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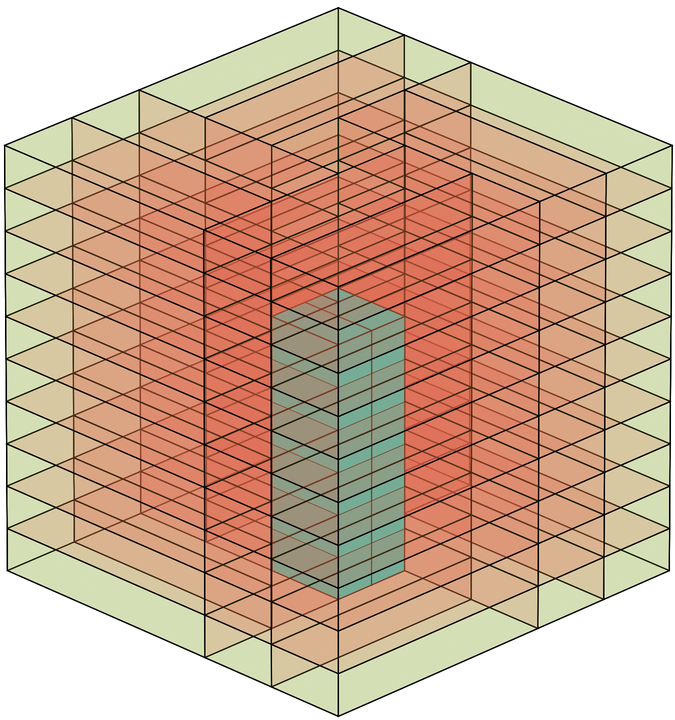
ISSN 1350-8539

M500 315



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Bricks

Kira Bhana and Tony Forbes

Given a supply of $1 \times 2 \times 6$ bricks, try packing 42 of them into an $8 \times 8 \times 8$ cube, a task which should not give you too much trouble. Or maybe you can pack 28 of these things into a $7 \times 7 \times 7$ cube—or prove that it cannot be done. More generally, what we are really after is the answer to the question:

> *What is the maximum number of* $1 \times 2 \times 6$ *bricks that you can pack into an* $n \times n \times n$ *cube,* $n = 4, 5, 6, \ldots$*?*

In the table we give some upper and lower bounds. Observe that when n is even the maximum packing number is determined exactly—the upper and lower bounds are the same.

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\lfloor n^3/12 \rfloor$	5	10	18	28	42	60	83	110	144	183	228	281	341	409	486	571
unused	4	5	0	7	8	9	4	11	0	1	8	3	4	5	0	7
upper bound	0	2	18	28	42	60	82	110	144	183	228	280	340	409	486	571
unused	64	101	0	7	8	9	16	11	0	1	8	15	16	5	0	7
lower bound	0	2	18	27	42	57	82	108	144	180	228	276	340	405	486	567
unused	64	101	0	19	8	45	16	35	0	37	8	63	16	53	0	55

The 'unused' lines indicate how many holes are left unfilled by the possibly hypothetical packing. The lower bounds are obtained as follows.

$\boldsymbol{n = 6k}$ Left for the reader.

$\boldsymbol{n = 6k + 2}$ Lay $3k^2 + 2k$ bricks flat on an $n \times n$ square to leave a 2×2 hole in a corner. Pile n of these structures vertically and drop $2k$ bricks into the $2 \times 2 \times n$ hole:

$$n(3k^2 + 2k) + 2k \;=\; \frac{n^3 - 8}{12} \;=\; \left\lfloor \frac{n^3}{12} \right\rfloor \quad \text{bricks,}$$

best possible.

$\boldsymbol{n = 4}$ Left for the reader.

$\boldsymbol{n = 6k + 4 \geq 10}$ Lay $8k$ bricks flat to make a wall 4 units thick that encloses an $(n-8) \times (n-8)$ square region. If $n > 10$, apply the first part of the $6(k-1)+2$ construction to the region. Thus an $n \times n$ square is covered except for a 2×2 hole. Pile n of these structures vertically and drop $2k$

bricks into the $2 \times 2 \times n$ hole:

$$\frac{n(n^2-4)}{12} + 2k \;=\; \frac{n^3-16}{12} \;=\; \left\lfloor \frac{n^3}{12} \right\rfloor - 1 \;\; \text{bricks},$$

leaving a volume of 16 unoccupied.

Since 16 exceeds the volume of a brick by 4, it is tempting to suggest that perhaps there is some smart arrangement which accommodates one extra. However, no such packing exists; $\lfloor n^3/12 \rfloor - 1$ is best possible, as we shall prove in Theorem 1, below.

$\boldsymbol{n = 5}$ See Problem 315.1 on page 11.

Odd $\boldsymbol{n \geq 7}$ Take the arrangement for the $(n-1)\times(n-1)\times(n-1)$ cube and clad three mutually orthogonal faces with as many bricks as possible. Thus, for example, $18 + 3 \cdot 3 = 27$ for $n = 7$, and $42 + 3 \cdot 5 = 57$ for $n = 9$.

In some cases we can improve on this construction.

$\boldsymbol{n = 11}$ Use 20 bricks to build a wall 2 units high and 5 units thick that encloses a $1 \times 1 \times 2$ hole. Pile five of these structures vertically and lay eight bricks on top: $5 \cdot 20 + 8 = 108$ bricks.

$\boldsymbol{n = 15}$ See Theorem 2, below.

$\boldsymbol{n = 17}$ Use 40 bricks to build a wall 2 units high and 5 units thick that encloses a $7 \times 7 \times 2$ hole into which place eight more bricks. Pile eight of these structures vertically and lay 21 bricks on top: $8 \cdot 48 + 21 = 405$ bricks.

The reader is invited to reduce the gaps between the lower and upper bounds given for odd n in the table.

Also, we would be very interested in the smallest k for which you can put $18k^3 + 9k^2 + 1$ bricks in a $(6k+1) \times (6k+1) \times (6k+1)$ cube. To show that this is a sensible request, put $k = 10$, say. The construction described above involving the cladding of three faces of a $60 \times 60 \times 60$ cube uses 18900 bricks and leaves an unused volume of 181, quite a lot more than sufficient for one extra brick.

If it helps, there is a child's toy marketed under the name Tumbling Tower that consists of fifty-four $1 \times 2 \times 6$ bricks neatly fashioned out of wood and packaged in a $6 \times 6 \times 18$ cardboard and plastic box. Although we suspect its intended purpose is not to investigate the filling of cubical bins with small cuboids, we did actually find it useful.

Finally, we have the following results, which show that the trivial upper bound cannot be attained for $n \equiv 4 \pmod 6$ and $n \equiv 3 \pmod{12}$.

Theorem 1 *Let k be a positive integer and let $n = 6k + 4$. The maximum number of $1 \times 2 \times 6$ bricks that you can pack into an $n \times n \times n$ cube is $\lfloor n^3/12 \rfloor - 1$.*

Proof We have already shown how to pack the $n \times n \times n$ cube with $\lfloor n^3/12 \rfloor - 1$ bricks. So we only need to prove that $\lfloor n^3/12 \rfloor$ is impossible.

We think it is safe to assume that a brick in the packing must be orientated so that each of its six faces lies on one of the $3(n + 1)$ grid-planes that partition the cube into n^3 subcubes.

The proof involves polynomials in three complex variables. It might be helpful to follow the argument with $n = 10$.

Let the cube occupy $[0, n] \times [0, n] \times [0, n]$ in Euclidean 3-dimensional space, and suppose it is packed with $\lfloor n^3/12 \rfloor$ bricks to leave four of its subcubes unoccupied. The location of a compact set of points S is the $(a, b, c) \in S$ that minimizes each of the coordinates a, b and c.

Associate a subcube located at (a, b, c) with the polynomial $x^a y^b z^c$, where x, y and z are complex variables. Then the sum of the n^3 polynomials associated with the n^3 subcubes is

$$C(x, y, z) \;=\; \sum_{a=0}^{n-1} \sum_{b=0}^{n-1} \sum_{c=0}^{n-1} x^a y^b z^c.$$

A brick located at (a, b, c) is represented by a polynomial

$$x^a y^b z^c \, B_j(x, y, z),$$

where $B_j(x, y, z)$, $j \in \{1, 2, \ldots, 6\}$ is one of

$$\begin{aligned}
B_1(x, y, z) &= (1 + x + x^2 + x^3 + x^4 + x^5)(1 + y),\\
B_2(x, y, z) &= (1 + x + x^2 + x^3 + x^4 + x^5)(1 + z),\\
B_3(x, y, z) &= (1 + y + y^2 + y^3 + y^4 + y^5)(1 + x),\\
B_4(x, y, z) &= (1 + y + y^2 + y^3 + y^4 + y^5)(1 + z),\\
B_5(x, y, z) &= (1 + z + z^2 + z^3 + z^4 + z^5)(1 + x),\\
B_6(x, y, z) &= (1 + z + z^2 + z^3 + z^4 + z^5)(1 + y),
\end{aligned}$$

depending on the orientation of the brick in the cube. For example, $B_6(x, y, z)$ corresponds to a brick standing upright with its side of length 2 in the direction of the y-axis. The polynomial represents the 12 subcubes occupied by the brick, on the assumption that it is located at $(0, 0, 0)$. Shifting the brick to (a, b, c) corresponds to multiplying $B_6(x, y, z)$ by $x^a y^b z^c$.

The sum of the polynomials associated with the bricks in the packing is

$$B(x,y,z) \;=\; \sum_{j=1}^{6} P_j(x,y,z) B_j(x,y,z)$$

for some polynomials $P_1(x,y,z)$, $P_2(x,y,z)$, ..., $P_6(x,y,z)$. Thus $B(x,y,z)$ is the sum of $\lfloor n^3/12 \rfloor$ expressions of the form $x^a y^b z^c B_j(x,y,z)$ for various (a,b,c) and various $j \in \{1,2,\ldots,6\}$.

But there are also four points corresponding to the unused subcubes. Assuming they occur at distinct locations

$$(a_1,b_1,c_1), \quad (a_2,b_2,c_2), \quad (a_3,b_3,c_3), \quad (a_4,b_4,c_4),$$

the sum of their associated polynomials is

$$U(x,y,z) \;=\; \sum_{h=1}^{4} x^{a_h} y^{b_h} z^{c_h}.$$

Note that $U(x,y,z)$ depends on the parameters a_h, b_h, c_h.

For the assumed packing, there must exist polynomials $P_1(x,y,z)$, $P_2(x,y,z)$, ..., $P_6(x,y,z)$ and point coordinates $a_h, b_h, c_h \in \{0,1,\ldots,n-1\}$, $h = 1, 2, 3, 4$, such that

$$C(x,y,z) \;=\; B(x,y,z) + U(x,y,z) \quad \text{for all complex } x,y,z. \tag{1}$$

Now for the clever part. Put

$$x \;=\; y \;=\; z \;=\; \rho \;=\; \frac{1}{2} + \frac{\sqrt{3}\,i}{2},$$

a primitive 6th root of 1. Then, observing that $1+\rho+\rho^2+\rho^3+\rho^4+\rho^5 = 0$, we have

$$\begin{aligned} C(\rho,\rho,\rho) \;&=\; \sum_{a=0}^{n-1}\sum_{b=0}^{n-1}\sum_{c=0}^{n-1} \rho^a\rho^b\rho^c \;=\; \left(1+\rho+\rho^2+\cdots+\rho^{n-1}\right)^3 \\ &=\; \left(1+\rho+\rho^2+\rho^3\right)^3 \\ &=\; -\,3\sqrt{3}\,i \end{aligned}$$

and

$$B(\rho,\rho,\rho) \;=\; 0 \quad \text{for every choice of the polynomials } P_j.$$

We have annihilated the bricks—so we do not have to worry about where they are. To deal with the unused part, we see that

$$U(\rho,\rho,\rho) \;=\; \sum_{h=1}^{4} \rho^{a_h}\rho^{b_h}\rho^{c_h}$$

is the sum of four powers of ρ and therefore cannot have absolute value greater than 4. However $|C(\rho,\rho,\rho)| = 3\sqrt{3} > 5$. Hence for any choice of (a_h, b_h, c_h), $h = 1, 2, 3, 4$, we have $U(\rho,\rho,\rho) \neq C(\rho,\rho,\rho)$ and, recalling that $B(\rho,\rho,\rho) = 0$,

$$C(\rho,\rho,\rho) \;\neq\; B(\rho,\rho,\rho) + U(\rho,\rho,\rho),$$

contradicting (1). □

Theorem 2 *Let k be a positive integer and let $n = 12k + 3$. You cannot pack $\lfloor n^3/12 \rfloor$ $1 \times 2 \times 6$ bricks into an $n \times n \times n$ cube.*

Proof Assume the cube occupies $[0,n] \times [0,n] \times [0,n]$ and it is packed with $\lfloor n^3/12 \rfloor$ bricks to leave three of its subcubes unoccupied.

We employ the same method as in Theorem 1. With $C(x,y,z)$, $B(x,y,z)$ and $U(x,y,z)$ defined as before,

$$\begin{aligned}
C(\rho,\rho,\rho) &= \sum_{a=0}^{n-1}\sum_{b=0}^{n-1}\sum_{c=0}^{n-1} \rho^a\rho^b\rho^c \\
&= \left(1 + \rho + \rho^2 + \cdots + \rho^{12k+2}\right)^3 \\
&= \left(1 + \rho + \rho^2\right)^3 \;=\; -8, \\
B(\rho,\rho,\rho) &= 0, \\
U(\rho,\rho,\rho) &= \sum_{h=1}^{3} \rho^{a_h}\rho^{b_h}\rho^{c_h},
\end{aligned}$$

where $\rho = 1/2 + \sqrt{3}\,i/2$ and each of a_h, b_h, c_h, $h = 1, 2, 3$, can take any value in $\{0, 1, \ldots, n-1\}$. But for any choice of these parameters,

$$|U(\rho,\rho,\rho)| \;\le\; 3 \;<\; 8 \;=\; |C(\rho,\rho,\rho)|.$$

Hence

$$C(\rho,\rho,\rho) \;\neq\; B(\rho,\rho,\rho) + U(\rho,\rho,\rho)$$

and therefore the packing does not exist. □

Solution 312.6 – 53 bricks

> You cannot fit 54 $1 \times 1 \times 4$ bricks into a $6 \times 6 \times 6$ box. If you can devise a simple proof, we would like to see it. What about 53 bricks?

Tony Forbes

We show that you cannot pack 53 bricks into a $6 \times 6 \times 6$ cube. There is actually an easy way to prove this by partitioning the cube into 27 coloured $2 \times 2 \times 2$ subcubes. However, in my opinion the somewhat more complicated proof I offer, which is similar to that of Theorem 1 on page 3, is far too interesting to be ignored. We might as well deal with the general case where the cube has side congruent to 2 modulo 4.

Theorem 1 *You cannot pack* $(4m + 2)^3/4 - 1$ *bricks of size* $1 \times 1 \times 4$ *into a* $(4m + 2) \times (4m + 2) \times (4m + 2)$ *cube.*

Proof Position the cube to occupy $[0, 4m + 2] \times [0, 4m + 2] \times [0, 4m + 2]$ in Euclidean 3-dimensional space. Suppose $(4m + 2)^3/4 - 1$ bricks are packed in the cube to leave 4 units unoccupied.

A point (a, b, c) is represented by the monomial expression $x^a y^b z^c$ in variables x, y, z, The whole cube is represented by the polynomial

$$C(x, y, z) = \sum_{a=0}^{4m+1} \sum_{b=0}^{4m+1} \sum_{c=0}^{4m+1} x^a y^b z^c.$$

A brick polynomial is one of

$$\begin{aligned} B_1(x, y, z) &= 1 + x + x^2 + x^3, \\ B_2(x, y, z) &= 1 + y + y^2 + y^3, \\ B_3(x, y, z) &= 1 + z + z^2 + z^3, \end{aligned}$$

depending on its orientation. The bricks in the packing are represented by

$$B(x, y, z) = \sum_{r=1}^{3} P_r(x, y, z) B_r(x, y, z),$$

where $P_1(x, y, z)$, $P_2(x, y, z)$ and $P_3(x, y, z)$ are polynomials. The polynomial representing the holes at (a_h, b_h, c_h), $h = 1, 2, 3, 4$, is

$$U(x, y, z) = \sum_{h=1}^{4} x^{a_h} y^{b_h} z^{c_h}.$$

For the supposed packing, there must exist polynomials $P_r(x, y, z)$, $r \in \{1, 2, 3\}$, and point coordinates $a_h, b_h, c_h \in \{0, 1, \ldots, 4m+1\}$, $h = 1, 2, 3, 4$, such that

$$C(x, y, z) = B(x, y, z) + U(x, y, z) \quad \text{for all } x, y, z \in \mathbb{H}, \tag{1}$$

where $\mathbb{H}$ is the ring of quaternions.

We represent a quaternion by an expression of the form $\alpha + \beta i + \gamma j + \delta k$, where α, β, γ, δ are real numbers and i, j and k are the basis elements. Addition is performed by doing each component separately:

$$\begin{aligned} \alpha_1 + \beta_1 i + \gamma_1 j + \delta_1 k &+ \alpha_2 + \beta_2 i + \gamma_2 j + \delta_2 k \\ &= (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2)i + (\gamma_1 + \gamma_2)j + (\delta_1 + \delta_2)k. \end{aligned}$$

Multiplication is done in the usual way,

$$\begin{aligned} (\alpha_1 + \beta_1 i + \gamma_1 j + \delta_1 k)&(\alpha_2 + \beta_2 i + \gamma_2 j + \delta_2 k) \\ &= \alpha_1\alpha_2 + \alpha_1\beta_2 i + \alpha_1\gamma_2 j + \alpha_1\delta_2 k \\ &\quad + \beta_1\alpha_2 i + \beta_1\beta_2 ii + \beta_1\gamma_2 ij + \beta_1\delta_2 ik \\ &\quad + \gamma_1\alpha_2 j + \gamma_1\beta_2 ji + \gamma_1\gamma_2 jj + \gamma_1\delta_2 jk \\ &\quad + \delta_1\alpha_2 k + \delta_1\beta_2 ki + \delta_1\gamma_2 kj + \delta_1\delta_2 kk, \end{aligned}$$

which is then simplified by Table 1. Using Table 1, we see that $B_1(i, j, k) = B_2(i, j, k) = B_3(i, j, k) = 0$ and therefore

$$B(i, j, k) = 0. \tag{2}$$

Also

$$\begin{aligned} C(i, j, k) &= \sum_{a=0}^{4m+1} \sum_{b=0}^{4m+1} \sum_{c=0}^{4m+1} i^a j^b k^c \\ &= (1+i)(1+j)(1+k) = (1+i+j+k)(1+k) \\ &= 1 + i + j + k + k - j + i - 1 = 2i + 2k. \end{aligned} \tag{3}$$

Table 1: Quaternion multiplication

$\times$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Table 2: Hole coordinates

a mod 2	0	0	0	0	1	1	1	1
b mod 2	0	0	1	1	0	0	1	1
c mod 2	0	1	0	1	0	1	0	1
$i^a j^b k^c$	± 1	$\pm k$	$\pm j$	$\pm i$	$\pm i$	$\pm j$	$\pm k$	± 1

Now consider the holes. We have

$$U(i,j,k) \;=\; \sum_{h=1}^{4} i^{a_h} j^{b_h} k^{c_h}.$$

Apart from a plus or minus sign, $i^a j^b k^c$ depends only on the parities of the coordinates (a, b, c) as indicated in Table 2, which clearly shows that each term of $U(i, j, k)$ must be one of the eight elements of the set

$$R = \{1, -1, i, -i, j, -j, k, -k\}.$$

For (1) to be satisfied, we must have

$$U(i,j,k) \;=\; 2i + 2k. \tag{4}$$

by (2) and (3). The only way to make this quantity from four elements of R, is $i + i + k + k$, and we may assume without loss of generality that

$$i^{a_1} j^{b_1} k^{c_1} \;=\; i^{a_2} j^{b_2} k^{c_2} \;=\; i, \qquad i^{a_3} j^{b_3} k^{c_3} \;=\; i^{a_4} j^{b_4} k^{c_4} \;=\; k.$$

Therefore

$$\begin{aligned}(a_1, b_1, c_1),\ (a_2, b_2, c_2) \;&\equiv\; (0,1,1) \text{ or } (1,0,0) \pmod 2,\\ (a_3, b_3, c_3),\ (a_4, b_4, c_4) \;&\equiv\; (0,0,1) \text{ or } (1,1,0) \pmod 2.\end{aligned}$$

Now remove the bricks, reflect them in the plane $x = y$ and return them to the cube. The hole that previously occupied position (a_h, b_h, c_h) is now at (b_h, a_h, c_h), $h = 1$, 2, 3, 4. But then

$$\begin{aligned}(b_1, a_1, c_1),\ (b_2, a_2, c_2) \;&\equiv\; (0,1,0) \text{ or } (1,0,1) \pmod 2,\\ (b_3, a_3, c_3),\ (b_4, a_4, c_4) \;&\equiv\; (0,0,1) \text{ or } (1,1,0) \pmod 2,\end{aligned}$$

and it follows that

$$i^{b_1} j^{a_1} k^{c_1} = \pm j, \quad i^{b_2} j^{a_2} k^{c_2} = \pm j, \quad i^{b_3} j^{a_3} k^{c_3} = \pm k, \quad i^{b_4} j^{a_4} k^{c_1} = \pm k,$$

which contradicts (4). □

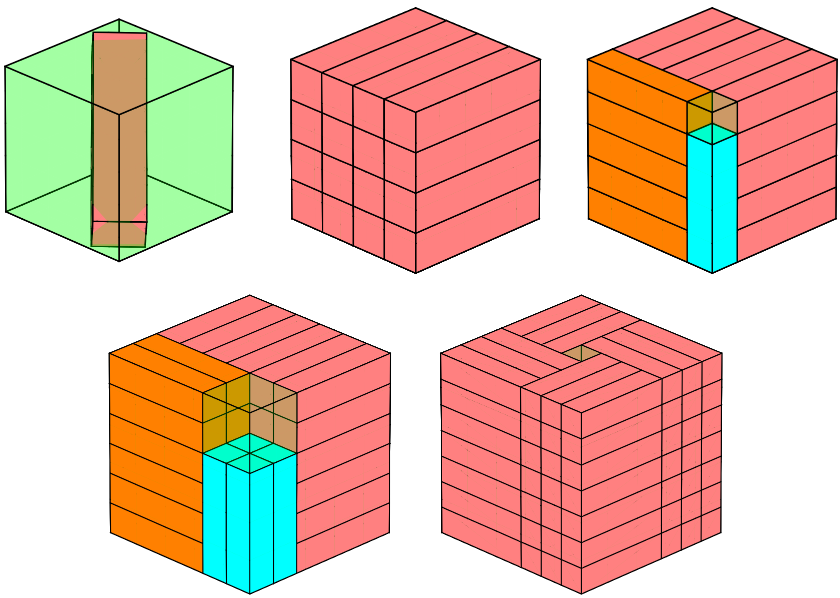
What makes this proof work seems to be the non-symmetry of (3) under permutations of i, j and k. For instance, putting $x = j$ and $y = i$ gives

$$C(j, i, k) \;=\; (1 + j)(1 + i)(1 + k) \;=\; (1 + i + j - k)(1 + k) \;=\; 2 + 2i,$$

and again we can get a contradiction by a suitable transformation of the bricks.

As can be seen from the results summarized in the table below, for the general problem of finding optimal packings of $1 \times 1 \times 4$ bricks in n-sided cubes, the only case where the trivial upper bound is not attained is $n \equiv 2 \pmod 4$.

Cube side, n	Maximum number of bricks	Holes
$n = 0, 1, 2, 3$	0	n^3
$n \equiv 0 \pmod 4$, $n \geq 4$	$n^3/4$	0
$n \equiv 1 \pmod 4$, $n \geq 5$	$\lfloor n^3/4 \rfloor$	1
$n \equiv 2 \pmod 4$	$n^3/4 - 2$	8
$n \equiv 3 \pmod 4$, $n \geq 7$	$\lfloor n^3/4 \rfloor$	3



Sums involving tangents

Tommy Moorhouse

In M500 **312** David Sixsmith poses a question about the convergence of certain series involving tangents, such as

$$t_p = \sum_{n=1}^{\infty} \frac{\tan n}{n^p}$$

where p is an integer. My intention here is to consider in a 'hand waving' way the convergence of such sums, using elementary results from number theory and analysis. It strikes me that a central obstruction to convergence is the fact that tan is periodic on the real axis and diverges at $n = (2k+1)\pi/2$ for integer k. Let's consider this first.

Dirichlet's approximation theorem Dirichlet proved that any irrational number θ can be approximated arbitrarily closely by rational numbers. There is a pleasing proof of this theorem in [Apostol] and the conclusion is that given $\epsilon > 0$ there are integers h and k (which can be chosen to be relatively prime) such that $|h - k\theta| < \epsilon$. This means that the fraction h/k can be made as close as we like to θ, and since ϵ can be made arbitrarily small we see from the case $\theta = \pi/2$ that $\tan(h)$ can be made arbitrarily large by a judicious choice of h.

The case t_1 Dirichlet's theorem does not seem to help us immediately. How large does h have to be to get a large value of $\tan(h)$? How dense are the relevant values of h among the integers? Dirichlet's theorem does not give a value for h, although it can be re-expressed in the following form to give some information about k:

$$|h - k\theta| < \frac{1}{k}.$$

It would be useful to find a pair (h, k) with 'small' k close to saturating the inequality, and this can be done by appealing to the theory of continued fractions (see [Burton] for an introduction). Specifically, it can be shown that if $C_n = p_n/q_n$ is one of the convergents of θ arising from its continued fraction then

$$|p_n - q_n\theta| < \frac{1}{q_n}$$

and any other pair (h, k) satisfying this bound has $k > q_n$. We can use this to show that t_1 does not converge. For consider the subseries

$$\sum_{p_n} \frac{\tan(p_n)}{p_n}$$

where p_n is the numerator of a convergent with denominator q_n odd. Each term $\tan(p_n)$ is close to

$$\tan\left(\frac{q_n\pi}{2} + \frac{1}{q_n}\right) \;\geq\; q_n - \frac{1}{q_n} \;>\; q_n - 1$$

and so

$$\frac{\tan(p_n)}{p_n} \;>\; \frac{q_n}{p_n} - \frac{1}{p_n} \;=\; \frac{2}{\pi} - O\left(\frac{1}{q_n}\right).$$

There are infinitely many such terms (I think!), which means that t_p has a divergent subsequence and so is not convergent.

Thoughts on the general case Some of the standard tests for convergence depend on knowledge of the size of successive terms in the sequence being summed, and in this case the question is complicated by the unpredictability of $\tan(n)$. While the case t_1 has been resolved (in outline at least) it isn't clear that we can use a similar method to decide the other t_p.

References

[Apostol] Tom Apostol, *Modular Functions and Dirichlet Series in Number Theory*, 2nd Ed., Springer, 1990, Chapter 7.

[Burton] David Burton, *Elementary Number Theory*, 3rd Ed., McGraw Hill, 1995.

Problem 315.1 – Rectangles in a square

Tony Forbes

How many $1 \times (n+1)$ rectangles can you fit in an $n \times n$ square?

Obviously fitting any at all might be a bit difficult when $n = 1$ or 2. But as n increases the difference between n and $\sqrt{2}\,n$ becomes more and more significant. There is I think a crossover point at

$$n \;=\; \frac{2}{\sqrt{2}-1} \;=\; 4.8284,$$

where you can just squeeze in one $(n+1) \times 1$ rectangle along the diagonal.

Solution 311.5 – Colours and shapes

There are 9 objects. They could be a child's bricks, but their exact nature need not concern us. Each has one of 3 colours, {red, blue, green} say, and one of 3 shapes, {triangle, square, pentagon}. All 9 distinct combinations are represented. They are to be arranged in a circle such that two adjacent objects differ in colour or in shape but not both and not neither. For example,

(R3, R4, R5, B5, B3, B4, G4, G5, G3).

How many ways?

Ted Gore

The possible combinations of colour and shape can be represented as a two dimensional grid.

	3	4	5
R	a	b	c
B	d	e	f
G	g	h	i

I have added labels a, b, …, i to generalise the question and to simplify notation. Using the arrangement of rows and columns above, the example in the question would translate to $abcfdehig$.

The rules ensure that we can move from square to square only in a horizontal or vertical direction, similar to the movement of a rook on a chessboard. We can place the first term of the ring in square a. This is the starting point and end point of the ring.

From a, the first move can only be to one of $\{b, c, d, g\}$ and the last move must be to a from a different member of the same set.

I will use b and d as an example. A path from b to d is a sequence of moves conforming to the rules. A path from d to b is the same as a path from b to d.

If C is the number of colours and S the number of shapes, there will be

$$\frac{(C+S-2)(C+S-3)}{2}$$

starting/ending pairs (six for the 3×3 grid).

Let the set

$$Q_{b,d} \;=\; \{b, c, d, e, f, g, h, i\} \setminus \{b, d\} \;=\; \{c, e, f, g, h, i\}.$$

For the 3×3 grid, there are six such sets. Then the sequence b followed by a permutation from $Q_{b,d}$ followed by d is a path from b to d if each move in the sequence is valid.

Each of these paths, preceded by a move from a and succeeded by a move back to a, constitutes a Hamiltonian cycle.

Let $P(x, y)$ be the number of paths from x to y.

The following results were produced by a computer program checking all possible valid paths. For the 3×3 grid,

$$P(b,c) = P(b,d) = P(b,g) = P(c,d) = P(c,g) = P(d,g) = 8$$

so that the total number of paths is 48.

Adding a further colour (Y) and further squares $\{j, k, m\}$ to make a 4×3 grid with 10 starting/ending pairs gives these results:

$$\begin{aligned} P(b,c) &= 180, \\ P(b,d) &= P(b,g) = P(b,j) = P(c,d) = P(c,g) = P(c,j) = 156, \\ P(d,g) &= P(d,j) = P(g,j) = 150, \end{aligned}$$

to give a total of 1566.

From the results for the 3×3 and 4×3 grids it appears that $P(x, y)$ is the same for all pairs that start and end on the first row, is the same for all pairs that start and end on the first column, and is the same for all pairs that start on one axis and end on the other, so that swapping two rows or two columns does not change the total number of paths.

In order to answer the question we need to find the number of ways to map the colour/shape combinations onto the grid.

If we count the path from x to y as two colour/shape sequences (from x to y and also y to x) then we need to take twice the number of paths.

Additionally we should take into account the sequences that arise from changing the order of rows and columns.

There are $C!$ ways to order the rows and $S!$ ways to order the columns.

If T is the total number of paths through a grid for C colours and S shapes (48 for 3×3 and 1566 for 4×3) then the number of rings is $2TC!S!$. For the 3×3 grid this gives 3456 rings and for the 4×3 grid 451008.

An important step in the solution is to find the number of Hamiltonian cycles in the given grid. This is the term T in the equation above.

I had not found a way to do this except by examining every permutation of moves for validity, a procedure that my computer could not realise in a reasonable time for any grid greater than the 4×3.

While searching for a formula for the number of cycles I found a Wikipedia article for 'rook's graph'. The article states that for an $n \times m$ chessboard, the rook's graph is the Cartesian product $K_n \square K_m$.

My thanks to TF for providing the following results.

(n,m)	$(n,1)$	(2,2)	(3,2)	(3,3)	(4,2)	(4,3)	(4,4)	(5,2)	(5,3)	(6,2)
Hamiltonian cycles	$\left\lfloor \dfrac{(n-1)!}{2} \right\rfloor$	1	3	48	30	1566	284112	480	126120	12000

TF subsequently found this formula for the number of Hamiltonian cycles in rook$(n,2)$:

$$\frac{1}{2} \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-k-1}{k-1} \frac{n!\,(n-k-1)!}{k!}.$$

Without the half, it is the generator of entry A089039 in *The On-Line Encyclopedia of Integer Sequences*, where it is described as 'The number of circular permutations of $2n$ people consisting of n married couples, such that no one sits next to a person of the opposite sex who is not his or her spouse.' For example, there are 430920 Hamiltonian cycles for rook(7,2). He also found the following values for rook$(n,3)$ from the OEIS database:

1, 3, 48, 1566, 126120, 18153720, 4357332000, 1619499374640, 883124275824000, 677267024315091840, 706022078404964428800, 972890835488032591468800, 1731258722423272253052441600, 3900512495412914495014418918400, ...

However, it seems that apart from $(n,1)$ and $(n,2)$ there is no general formula for (n,m).

Problem 315.2 – Points

Graham Lovegrove

Given n points in the Euclidean plane, must there exist three of them that define a triangle with an angle in the range $[180 - 360/n,\ 180]$ degrees? Note that the lower bound is attained whenever the points are the vertices of a regular n-gon.

Solution 309.7 – Limit

Show that

$$\frac{x\sin y - y\sin x}{x\cos y - y\cos x} \;\to\; \tan(x - \arctan x) \qquad \text{as } y \to x.$$

Peter Fletcher

Let the given expression be M and let $y = x + h$. Then

$$\begin{aligned} M &= \frac{x\sin(x+h) - (x+h)\sin(x)}{x\cos(x+h) - (x+h)\cos(x)} \\ &= \frac{x\sin(x)\cos(h) + x\cos(x)\sin(h) - (x+h)\sin(x)}{x\cos(x)\cos(h) - x\sin(x)\sin(h) - (x+h)\cos(x)} \\ &= \frac{(1-\frac{h^2}{2!}+\cdots)x\sin(x)+(h-\frac{h^3}{3!}+\cdots)x\cos(x)-(x+h)\sin(x)}{(1-\frac{h^2}{2!}+\cdots)x\cos(x)-(h-\frac{h^3}{3!}+\cdots)x\sin(x)-(x+h)\cos(x)} \\ &= \frac{(-\frac{h^2}{2!}+\frac{h^4}{4!}-\cdots)x\sin(x) + (h-\frac{h^3}{3!}+\cdots)x\cos(x) - h\sin(x)}{(-\frac{h^2}{2!}+\frac{h^4}{4!}-\cdots)x\cos(x) - (h-\frac{h^3}{3!}+\cdots)x\sin(x) - h\cos(x)}. \end{aligned}$$

Now we divide top and bottom by h,

$$M = \frac{(-\frac{h}{2!}+\frac{h^3}{4!}-\cdots)x\sin(x) + (1-\frac{h^2}{3!}+\cdots)x\cos(x) - \sin(x)}{(-\frac{h}{2!}+\frac{h^3}{4!}-\cdots)x\cos(x) - (1-\frac{h^2}{3!}+\cdots)x\sin(x) - \cos(x)},$$

and let $h \to 0$,

$$L = \lim_{h\to 0}(M) = \frac{x\cos(x) - \sin(x)}{-x\sin(x) - \cos(x)} = \frac{\sin(x) - x\cos(x)}{\cos(x) + x\sin(x)}.$$

Now we divide the top and bottom of L by $\sqrt{1+x^2}$ to give

$$L = \frac{\dfrac{1}{\sqrt{1+x^2}}\sin(x) - \dfrac{x}{\sqrt{1+x^2}}\cos(x)}{\dfrac{1}{\sqrt{1+x^2}}\cos(x) + \dfrac{x}{\sqrt{1+x^2}}\sin(x)}.$$

Let $\theta = \arctan(x)$, so $\tan(\theta) = x$. This means $\sin(\theta) = x/\sqrt{1+x^2}$ and $\cos(\theta) = 1/\sqrt{1+x^2}$ and

$$\begin{aligned} L &= \frac{\sin(x)\cos(\theta) - \cos(x)\sin(\theta)}{\cos(x)\cos(\theta) + \sin(x)\sin(\theta)} \\ &= \frac{\sin(x-\theta)}{\cos(x-\theta)} = \tan(x-\theta) = \tan(x - \arctan(x)). \end{aligned}$$

Solution 313.3 – Tetrahedron

A tetrahedron has vertices A, B, C, D, and

$$\angle BAD \;=\; \angle BAC \;=\; \angle CAD \;=\; 90^\circ.$$

Show that the face areas $\triangle BCD$, $\triangle BAC$, $\triangle CAD$ and $\triangle DAB$ satisfy

$$(\triangle BCD)^2 \;=\; (\triangle BAD)^2 + (\triangle BAC)^2 + (\triangle CAD)^2.$$

Stuart Walmsley

The choice of labels for the vertices is changed so that certain equations which occur in the solution can be expressed in their familiar form. Explicitly, the unique vertex A is replaced by D, so that the problem now reads:

A tetrahedron with vertices A, B, C, D has angles

$$\angle ADB \;=\; \angle ADC \;=\; \angle BDC \;=\; 90^\circ.$$

Show that the face areas $\triangle ABC$, $\triangle ADB$, $\triangle ADC$, $\triangle BDC$ are given by

$$(\triangle ABC)^2 \;=\; (\triangle ADB)^2 + (\triangle ADC)^2 + (\triangle BDC)^2.$$

Further, areas of triangles are most conveniently expressed in terms of the lengths of their sides. Accordingly, the following choice is made:

$$AB \;=\; c, \qquad AC \;=\; b, \qquad BC \;=\; a,$$

$$AD \;=\; p, \qquad BD \;=\; q, \qquad CD \;=\; r.$$

Three of the triangles are right-angled. By Pythagoras, the lengths are related:

$$a^2 \;=\; q^2 + r^2, \qquad b^2 \;=\; p^2 + r^2, \qquad c^2 \;=\; p^2 + q^2$$

and the squares of the areas are

$$(\triangle ADB)^2 = \frac{p^2q^2}{4}, \qquad (\triangle ADC)^2 = \frac{p^2r^2}{4}, \qquad (\triangle BDC)^2 = \frac{q^2r^2}{4}.$$

In this way, the problem is to show that

$$(\triangle ABC)^2 \;=\; \frac{1}{4}\left(p^2q^2 + p^2r^2 + q^2r^2\right).$$

No dimensions of the triangle ABC are specified. The basic formula for the square of the area of a triangle is

$$\frac{1}{4}\left(\text{base}\times(\text{perpendicular height})\right)^2.$$

For the triangle ABC with side lengths a, b, c, this can be shown to be, for example,

$$(\triangle ABC)^2 = \frac{1}{4}\,a^2b^2\sin^2\gamma,$$

where γ is the included angle between a and b. This may be rewritten

$$(\triangle ABC)^2 = \frac{1}{4}\,a^2b^2\left(1-\cos^2\gamma\right).$$

From the cosine rule,

$$\cos^2\gamma = \frac{\left(a^2+b^2-c^2\right)^2}{4a^2b^2},$$

this becomes

$$\begin{aligned}(\triangle ABC)^2 &= \frac{1}{16}\left(4a^2b^2-(a^2+b^2-c^2)^2\right)\\ &= \frac{1}{16}\left(4a^2b^2-(a^4+b^4+c^4+2a^2b^2-2a^2c^2-2b^2c^2)\right)\\ &= \frac{1}{16}\left((2a^2b^2+2a^2c^2+2b^2c^2)-(a^4+b^4+c^4)\right)\\ &= \frac{1}{16}\left(4(a^2b^2+a^2c^2+b^2c^2)-(a^2+b^2+c^2)^2\right).\end{aligned}$$

Expanding in terms of p, q, r,

$$\begin{aligned}&4(a^2b^2+a^2c^2+b^2c^2)\\ &\quad= 4\left((q^2+r^2)(p^2+r^2)+(q^2+r^2)(p^2+q^2)+(p^2+r^2)(p^2+q^2)\right)\\ &\quad= 4(p^4+q^4+r^4)+12(p^2q^2+p^2r^2+q^2r^2)\end{aligned}$$

and

$$\begin{aligned}\left(a^2+b^2+c^2\right)^2 &= 4\left(p^2+q^2+r^2\right)^2\\ &= 4(p^4+q^4+r^4)+8(p^2q^2+p^2r^2+q^2r^2).\end{aligned}$$

Then

$$4(a^2b^2+a^2c^2+b^2c^2)-\left(a^2+b^2+c^2\right)^2 = 4(p^2q^2+p^2r^2+q^2r^2).$$

And therefore

$$(\triangle ABC)^2 \;=\; \frac{p^2q^2 + p^2r^2 + q^2r^2}{4},$$

which is the required result; that is,

$$(\triangle ABC)^2 \;=\; (\triangle ADB)^2 + (\triangle ADC)^2 + (\triangle BDC)^2,$$

or with the labels given in the original form of the problem, in which A and D are interchanged:

$$(\triangle BCD)^2 \;=\; (\triangle BAD)^2 + (\triangle BAC)^2 + (\triangle CAD)^2.$$

Problem 315.3 – Cyclic polygon area

Tony Forbes

Show that for large n, the area of a cyclic polygon of n sides with lengths e_1, e_2, ..., e_n is given approximately by the formula

$$\frac{e}{\pi}\sqrt{s_n^{4-n}\prod_{i=1}^{n}(s_n - e_i)}, \qquad \text{where } s_n = \frac{e_1 + e_2 + \cdots + e_n}{2},$$

provided that the e_i do not deviate too far from equality.

Observe that if $e = \pi$, this is Brahmagupta's formula for the area of a cyclic quadrilateral when $n = 4$ and when $n = 3$ it reduces to Heron's formula for triangles.

Problem 315.4 – Cylinder in a cube

What is the largest r such that a cylinder of radius r and length 6 will fit in a cube of side 5?

Obviously this is one of an infinite number of similar problems. I (TF) stumbled upon this particular instance while I was doing something else. It interested me because the answer seems to be 1.0 or thereabouts.

Problem 315.5 – Convexity

There is a finite set of ordered pairs, $S = \{(x_i, y_i) : i = 1, 2, \ldots, n\}$, say. Find a simple test to determine whether the elements of S define the Cartesian coordinates of the vertices of a convex polygon.

M500 Revision Weekend May 2023

Judith Furner

The 47th M500 Revision Weekend took place at Kents Hill Training and Conference Centre over 12th – 14th May 2023. We ran 11 undergraduate maths modules and five postgraduate maths modules. We ran one Science module – SM380 – for the first time. We also provided a room for a tutor group for students of M832 as no tutor was available. We had 161 students—of those 143 were residential and 18 non-residential. This was an increase on 2022, and although we could have done with more it was pleasing that there was an increase. There were 19 tutors, who were mostly entirely satisfactory. There were seven M500 'staff': Angela Allsopp, Paul Cooper, Chris Furner, Judith Furner, Charlotte Connolly, Dorothy Leddy and Milena Dragic. Once again we made a loss, although it was considerably less than it was in 2022. Apart from anything else, the rail strike meant that some tutors' travel expenses were considerably higher. All feedback was passed on to tutors, and an anonymised version was sent to students.

The feedback was very good—nearly all concerned clearly had an excellent weekend. We received 139 forms out of a potential 149. Paul and Angela did a splendid job of delivering feedback forms to tutorial classes, and retrieved them, along with the lanyards, from students on the Sunday afternoon. If you came to the Weekend, and find that you managed to take your lanyard home, I would be grateful if you would return it (the Revision Weekend in May 2024 is an acceptable time).

Many of our tutors have been working with us for thirty or forty years, and we pride ourselves on offering the best of the best to our students. Nearly all students seemed to agree with our views. Of the 112 comments, 100 were 'excellent', 10 'good' and 2 'average'. Comparisons are invidious, but it is a great delight to see some tutors, year after year, being marked 'excellent' by 100 per cent of their tutees. Typical comments were 'first-rate tuition', 'great communication', 'pitched it at our level and speed', 'patient and thorough with explanations', 'helped cement my understanding of more tricky subjects', 'he was incredible, we were SO lucky to have him. Engaging and fun. Get him back', 'very knowledgeable and experienced with facilitation and also adaptable to the group's needs. Lovely person too', 'hugely helpful, thanks very much', 'easy to follow. Good selection of questions and information on exam strategy', 'organised, well-prepared, clear and supportive', 'great tutor. He got through all the material, answered all our questions and was very focused on exam technique'.

We were delighted that of the 111 responses to Weekend organization on the feedback form, 98 were 'excellent', 10 'good' and 3 'average'. I am of course particularly interested in these last three, and the one who commented said 'Very patchy information, but a lot of it, quite confusing, not enough clarity'. She/he did not give a name, which is a shame: I am always keen to follow up such comments. Others were happy with the information provided, so maybe it is horses for courses. Several students said that they would be returning, which is always pleasing, and we were asked 'please don't put another real life event online' (my thought entirely). Students said 'it was lovely to meet other students', and one of my favourite comments was 'overall I am very happy'.

Revision Weekend May 2023 – Feedback forms from students

		Excellent	Good	Average	Avg–poor	Poor	Total
Tutors	Number	100	10	2	0	0	112
	Percent	89	9	2	0	0	
Classrooms	Number	35	41	22	4	3	105
	Percent	33	39	21	4	3	
Accommodation	Number	39	38	20	4	4	105
	Percent	37	36	19	4	4	
Food and drink	Number	38	55	12	5	0	110
	Percent	35	50	11	5	0	
Refreshments	Number	25	21	10	0	1	57
	Percent	44	37	18	0	2	
Bar	Number	15	22	11	2	0	50
	Percent	30	44	22	4	0	
Talk	Number	12	8	1	0	0	21
	Percent	57	38	5	0	0	
Weekend organization	Number	98	10	3	0	0	111
	Percent	88	9	3	0	0	

This is of course very much a team effort and I am reliant on the M500 team, who work so hard over the Weekend. They came in for much praise, for being friendly, helpful and available. We were thanked for our 'tremendous service' and some comments were 'all the M500 and OU staff and volunteers were extremely welcoming and friendly. They answered many questions for me which could really only be asked in a face to face setting', 'Thank you Judith and all your helpers. I am already looking forward to next year'. Others appreciated the availability of the OU staff, handbooks and other literature. One student said 'Would have liked some quiz or other collective activity on the Saturday evening (otherwise non-drinkers don't go to the bar and others may be shy)'.

The venue was generally appreciated, although there were many comments about the temperature: too hot, too cold, impossible to change, windows don't open, and so on. Room temperature adjustment may be simple, but as I pointed out to the venue, our delegates are mathematicians, not engineers. There were only a few comments about lack of whiteboards (old hands may remember this being an issue in the past). Students were generally pleased with the choice in the restaurant, although vegan food was sometimes found to be lacking. It seems that people's expectations vary widely. There was universal criticism of the coffee provided at breaks, and the fact that both snacks and milk ran out. The bar was less satisfactory: the service was described as slow and expensive. Students were generally united in their comments about the staff, typical was 'all staff were brilliant and polite'.

Dates for your diary The Winter Weekend takes place over the 12th – 14th January 2024, and the Revision Weekend over the 10th – 12th May 2024. Both events take place at Kents Hill. The Winter Weekend is essentially a fun weekend, where we enjoy all sorts of mathematical puzzles and games, not to mention a certain amount of problem-solving and deep thinking. Those of you who have ever been Mel's tutees will be delighted to hear that he will be doing a session for us. In January of this year he put a list of six items on the board. We had so many digressions, questions, and general discussion that when we were reluctantly called to supper we noticed that we had only covered three of the six. "Jolly good," said Mel, "That will do us for next year." Bookings are not yet open for these events, but keep an eye on the M500 website (which is not always updated as quickly as might be (note use of the passive voice)). It is, however, useful to have the dates in your diary to ensure that you don't inadvertently book another event.

Problem 315.6 – Prism

Tamsin Forbes

Is it possible to determine the n of a regular n-sided prism from a single side elevation parallel to a vertical face? Admittedly it might be difficult to distinguish between $n = 3$ and $n = 4$, but what about larger n?

Contents

04.00

Problem 315.7 – Sum

For $r = 3, 4, \ldots$, show that

$$\frac{1}{1^2+(r-1)^2} + \frac{1}{(r+1)^2+(2r-1)^2} + \frac{1}{(2r+1)^2+(3r-1)^2} + \ldots = \frac{\pi}{2(r-2)r} \tanh \frac{(r-2)\pi}{2r}.$$

For example, when $r = 3$ this is $\displaystyle\frac{1}{1^2+2^2} + \frac{1}{4^2+5^2} + \frac{1}{7^2+8^2} + \ldots.$

Front cover Bricks, transparent, pile of; see page 1.