ 7 6 5

West: ♠ A K Q 4 ♡ A K Q ♢ A K Q ♣ A K Q

East: ♠ J 10 9 ♡ J 10 9 ♢ J 10 9 ♣ J 10 9 8

South: ♠ 8 7 6 5 ♡ 4 3 2 ♢ 4 3 2 ♣ 4 3 2

By the way, the best hand I ever picked up was

♠ A K 10 x x x x x x x ♡ A x ♢ A ♣ –

I ended in 7♠ and much debate ensued at the club as to the correct bidding sequence to find 7NT, which makes. There is a system of cue bidding which would allow this but it is not easy to get your partner to do this with the hand above.

Solution 310.8 – Sum

Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n^3 + n^4} \;=\; \frac{\pi^2}{3} - 3.$$

Henry Ricardo

The partial fraction decomposition of the nth term of the series is

$$\frac{1}{n^2 + 2n^3 + n^4} \;=\; \frac{1}{n^2(n+1)^2} = -\frac{2}{n} + \frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{2}{n+1},$$

so that

$$\begin{aligned}\sum_{n=1}^{N} \frac{1}{n^2 + 2n^3 + n^4} \;&=\; -2\sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+1}\right) + \sum_{n=1}^{N}\frac{1}{n^2} + \sum_{n=1}^{N}\frac{1}{(n+1)^2} \\ &=\; -2\left(1 - \frac{1}{N+1}\right) + \sum_{n=1}^{N}\frac{1}{n^2} + \sum_{n=1}^{N}\frac{1}{(n+1)^2}. \qquad (1)\end{aligned}$$

As $N \to \infty$, the first term of (1) tends to -2, the second to $\pi^2/6$ ($= \zeta(2)$), and the last to $(\pi^2/6 - 1)$, so that the infinite series converges to

$$-2 \;+\; \pi^2/6 \;+\; (\pi^2/6 - 1) \;=\; -3 \;+\; \pi^2/3.$$

04.00

Problem 313.5 – Integers

Tony Forbes

For which x is $\sqrt{24x+1}$ an integer?

Problem 313.6 – APR

What is the APR of a loan of £1000 that is never repaid?

Front cover Lines generating a vector field on a sphere; see page 2.