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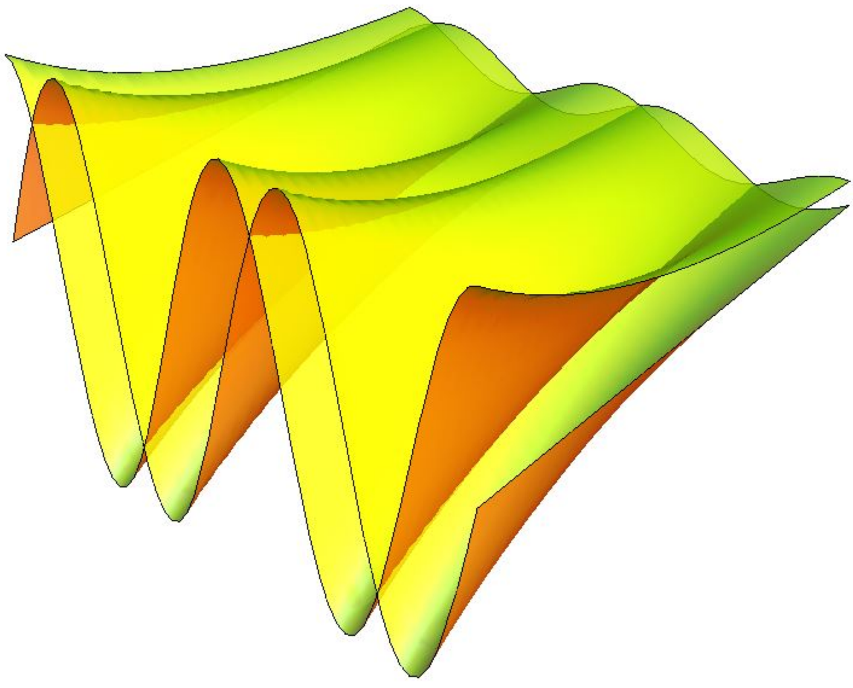
# M500 314



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**at Kents Hill Park Conference Centre, Milton Keynes.**

For details, pricing and a booking form, please refer to the M500 web site.

`m500.org.uk/winter-weekend/`

# Solution 311.2 – Triangle-free regular graphs

> Prove that there exists a $k$-regular graph with $2k+1$ vertices and girth at least 4 only when $k$ is 0 or 2. Or find a counter-example.

## Dave Clark

A $k$-regular graph with $2k+1$ vertices has $k(2k+1)/2$ edges. Obviously this must be an integer, and as $2k+1$ is odd for every integer $k$, we'll only have $k(2k+1)/2$ being an integer when $k$ is even.

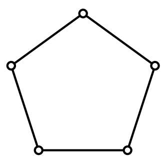
Suppose we have a $k$-regular graph with $2k+1$ vertices and girth at least 4 (i.e. triangle-free). Let $v$ be any vertex, let $X_1$ be the set of vertices which are neighbours of $v$, and let $X_2$ be the set of vertices other than $v$ which are neighbours of vertices in $X_1$. As $v$ has degree $k$, there must be exactly $k$ vertices in $X_1$, and triangle-freedom means there cannot be any edges joining vertices in $X_1$ to each other, so each vertex in $X_1$ has exactly $k-1$ neighbours in $X_2$.

Since there must be at least $k-1$ vertices in $X_2$, notice that $v$, $X_1$ and $X_2$ together account for at least $1+k+(k-1)=2k$ vertices. The last remaining vertex must have $k$ neighbours, and it cannot be a neighbour of $v$ (else it would be in $X_1$), and we have so far counted only $k-1$ vertices in $X_2$, so it must have at least one neighbour in $X_1$ and it is therefore also in $X_2$, so there are $k$ vertices in $X_2$ altogether.

There are $k$ edges connecting $v$ to vertices in $X_1$, and $k(k-1)$ edges connecting vertices in $X_1$ to vertices in $X_2$, but the $k$ vertices in $X_2$ all have degree $k$, and triangle-freedom means that none of them is a neighbour of $v$, so there must be exactly $k(2k+1)/2-k-k(k-1)=k/2$ edges joining vertices in $X_2$ to each other. However, triangle-freedom also means that two vertices in $X_2$ cannot be neighbours if they have a common neighbour in $X_1$, and each vertex in $X_1$ has $k-1$ neighbours in $X_2$, which is all but one of the vertices in $X_2$, so there can be at most one edge joining vertices in $X_2$ to each other. Thus $k/2 \le 1$, so $k \le 2$. We've already seen that $k$ must be even, so the only possibilities are when $k$ is 0 or 2.

As it happens, a unique $k$-regular graph with $2k+1$ vertices and girth at least 4 exists for $k=0$ and for $k=2$.

0-regular graph with 1 vertex



2-regular graph with 5 vertices

# The Thue–Morse sequence is aperiodic: A proof

## Martin Hansen

Although first investigated by the French mathematician Eugène Prouhet in 1851, the Thue–Morse sequence is named after the Norwegian, Axel Thue, who used it in 1906 as a foundation stone for the branch of mathematics *Combinatorics on Words*, and the American, Marston Morse, who applied it in 1921 to great effect in *differential geometry.* It can be defined in several ways. The following is particularly suited to introducing the sequence.

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**Definition : Generation of the Thue–Morse Word**

The right-sided infinite Thue–Morse word may be generated in an iterative fashion by starting with an initial letter $a$ and then repeatedly applying the substitution $\theta_{\mathcal{TM}}$ given by $a \to ab$, $b \to ba$. The finite $n^{\text{th}}$ right-sided Thue–Morse word is defined as $\mathcal{TM}_n = \theta^n_{\mathcal{TM}}(a)$.

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The table below gives the first few of the finite Thue–Morse words

| $n$ | $\mathcal{TM}_n = \theta^n_{\mathcal{TM}}(a)$ $(a \to ab,\, b \to ba)$ | $\lvert a\rvert$ | $\lvert b\rvert$ | $\lvert\mathcal{TM}_n\rvert$ |
|---|---|---|---|---|
| 0 | $a$ | 1 | 0 | 1 |
| 1 | $ab$ | 1 | 1 | 2 |
| 2 | $abba$ | 2 | 2 | 4 |
| 3 | $abbabaab$ | 4 | 4 | 8 |
| 4 | $abbabaabbaababba$ | 8 | 8 | 16 |
| 5 | $abbabaabbaababbabaababbaabbabaab$ | 16 | 16 | 32 |

In the table, observe that each preceding word occurs at the start of every subsequent word. In other words, the infinite Thue–Morse word is the fixed point of an iterative process. What makes the Thue–Morse sequence interesting is the fact that it is clearly not a chaotic random jumble of the letters $a$ and $b$. It seems to have pattern, but that pattern is hard to pin down.

The full bi-infinite Thue–Morse word extends indefinitely to both the left and the right. It has two remarkable properties that in combination are what earn it the label 'aperiodic'. The first is that if a duplicate copy of the word is made, and translated left or right over the first, there is no position at which the two pieces align other than if no translation is applied at all. However, the following bi-infinite sequence also has this property,

$$\ldots aaaaaaaabaaaaaaaa \ldots .$$

This is an example of a sequence that is non-periodic but it is not aperiodic. To be aperiodic there is the additional requirement that the sequence contain no arbitrarily large periodic part.

### Incidence Matrix and Perron–Frobenius Eigenvalue

Mathematically, this second requirement demands the substitution be primitive. To explain this property, we first need to establish what the incidence matrix of a substitution is. Let's look at an example where there are no parts that can be confused with each other. Consider the substitution,

$$a \to abaaaba, \qquad b \to abbb.$$

This has incidence matrix

$$\begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix}$$

because, in the substitution, $a$ is replaced with five $a$ and two $b$ whereas $b$ is replaced with one $a$ and three $b$. The matrix carries frequency information but the order of the letters within the substitution is lost. Undergraduate matrix algebra is now used to obtain the characteristic polynomial and the eigenvalues. For the above,

$$\lambda^2 - 8\lambda + 13 \;=\; 0 \qquad \text{with } \lambda = 4 \pm \sqrt{3}.$$

To be primitive the incident matrix (or a positive integer power of that matrix) must have all positive entries. The above matrix satisfies this requirement and so is the primitive matrix of a primitive substitution. Such a matrix has a largest positive eigenvalue (by Perron–Frobenius theory) called the Perron–Frobenius eigenvalue. For our example, $\lambda_{\mathrm{PF}} = 4 + \sqrt{3}$. In general, if this special eigenvalue is irrational, we immediately know that the substitution is aperiodic. This is because of Theorem 1, which formally tidies up the above discussion.

**Theorem 1 : Aperiodic Proof (Baake and Grimm, 2013)**

Let $\theta$ be a primitive substitution on the finite alphabet $\mathcal{A}_n = \{a_1, a_2, \ldots, a_n\}$ with incidence matrix $\mathbf{M}_\theta$, and let $w$ be a bi-infinite word of $\theta$. If the Perron–Frobenius eigenvalue of $\mathbf{M}_\theta$ is irrational, then $w$ is aperiodic.

*Proof*

See *Aperiodic Order: Volume 1, A Mathematical Invitation*, by Michael Baake and Uwe Grimm, page 89. □

Unfortunately, Theorem 1 does not allow us to lazily deduce that the Thue–Morse word is aperiodic. Although the Thue–Morse substitution satisfies the requirement of the theorem that it be primitive, the incidence matrix gives rise to an integer eigenvalue rather than one that is irrational. To be specific,

$$\mathbf{M}_{\mathcal{TM}} \;=\; \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \;\Rightarrow\; \lambda^2 - 2\lambda = 0 \;\Rightarrow\; \lambda_{\mathrm{PF}} = 2.$$

The integer eigenvalue means that our desire to prove the Thue–Morse sequence is aperiodic is going to be a 'living on our wit and cunning' affair. Along the way we shall make use of another definition of the Thue–Morse words.

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**Definition : The Thue-Morse Words by Concatenation**

$$\mathcal{TM}_n \;=\; \mathcal{TM}_{n-1}\,\overline{\mathcal{TM}_{n-1}}$$

for integer $n \geq 1$, with $\mathcal{TM}_0 = a$.

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For example,

$$\mathcal{TM}_4 \;=\; \mathcal{TM}_3\,\overline{\mathcal{TM}_3} \;=\; abbabaab\overline{abbabaab} \;=\; abbabaabbaababba.$$

## The Thue–Morse Language Table

We are now almost ready to start working through the steps that will result in a proof that the Thue–Morse word is indeed aperiodic. However, there is one more idea to be grasped before starting in earnest. It is to consider how many subwords of various lengths can occur in the Thue–Morse word. To help explain this here is the start of the right-sided infinite word once again,

$$abbabaabbaababbabaababbaabbabaab\ldots.$$

Looking carefully, notice that the subword *aaa* does not appear. This is an example of a cube. Whilst *abab* can be found (an example of a square) *ababab*, another cube, can not. A substitution's language table is the start of an effort to tabulate which subwords can occur. It can be drawn up by hand, but often software is used. Here is the start of such a table for the Thue–Morse word. This table will be useful later on.

| length | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... |
|---|---|---|---|---|---|---|---|---|
| | *a* | *aa* | *aab* | *aaba* | *aabab* | *aababb* | *aababba* | ... |
| | *b* | *ab* | *aba* | *aabb* | *aabba* | *aabbaa* | *aabbaab* | ... |
| | | *ba* | *abb* | *abaa* | *abaab* | *aabbab* | *aabbaba* | ... |
| $\mathcal{TM}$ | | *bb* | *baa* | *abab* | *ababb* | *abaaba* | *abaabab* | ... |
| words | | | *bab* | *abba* | *abbaa* | *abaabb* | *abaabba* | ... |
| | | | *bba* | *baab* | *abbab* | *ababba* | *ababbaa* | ... |
| $a \to ab$ | | | | *baba* | *baaba* | *abbaab* | *ababbab* | ... |
| $b \to ba$ | | | | *babb* | *baabb* | *abbaba* | *abbaaba* | ... |
| | | | | *bbaa* | *babaa* | *baabab* | *abbaabb* | ... |
| | | | | *bbab* | *babba* | *baabba* | *abbabaa* | ... |
| | | | | | *bbaab* | *babaab* | *baababb* | ... |
| | | | | | *bbaba* | *babbaa* | *baabbaa* | ... |
| | | | | | | *babbab* | *baabbab* | ... |
| | | | | | | *bbaaba* | *babaaba* | ... |
| | | | | | | *bbaabb* | *babaabb* | ... |
| | | | | | | *bbabaa* | *babbaab* | ... |
| | | | | | | | *babbaba* | ... |
| | | | | | | | *bbaabab* | ... |
| | | | | | | | *bbaabba* | ... |
| | | | | | | | *bbabaab* | ... |
| № of words | 2 | 4 | 6 | 10 | 12 | 16 | 20 | ... |

## The Proof

This builds steadily over the next three pages as we establish three lemmas and define what it means to be strongly cube free. We then show that the (infinite) Thue–Morse word is strongly cube-free and so aperiodic.

**Lemma 1 : Exclusion from $\mathcal{TM}$ of *aaa* and *bbb***

The subwords *aaa* and *bbb* cannot occur in the Thue–Morse word.

*Proof (by contradiction)*

Suppose by way of deriving a contradiction that the three letter subword $\ldots a\boldsymbol{a}a\ldots$ has occurred in $\mathcal{TM}$. Focus on the middle $\boldsymbol{a}$. We can argue that this $\boldsymbol{a}$ must have come from the iteration in a previous word of a letter $b$ because if it had come from a letter $a$ then one of the letters adjacent to $\boldsymbol{a}$ would be $b$, which neither is. But if it had come from a letter $b$ then, again, one of the letters adjacent to $\boldsymbol{a}$ would be $b$, which neither is. The only conclusion is that the original assumption, that $\dots aaa \dots$ could occur in the Thue–Morse word is false. By a similar argument, $\dots bbb \dots$ can also not occur in $\mathcal{TM}$. □

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**Lemma 2 : Exclusion from $\mathcal{TM}$ of *ababa* and *babab***

The subwords *ababa* and *babab* cannot occur in the Thue–Morse word.

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*Proof (by contradiction)*

Assume that the five letter subword $\dots ababa \dots$ has occurred in $\mathcal{TM}$. With a view to desubstitution this word can be placed into letter pair brackets in two ways, either as $\dots(ab)(ab)(a\dots$ or $\dots a)(ba)(ba)\dots$, which we consider in turn.

CASE 1 : $\dots(ab)(ab)(a\dots$

The first bracketed pair desubstitutes to $a$, as does the second. In order to desubstitute the third bracket the subsequent letter must be $b$. We then have the following.

$$\begin{array}{ccccl} \dots & (ab) & (ab) & (ab) & \dots \\ & \downarrow & \downarrow & \downarrow & \text{desubstitution} \\ \dots & a & a & a & \dots \end{array}$$

However, from Lemma 1, it is known that $\dots aaa \dots$ can not occur and so we deduce that $\dots(ab)(ab)(a\dots$ can not occur.

CASE 2 : $\dots a)(ba)(ba)\dots$

The last bracketed pair desubstitutes to $b$, as does the bracket before. In order to desubstitute the first bracket the previous letter must be $b$. We then have the following.

$$\begin{array}{ccccl} \dots & (ba) & (ba) & (ba) & \dots \\ & \downarrow & \downarrow & \downarrow & \text{desubstitution} \\ \dots & b & b & b & \dots \end{array}$$

However, from Lemma 1, it is known that $\dots bbb \dots$ can not occur and so we deduce that $\dots a)(ba)(ba)\dots$ can also not occur.

It has thus been shown that neither *ababa* nor *babab* are subwords of $\mathcal{TM}$. □

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**Lemma 3 : The *aa*, *bb* constraint on $\mathcal{TM}$ subwords**

Any subword of the Thue–Morse word that contains five letters or more, must contain *aa* or *bb*.

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*Proof*

With only two letters to play with, the only way to write down a subword of length five without *aa* or *bb* occurring is to alternate the occurrences of *a* and *b*. This can be done in two ways, either *ababa* or *babab*. However, Lemma 2 tells us that neither of these is legal. Therefore any subword of five letters must contain either *aa* or *bb*. Any subword of more than five letters must contain five letter subwords. Such subwords must contain *aa* or *bb* and therefore so must any subword of more than five letters. □

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**Definition : Cube-free words**

For a non-empty finite word $u$, let $u_0$ and $u_z$ denote the first and last letters of $u$, respectively.
A weak cube is a word of the form $uuu_0$ (or, equivalently, $u_z uu$).
A word $w$ is strongly cube-free if it does not contain any weak cubes.

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**Theorem 2 : Strongly Cube-free**

The Thue–Morse word is strongly cube-free.

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*Proof (by contradiction)*

Let us assume the opposite of the claimed result, that the Thue–Morse word, $\mathcal{TM} = w_0 w_1 w_2 \ldots$, contains at least one subword, $u$, such that $uuu_0$ is a subword of $\mathcal{TM}$. Consider the minimal length of $u$ and denote that minimal length $l$. In other words, $|u| = l$. Clearly, $u$ cannot be a single letter $a$ or a single letter $b$ for then $uuu_0$ would be $aaa$ or $bbb$ respectively, both of which, from Lemma 1, are illegal. Nor can $u$ be either of the double letter subwords $aa$ or $bb$ for the same reason. So, the minimum length that $u$ can be is two and, even then, only $ab$ or $ba$ are possibilities. However, if $u$ was $ab$ or $ba$ then $uuu_0$ would be $ababa$ or $babab$ respectively, both of which, by Lemma 2 are illegal. So $|u| \geq 3$ and $|uuu_0| \geq 7$. This implies that $uuu_0$ can only be found in a Thue–Morse word of seven or more letters. The first such word is $\mathcal{TM}_3 = abbabaab$. Notice that the occurrence of $bb$ in this word is at $w_1$ and $aa$ at $w_5$. The manner of generating subsequent words using

$$\mathcal{TM}_n = \mathcal{TM}_{n-1}\overline{\mathcal{TM}_{n-1}}$$

means both that there will always be at least two $aa$ or $bb$ and that all such occurrences are at positions $w_x$, where $x$ is an odd number. Now consider the parity of $l$.

CASE 1 : $l$ is odd

If $|u| = 3$, then we can draw up the following table of $u$ and $uuu_0$ and show by desubstitution that none of the six possible $uuu_0$ can be in the Thue–Morse word. Alternatively it can be seen by an inspection of the Thue–Morse language table, given earlier, that none of these $uuu_0$ are legal subwords.

| $u$ | $uuu_0$ |
|---|---|
| $aab$ | $aabaaba$ |
| $aba$ | $abaabaa$ |
| $abb$ | $abbabba$ |
| $baa$ | $baabaab$ |
| $bab$ | $babbabb$ |
| $bba$ | $bbabbab$ |

For $|u| \geq 5$, we know from Lemma 3 that $u$ contains $aa$ or $bb$ at least twice (as indeed does $\mathcal{TM}$). Where they occur in $\mathcal{TM}$ (and there will be an infinite number of such occurrences) must be at odd positions. Thus the distances between all occurrences of $aa$ and $bb$ are even. Crucially, one of these distances must be $l$. However, we are looking at the case where $l$ is odd and so this contradicts that assumption. To summarise, the assumption that $l$ is odd has led to a contradiction and so that assumption cannot be correct.

CASE 2 : $l$ is even

Recall that in the Thue–Morse word deleting every other letter in an odd position results in the Thue–Morse word. So if a subword $u = w_0w_1w_2 \ldots w_{l-1}$ (of even length) gives a $uuu_0$ in $\mathcal{TM}$ then so too must $u' = w_0w_2 \ldots w_{l-2}$. This, however, is a shorter word, in contradiction to $l$ being minimal.

In overall conclusion, the Thue–Morse word is strongly cube-free. □

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**Corollary : The Thue–Morse Aperiodicity**

The Thue–Morse word is aperiodic.

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*Proof*

This follows immediately from Theorem 2. □

**Intuitive Understanding**

The proof tells us that, given any piece whatsoever (of any finite length) of the Thue–Morse word, that piece will never occur more than twice in succession. It may well occur singly or as a pair an infinite number of (spaced out) times in the word, but that is another story.

**Acknowledgement**

An outline sketch of this proof was suggested by the late Uwe Grimm when I knew him at the Open University in 2021. Discussions in 2023 with Dr Nicholas Korpelainen and Dr Petra Staynova at the University of Derby further added to the mix of ideas that have now crystallised in the proof presented.

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# Problem 314.1 – Cosines

## David Sixsmith

The cosine formula

$$a^2 \;=\; b^2 + c^2 - 2bc\cos A, \tag{1}$$

where $a, b, c$ are the sides of a triangle and $A$ is the angle opposite the side of length $a$, is well known. Equation (1) can be written in a symmetric 'Pythagorean' form

$$a^2 \;=\; (\alpha b + \beta c)^2 + (\beta b + \alpha c)^2. \tag{2}$$

An initial problem it to find $\alpha$ and $\beta$ as simple functions of the angle $A$. A more difficult problem is to find a direct geometric justification for equation (2), in other words, a derivation that does not use the cosine formula. (I have not had success with the latter.)

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# Problem 314.2 – Rational eigenvalues

## Tony Forbes

For which positive integers $a$ does the matrix

$$M \;=\; \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & a \\ 1 & b & c \end{bmatrix}$$

have non-zero determinant and rational eigenvalues for some positive integers $b$ and $c$?

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# Pythagoras' theorem revisited again

## Tony Forbes

My proof of the Pythagorean theorem in M500 **313** has been the subject of a certain amount criticism from various people to whom I have shown it. Leaving aside any questioning of the author's sanity, the main discussion was centered on my sledgehammer approach to the solution of a standard problem in high-school Euclidean geometry. Spread over three pages (albeit A5 ones) it is far too long-winded.

The power-series proof in M500 **313**, which involves computing (1) and then letting $n$ tend to infinity, can be done much more easily by dealing directly with infinite sums. However, I claim that my proof is of some interest (at least to me). It avoids tortuous geometric reasoning involving such lofty concepts as 'angles', 'straight lines' and the relations between them. And since there isn't one, we don't have to worry about our diagram truly representing the general case.

I was quite keen to see exactly what happens when you compute the finite sum

$$\left(\sum_{k=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!}\right)^2+\left(\sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\right)^2. \qquad (1)$$

For instance, when $n=4$ you get

$$\frac{131681894400+72576x^{10}-30240x^{12}+2160x^{14}-63x^{16}+x^{18}}{131681894400},$$

and you can plainly see that this is nearly equal to 1 when $|x|$ is not too large. What has happened is that all the small positive powers of $x$ have disappeared. For general $n$, as explained in M500 **313**, the non-constant part of the polynomial looks like this,

$$a_{2n+2}x^{2n+2}+a_{2n+4}x^{2n+4}+\cdots+a_{4n+2}x^{4n+2},$$

with the $a_k$ bounded by $\pm 2^k/k!$. Thus we are assured of rapid convergence to zero for any $x$. Just make $n$ go to $\infty$.

For the critics, I offer the suggested alternative proof of the theorem. Like the one in M500 **313**, there is no mention of triangles or trigonometry. We achieve considerable simplicity because we avoid the complicated details associated with manipulating finite sums. However, the simplicity is an illusion. We are not bothering to answer thorny questions concerning the convergence of any infinite power series that appears in our argument.

**Theorem 1 (Pythagoras)** *We have*

$$\left(\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}\right)^2 + \left(\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}\right)^2 \;=\; 1.$$

**Proof** Let

$$C(x) \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}, \qquad S(x) \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

Then

$$C^2(x) \;=\; \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k} \frac{x^{2j+2k}}{(2j)!(2k)!}.$$

Put $r = k + j$:

$$C^2(x) \;=\; \sum_{r=0}^{\infty}\sum_{j=0}^{r}(-1)^r \frac{x^{2r}}{(2j)!(2r-2j)!} \;=\; \sum_{r=0}^{\infty}(-1)^r \frac{x^{2r}}{(2r)!}\sum_{j=0}^{r}\binom{2r}{2j}.$$

The binomial sum can be evaluated: it is 1 when $r = 0$ and $2^{2r-1}$ when $r \geq 1$. Hence

$$C^2(x) \;=\; 1 + \frac{1}{2}\sum_{r=1}^{\infty}(-1)^r \frac{(2x)^{2r}}{(2r)!} \;=\; \frac{1 + C(2x)}{2}. \qquad (2)$$

Similarly,

$$\begin{aligned} S^2(x) \;&=\; \sum_{r=0}^{\infty}\sum_{j=0}^{r}(-1)^r \frac{x^{2r+2}}{(2j+1)!(2r-2j+1)!} \\ &=\; \sum_{r=0}^{\infty}(-1)^r \frac{x^{2r+2}}{(2r+2)!}\sum_{j=0}^{r}\binom{2r+2}{2j+1} \\ &=\; \frac{1}{2}\sum_{r=0}^{\infty}(-1)^r \frac{(2x)^{2r+2}}{(2r+2)!} \;=\; \frac{1 - C(2x)}{2} \end{aligned}$$

since the binomial sum is $2^{2r+1}$ for $r \geq 0$. Combining with (2), we have

$$C^2(x) + S^2(x) \;=\; 1. \qquad \square$$

If we permit the use of $i = \sqrt{-1}$, the previous argument can be expressed even more succinctly. Define

$$E(x) \;=\; \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

and observe that

$$\begin{aligned} E(x)E(y) \;&=\; \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^j}{j!}\frac{y^k}{k!} \;=\; \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{j=0}^{r} x^j y^{r-j} \binom{r}{j} \\ &=\; \sum_{r=0}^{\infty} \frac{1}{r!}\,(x+y)^r \;=\; E(x+y). \end{aligned} \tag{3}$$

Also one can easily verify the familiar identities

$$E(ix) \;=\; C(x) + iS(x), \qquad E(-ix) \;=\; C(x) - iS(x) \tag{4}$$

by splitting the sum for $E(\pm ix)$ into a part that does not explicitly involve $i$ and a part that does.

Now the proof of Theorem 1 is extremely straightforward:

$$\begin{aligned} C^2(x) + S^2(x) \;&=\; \big(C(x) + iS(x)\big)\big(C(x) - iS(x)\big) \\ &=\; E(ix)E(-ix) \;=\; E(0) \;=\; 1. \end{aligned}$$

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By looking at the power series expressions, we see immediately that $E(0) = C(0) = 1$, $S(0) = 0$. And I believe that $E(1) \approx 2.71828$ is an important mathematical constant.

By computing its power series to sufficiently many terms we can verify that $S(x)$ has a zero in the vicinity of $22/7$. So let us define a number, which we shall call $\pi$, by

$$S(\pi) \;=\; 0, \qquad 3.14 \;\le\; \pi \;\le\; 3.15. \tag{5}$$

This number has some interesting properties. Using Pythagoras' theorem we obtain $C(\pi) = \pm 1$, and by comparing with the value obtained from directly computing the power series, $C(\pi) \approx -1.0$, it is clear that we must select the negative sign; thus $C(\pi) = -1$ exactly. Then, by making use of (3) and (4), we have

$$E(i\pi) \;=\; -1, \qquad E(2i\pi) \;=\; 1,$$

$$\begin{aligned} E(x+2i\pi) &= E(x)E(2i\pi) = E(x), \\ C(x+2\pi) &= C(x), \qquad S(x+2\pi) = S(x). \end{aligned}$$

Thus we have shown that $C(x)$ and $S(x)$ are periodic with period $2\pi$. Moreover, it is not difficult to prove that whenever $j$ is an integer

$$\begin{aligned} S(j\pi) &= C\left(j\pi + \frac{\pi}{2}\right) = 0, \\ C(2j\pi) &= S\left(2j\pi + \frac{\pi}{2}\right) = 1, \\ C(2j\pi + \pi) &= S\left(2j\pi - \frac{\pi}{2}\right) = -1, \end{aligned}$$

and, again with the help of (3) and Pythagoras' theorem, we can obtain values at other rational multiples of $\pi$, such as

$$S\left(\frac{\pi}{4}\right) = C\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},$$

$$S\left(\frac{\pi}{3}\right) = C\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \qquad S\left(\frac{\pi}{6}\right) = C\left(\frac{\pi}{3}\right) = \frac{1}{2},$$

$$S\left(\frac{\pi}{5}\right) = \sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}, \quad C\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}.$$

The proofs are straightforward and left to the reader.

Finally, let us define yet another function by a power series:

$$\begin{aligned} D(x) &= \sum_{k=0}^{\infty} (-1)^k \binom{1/2}{k} \frac{x^{2k+1}}{2k+1} \\ &= x - \frac{x^3}{6} - \frac{x^5}{40} - \frac{x^7}{112} - \frac{5x^9}{1152} - \frac{7x^{11}}{2816} - \frac{21x^{13}}{13312} - \frac{11x^{15}}{10240} - \dots, \end{aligned}$$

valid for $-1 \le x \le 1$ (the coefficient of $x^{2k+1}$ is $O(k^{-5/2})$; see Problem 314.3 on page 16). By computing the power series to sufficient accuracy, you can verify that $D(1) \approx \pi/4$. Moreover, you probably recognise $D(x)$ as $\int_0^x \sqrt{1-u^2}\,du$, and so possibly one can argue that $4D(1)$ is the area of a unit circle—whatever that might mean. So we offer an interesting problem for you to solve.

Prove that $4D(1) = \pi$, where $\pi$ is defined by (5).

# Solution 308.4 – Group construction

> Let $G$ be a finite group of even order. Let $H$ be a subgroup of $G$ of order $|G|/2$. Let $A = G \setminus H$ and assume there exists an element $a$ of $A$ that commutes with every element of $A$. Show that $(A, \circ)$ is a group, where the operation $\circ$ is defined by
>
> $$x \circ y \;=\; xya.$$

## Tommy Moorhouse

**Preliminary lemmas** We are given a group $G$ of even order, with a subgroup $H$ of order $|G|/2$, and we need to show that $A = G \backslash H$ is a group when the group operation is

$$a_1 \circ a_2 \;=\; a_1 a_2 a$$

where $a \in A$ commutes with all elements of $A$ and the operation on the right is the group product in $G$. We will do this by checking the group axioms, but first we need some lemmas.

**Lemma 1** If $a_1 \in A$ then $a_1^{-1} \in A$, where the inverse is w.r.t. the group product in $G$.

**Proof** Suppose that $h \notin A$, so that $h \in H$, and further that $ha_1 = e$. Since $e \in H$ and $h^{-1} \in H$ ($H$ is a subgroup of $G$) we have $a_1 = h^{-1} \in H$, a contradiction.

**Lemma 2** Fix $a \in A$. The set $\{ha : h \in H\} = A$.

**Proof** Suppose that $h \in H, a \in A$ and $ha = h_1 \in H$. Multiplying the last expression from the left by $h^{-1}$ we have $a = h^{-1}h_1 \in H$, a contradiction. Thus $ha \in A$ for all $h \in H$. Now we show that as $h$ ranges over $H$ we get each element of $A$. For suppose $h_1 a = h_2 a$. Multiplying from the right by $a^{-1}$ we find $h_1 = h_2$. Each element of $H$ thus gives a different element of $A$, and the lemma is proved.

**Corollary** Fix $a_2$. Then if $a_1 \in A$ we have $a_1 = ha_2$ for some $h \in H$.

**Lemma 3** If $a_1 \in A, a_2 \in A$ then $a_1 a_2 \in H$.

**Proof** Suppose that $a_1 a_2 = a_3 \in A$. By the corollary to Lemma 2 $a_1 = ha_2^{-1}$ for some $h \in H$, so $a_3 = h$, a contradiction.

**Checking the group axioms** The axioms are listed below (in no particular order), together with a demonstration that $\{A, \circ\}$ satisfies each one.

**Axiom I: Closure** Let $\mathcal{G}$ be a group. If $g \in \mathcal{G}, g' \in \mathcal{G}$ then $gg' \in \mathcal{G}$.

**Closure in** $\{A, \circ\}$. For $a_1, a_2 \in A$ we have

$$\begin{aligned} a_1 \circ a_2 &= a_1 a_2 a \\ &= ha \in A \end{aligned}$$

for some $h \in H$. The second equality follows from Lemma 3, and the conclusion from Lemma 2.

**Axiom II: Identity** There is an element $e \in \mathcal{G}$ such that $eg = ge = g$ for all $g \in \mathcal{G}$

**Identity in** $\{A, \circ\}$. We have fixed $a$ to be an element of $A$ commuting with every other element of $A$. Given $a_1 \in A$ we consider the action of the element $a^{-1}$

$$\begin{aligned} a_1 \circ a^{-1} &= a_1 a^{-1} a \\ &= a_1. \end{aligned}$$

Since $a$ commutes with all elements of $A$ we easily see that $a^{-1} \circ a_1 = a_1$, so $a^{-1}$ is the required identity $e$ in $A$.

**Axiom III: Inverses** If $g \in \mathcal{G}$ there is a unique $g' \in \mathcal{G}$ such that $gg' = g'g = e$.

**Inverses in** $\{A, \circ\}$. Given $a_1 \in A$ consider the element $a_1^{-1}(a^{-1})^2$. This is in $A$ and

$$\begin{aligned} a_1 \circ a_1^{-1}(a^{-1})^2 &= a_1 a_1^{-1} a (a^{-1})^2 \\ &= a^{-1}, \end{aligned}$$

which is the identity in $\{A, \circ\}$. Uniqueness is readily established. Suppose that $a_1 \circ a_2 = a^{-1}$. Then $a_1 a_2 a = a^{-1}$ and $a_2 = a_1^{-1}(a^{-1})^2$.

**Axiom IV: Associativity** If $g \in \mathcal{G}, g' \in \mathcal{G}$ and $g'' \in \mathcal{G}$ then $(gg')g'' = g(g'g'')$, where the brackets indicate the order in which the products are to be evaluated.

**Associativity in $\{A, \circ\}$.** For $a_1, a_2, a_3 \in A$ we have

$$\begin{aligned}(a_1 \circ a_2) \circ a_3 &= (a_1 a_2 a) \circ a_3 \\ &= a_1 a_2 a a_3 a \\ &= a_1 a_2 a_3 a^2\end{aligned}$$

since $a$ commutes with $a_3$, while

$$\begin{aligned}a_1 \circ (a_2 \circ a_3) &= a_1 \circ (a_2 a_3 a) \\ &= a_1 a_2 a_3 a^2.\end{aligned}$$

This establishes that $\{A, \circ\}$ is a group.

---

# Problem 314.3 – Binomial coefficient

## Tony Forbes

Show that for large $n$,

$$\binom{1/2}{n} \sim \frac{(-1)^{n+1}}{2\sqrt{\pi}\, n^{3/2}}.$$

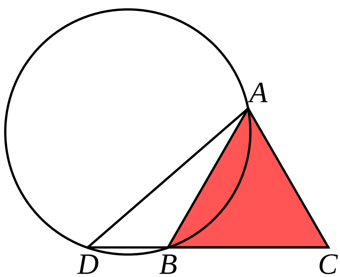
The expression on the left is interpreted as

$$\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\ldots\left(\frac{1}{2}-n+1\right)}{n!},$$

i.e. half choose $n$, the number of ways to select $n$ objects from half an object.

---

# Problem 314.4 – A triangle and a circle




A circle that passes through $A$ and $B$ of equilateral triangle $ABC$ meets $BC$ at $D$.

The length of $|AD|$ is 1.

What's the area of the circle?

---

# Solution 309.7 – Limit

Show that

$$\frac{x\sin y - y\sin x}{x\cos y - y\cos x} \;\to\; \tan(x - \arctan x) \qquad \text{as } y \to x.$$

## Richard Gould

Let $y = x + \delta$ then

$$\begin{aligned}
\lim_{y\to x}\frac{x\sin y - y\sin x}{x\cos y - y\cos x} &= \lim_{\delta\to 0}\frac{x\sin(x+\delta) - (x+\delta)\sin x}{x\cos(x+\delta) - (x+\delta)\cos x} \\
&= \lim_{\delta\to 0}\frac{x(\sin x\cos\delta + \cos x\sin\delta) - (x+\delta)\sin x}{x(\cos x\cos\delta - \sin x\sin\delta) - (x+\delta)\cos x} \\
&= \lim_{\delta\to 0}\frac{x\tan x\cos\delta + x\sin\delta - (x+\delta)\tan x}{x\cos\delta - x\tan x\sin\delta - (x+\delta)}.
\end{aligned}$$

Expanding Taylor series to $O(\delta^3)$ gives

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{x\tan x(1-\delta^2/2) + x(\delta - \delta^3/6) - (x+\delta)\tan x}{x(1-\delta^2/2) - x\tan x(\delta - \delta^3/6) - (x+\delta)} \\
&\qquad = \lim_{\delta\to 0}\frac{-\delta^2 x\tan x/3 + \delta(x - \tan x) - \delta^3 x/6}{-\delta^2 x/2 - \delta(x\tan x + 1) - \delta^3 x\tan x/6}.
\end{aligned}$$

Dividing through by $\delta$ and letting $\delta \to 0$ gives

$$\begin{aligned}
\lim_{y\to x}\frac{x\sin y - y\sin x}{x\cos y - y\cos x} &= \frac{x - \tan x}{-(x\tan x + 1)} \\
&= \frac{\tan x - x}{1 + x\tan x} = \tan(x - \arctan x),
\end{aligned}$$

as required.

*Note:* As is often the case, when given the solution, it can be helpful to work backwards as well as forwards to arrive at the solution. In this case an initial step of

$$\tan(x - \arctan x) = \frac{\tan x - x}{1 + x\tan x}$$

is particularly illuminating.

# Solution 311.6 – Square and dodecagon

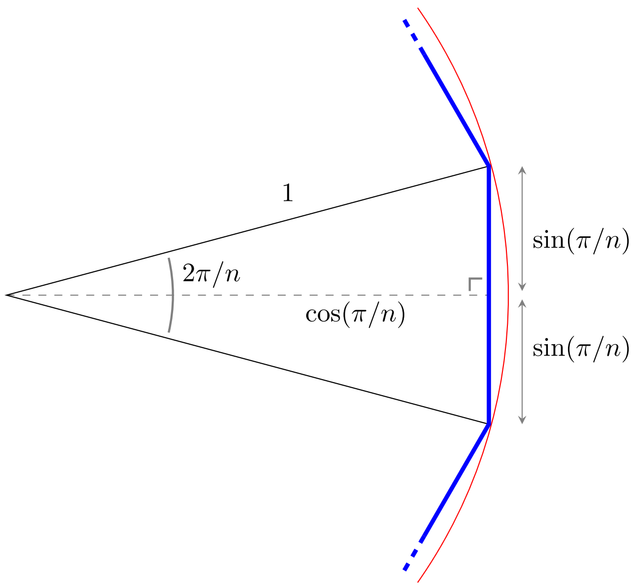
Show that the square and the dodecagon are the only regular polygons that have rational areas when inscribed in a unit circle.

## Dave Clark

We will use the following result, which is a form of Niven's Theorem.

**Lemma 1 (Niven)** *If* $\sin(q\pi)$ *is rational for some rational number* $q$ *then* $\sin(q\pi)$ *is* $-1$, $-\frac{1}{2}$, $0$, $\frac{1}{2}$, *or* $1$.

If a regular polygon with $n$ sides is inscribed in a unit circle, it is composed of $n$ segment triangles each with a central angle of $2\pi/n$.




The side length is $2\sin(\pi/n)$ and the apothem (distance from the centre of the circle to the side) is $\cos(\pi/n)$, so the area of each segment triangle is $\cos(\pi/n)\sin(\pi/n)$ and the area of the polygon is therefore

$$n\cos(\pi/n)\sin(\pi/n) \;=\; \frac{n}{2}\sin(2\pi/n).$$

This area is rational only when $\sin(2\pi/n)$ is rational, and Lemma 1 tells us that if $\sin(2\pi/n)$ is rational then it must be $-1$, $-\frac{1}{2}$, $0$, $\frac{1}{2}$, or $1$. In fact, since $2\pi/n$ is between 0 and $\pi$, and sine is positive over this range, $\sin(2\pi/n)$ must be $\frac{1}{2}$ or 1.

If $\sin(2\pi/n) = \frac{1}{2}$ then either $2\pi/n = \pi/6$, which gives $n = 12$, or $2\pi/n = 5\pi/6$ which doesn't give an integer $n$. If $\sin(2\pi/n) = 1$ then $2\pi/n = \pi/2$, which gives $n = 4$. This means the square ($n = 4$) and the dodecagon ($n = 12$) are the only such polygons with rational areas.

**Proof of Lemma 1**

Let $r = \frac{1}{2} - q$, and note that $\sin(q\pi) = \cos(r\pi)$, since $\sin(\theta) \equiv \cos(\frac{\pi}{2} - \theta)$.

If $q$ is rational then so is $r$, and we can write $r = a/b$ as a fraction in its lowest terms, with integers $a$ and $b$ coprime (no common factors) and $b$ positive. For $k = 0, 1, 2, \ldots$ consider the sequence of values $2\cos\left(2^k r\pi\right) = 2\cos\left(2^k a\pi/b\right)$. Since cosine is periodic with period $2\pi$, $2\cos\left(2^k a\pi/b\right)$ can take at most $2b$ distinct values, so this sequence must repeat after at most $2b$ terms.

If $\sin(q\pi)$ is rational then so is $\cos(r\pi)$, and we can write $2\cos(r\pi) = c/d$ as a fraction in its lowest terms, with integers $c$ and $d$ coprime (no common factors) and $d$ positive. Recall that

$$\cos(2\theta) \;\equiv\; 2\cos^2(\theta) - 1,$$

so

$$2\cos(2r\pi) \;=\; 2\left(2\cos^2(r\pi) - 1\right) \;=\; (2\cos(r\pi))^2 - 2 \;=\; \frac{c^2}{d^2} - 2 \;=\; \frac{c^2 - 2d^2}{d^2}.$$

This is also a fraction is its lowest terms, because if $p \mid d^2$ for some prime $p$ then $p \mid d$, and if we also have $p \mid (c^2 - 2d^2)$ then we must have $p \mid c^2$ and so $p \mid c$, but this contradicts $c$ and $d$ being coprime (no common factors) so $d^2$ and $(c^2 - 2d^2)$ must also be coprime. We can repeat this process using

$$2\cos\left(2^k r\pi\right) \;=\; \left(2\cos\left(2^{k-1} r\pi\right)\right)^2 - 2$$

to get an expression for $2\cos(2^2 r\pi)$ in lowest terms with $d^4$ as the denominator, and so on, with the denominator for $2\cos\left(2^k r\pi\right)$ expressed in lowest terms being $d^{2^k}$. But if $d > 1$ then $d^{2^k}$ is strictly increasing, which means the sequence can never repeat, which gives us a contradiction because we know the sequence can take at most $2b$ distinct values. So we must have $d = 1$.

If $d = 1$ then $\sin(q\pi) = \cos(r\pi) = c/(2d) = c/2$ for some integer $c$, but $-1 \le \sin(q\pi) \le 1$ so $\sin(q\pi)$ can only be $-1$, $-\frac{1}{2}$, 0, $\frac{1}{2}$, or 1. $\square$

# Problem 314.5 – Pascal and Bernoulli

**Tony Forbes**

Let $n$ be a positive integer and let $A$ be the Pascal triangle matrix of size $n+2$, where

$$A_{r,c} = \binom{r}{c},$$

with the usual convention that $\binom{r}{c} = 0$ whenever $c > r$. Form an $n \times n$ non-singular matrix $B$ by subtracting 1 from the positive entries of $A$ and discarding any rows and columns that consist entirely of zeros. For example, when $n = 13$, you should end up with

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 9 & 9 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 14 & 19 & 14 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 20 & 34 & 34 & 20 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 27 & 55 & 69 & 55 & 27 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 35 & 83 & 125 & 125 & 83 & 35 & 8 & 0 & 0 & 0 & 0 & 0 \\ 9 & 44 & 119 & 209 & 251 & 209 & 119 & 44 & 9 & 0 & 0 & 0 & 0 \\ 10 & 54 & 164 & 329 & 461 & 461 & 329 & 164 & 54 & 10 & 0 & 0 & 0 \\ 11 & 65 & 219 & 494 & 791 & 923 & 791 & 494 & 219 & 65 & 11 & 0 & 0 \\ 12 & 77 & 285 & 714 & 1286 & 1715 & 1715 & 1286 & 714 & 285 & 77 & 12 & 0 \\ 13 & 90 & 363 & 1000 & 2001 & 3002 & 3431 & 3002 & 2001 & 1000 & 363 & 90 & 13 \end{bmatrix}.$$

Now compute $B^{-1}$, the inverse of $B$. This is shown on the next page for our example with $n = 13$, and we see exhibited many striking features—for instance, the fractions $1/r$ appearing along the diagonal. But the most interesting (to me) is the appearance of the Bernoulli numbers, [`https://mathworld.wolfram.com/BernoulliNumber.html`],

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $B_k$ | 1 | $-\frac{1}{2}$ | $\frac{1}{6}$ | 0 | $-\frac{1}{30}$ | 0 | $\frac{1}{42}$ | 0 | $-\frac{1}{30}$ | 0 | $\frac{5}{66}$ | 0 | $-\frac{691}{2730}$ |

as the row sums of the matrix as well as in column 1, rows 1, 5, 7, 9, 11, 13. (The last one is a bit of a give-away. Anytime you see the integer 691 you can be sure that Bernoulli numbers are not far away.)

So that's the problem. *Where do these Bernoulli numbers come from?*

Thanks to Robin Whitty for suggesting this construction at a London South Bank University Maths Study Group meeting, `https://www.theoremoftheday.org/MathsStudyGroup/index.html`. A convenient way of computing $B_k$ is to use the formula

$$B_0 \;=\; 1, \qquad B_k \;=\; -\frac{1}{k+1}\sum_{j=0}^{k-1}\binom{k+1}{j}B_j, \quad k \geq 1.$$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{3} & -\frac{5}{6} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{4} & \frac{3}{4} & -\frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{30} & -\frac{1}{3} & \frac{5}{6} & -\frac{7}{10} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{12} & -\frac{1}{12} & -\frac{5}{12} & \frac{11}{12} & -\frac{2}{3} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{42} & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{2} & 1 & -\frac{9}{14} & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{12} & \frac{1}{12} & \frac{7}{24} & -\frac{7}{24} & -\frac{7}{12} & \frac{13}{12} & -\frac{5}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{30} & -\frac{2}{9} & \frac{2}{9} & \frac{7}{15} & -\frac{7}{15} & -\frac{2}{3} & \frac{7}{6} & -\frac{11}{18} & \frac{1}{9} & 0 & 0 & 0 & 0 \\
\frac{3}{20} & -\frac{3}{20} & -\frac{1}{2} & \frac{1}{2} & \frac{7}{10} & -\frac{7}{10} & -\frac{3}{4} & \frac{5}{4} & -\frac{3}{5} & \frac{1}{10} & 0 & 0 & 0 \\
\frac{5}{66} & \frac{1}{2} & -\frac{1}{2} & -1 & 1 & 1 & -1 & -\frac{5}{6} & \frac{4}{3} & -\frac{13}{22} & \frac{1}{11} & 0 & 0 \\
-\frac{5}{12} & \frac{5}{12} & \frac{11}{8} & -\frac{11}{8} & -\frac{11}{6} & \frac{11}{6} & \frac{11}{8} & -\frac{11}{8} & -\frac{11}{12} & \frac{17}{12} & -\frac{7}{12} & \frac{1}{12} & 0 \\
-\frac{691}{2730} & -\frac{5}{3} & \frac{5}{3} & \frac{33}{10} & -\frac{33}{10} & -\frac{22}{7} & \frac{22}{7} & \frac{11}{6} & -\frac{11}{6} & -1 & \frac{3}{2} & -\frac{15}{26} & \frac{1}{13}
\end{bmatrix}.$$

# Buffalo

## Jeremy Humphries

I acknowledge that M500 is a mathematics magazine, but mathematicians will sometimes go in for a bit of wordplay. Martin Gardner dealt mainly with recreational mathematics, but occasionally he would devote a *Scientific American* column to wordplay, sometimes, but not always, involving his legendary acquaintance Dr I. J. Matrix. Therefore, we can proceed.

Buffalo buffalo Buffalo buffalo buffalo buffalo Buffalo buffalo. Or so I've been told. It may be true, or it may be casting aspersions on certain even-toed ungulates of the Family Bovidae. But either way it makes grammatical sense. It relies on the fact that there are multiple meanings for the word 'B/buffalo'.

First, 'Buffalo' is, among several other places of the same name, the city in New York State where Ruby Keeler and Clarence Nordstrom shuffled off to in 42nd Street.

`https://www.youtube.com/watch?v=gqYkXQZAhTo`

If you are reading this on screen as a pdf, then you can click the link to see the routine. And of course you can copy the link to your browser by hand from the printed magazine, if you like. Consequently 'Buffalo' can be used adjectivally to describe residents or natives of that place.

Then there is the name 'buffalo', commonly used for the wild cattle more formally called American bison, and which can have the plural form 'buffalo'.

Then there is the verb 'to buffalo', meaning to bully, intimidate, confuse and so on.

So – Buffalo buffalo Buffalo buffalo buffalo buffalo Buffalo buffalo. What does it mean? It means that bison from Buffalo that are bullied by bison from Buffalo bully bison from Buffalo themselves.

Now – there's nothing special about eight B/buffalos in a sentence. It is the case that a sentence consisting of any number of B/buffalos, from one upwards, can make sense. Also you don't actually need the Buffalo location to do this stuff. The animal name together with the verb will suffice. But when you do add the location you get longer strings for essentially the same meaning, so it looks more impressive.

And a sentence consisting of a particular number of B/buffalos can make sense in more than one way. For instance, here's another eight-word one:

Buffalo buffalo buffalo Buffalo buffalo Buffalo buffalo buffalo.

That means that bison from Buffalo bully bison from Buffalo that are already bullied by bison from Buffalo.

So, I thought I would write a limerick. It's not too difficult to get the basic words, but it's a bit trickier, if you are using the 'Buffalo' location, to put in some necessary capitals, in addition to the ones starting the lines. And either way it's a bit tricky to work out the meaning of what you are saying. One way to devise the limerick, incorporating the 'Buffalo' location, is like this:

Buffalo buffalo Buffalo
Buffalo Buffalo buffalo
Buffalo buffalo
Buffalo Buffalo
Buffalo buffalo buffalo.

In non-verse form, without the line-leading capitals, that would be:

Buffalo buffalo Buffalo buffalo Buffalo buffalo buffalo buffalo buffalo Buffalo buffalo buffalo buffalo.

It means that bison from Buffalo, that are bullied by bison from Buffalo that bison from Buffalo bully, themselves bully bison from Buffalo that bison from nowhere in particular bully.

So, there we go. Since a 'B/buffalo' string of any length can make sense, presumably there is an iterative technique or formula that will produce and explain such strings. We leave the production of such a thing as an exercise for the reader.

## Problem 314.6 – Square

We won't bother with a diagram for this high-school geometry problem, and we insist that you don't include one in your answer.

There is a square with vertices $A$, $B$, $C$, $D$, $AB \| CD$, $BC \| AD$, point $E$ bisects $AD$, and $F$ is the point where a line through $C$ meets $BE$ at 90 degrees. Show that $|AB| = |DF|$.

## Problem 314.7 – Exponential function equation

Find all solutions of

$$e^{\pi x} = x^2.$$

# Sylow

## Jeremy Humphries

When we were doing Group Theory back in the early days of the OU, we were introduced to Sylow's theorems, named for the Norwegian mathematician Ludvig Sylow. We didn't have Google to enable us to listen to the pronunciation of his name, so we had to speculate. My friend Steve Ainley wrote a limerick about the situation.

Actually we were pretty sure that it was pronounced approximately as Mrs Sylow's name is pronounced in the limerick, and a Google enquiry today backs that up. But something else Google tells us today is that there was no Mrs Sylow, as Ludvig never married. And of course Sylow (1832–1918) predates the Li-lo (UK trademark registered 1944), and without doing any research we were pretty confident that was the case. Still, don't let facts get in the way of a good poem. Here we go:

A group theoretician named Sylow
Was afloat with his wife on a Lylow.
  In the heat Mrs Sylow
  Decided to pylow –
They didn't come back for a whylow.

A good source of names not pronounced as they look is the British aristocracy. The Earldoms of Wemyss and of March are technically distinct, but have been held by the same person since 1826, each holder being conventionally styled the Earl of Wemyss and March. Here, for poetic necessity, I have reversed the order.

The Earldom of March and of Wemyss
Is pronounced not the way that it semyss.
  To think that it's Wemyss
  Would be a false premyss,
And not what authority demyss.

---

# Problem 314.8 – Power series coefficients

## Tony Forbes

Show that if $\sqrt{1-x}$ is expanded as a power series $\sum_{k=0}^{\infty} a^k x^k$, then the coefficients $a_k$ are integer multiples of powers of 1/2.

# Problem 314.9 – Bernoulli sum

## Tony Forbes

Show that for any non-negative integer $r$,

$$\sum_{j=0}^{r} 2^{2j}(2^{2j}-1)\binom{2r}{2j}B_{2j} \;=\; 2r,$$

where, $B_k$ is the $k$-th Bernoulli number (see Problem 314.5) on page 20.

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# Problem 314.10 – Limit

Show that $n/(n!)^{1/n} \to e$ as $n \to \infty$.

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# Solution 311.7 – Grand slam

## Tony Forbes

> As any bridge player knows, it is possible to make a grand slam with a combined holding of only 5 high-card points. Devise a deal where it is possible to make a grand slam in a suit with no card higher than a 10.

I showed the problem to a few bridge players. “No way!” was the usual response, “Blah blah ace of trumps blah blah.” But of course they were assuming competent defence.

Dealer West.

North: ♠xxxx ♡– ♢– ♣xxxxxxxxx

West: ♠AQ ♡AKQJx ♢AKQJ ♣QJ East: ♠KJ ♡xxxx ♢xxxxx ♣AK

South: ♠xxxxx ♡xxxx ♢xxxx ♣–

The bidding: 4NT, pass, 5♢, pass, 6NT, pass, pass, 7♠, dbl, all pass.

The play: West ♢, ♠, ♢, ♢; North ♣, ♠J (!), ♠ (a cunning underruff laid on the table to obscure the club convinces West that a spade was led), ♠Q (!); West ♢, ♠, ♢, ♢; North ♣, ♠K (!), ♠, ♠A (!); West ♣, ♣, ♣, ♠. At this point the revokes at trick 2 (not to mention trick 4) are exposed. South ♢, ♢, ♠, ♢; North ♣, ♣, ♠, ♣; South ♢, ♢, ♠, ♢; North & South get the rest, 11 tricks plus 2 transferred for the revoke, 7♠$^{\times}$ made.

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04.00

**Front cover** Real and imaginary parts of $E(x+iy)$; see page 10.