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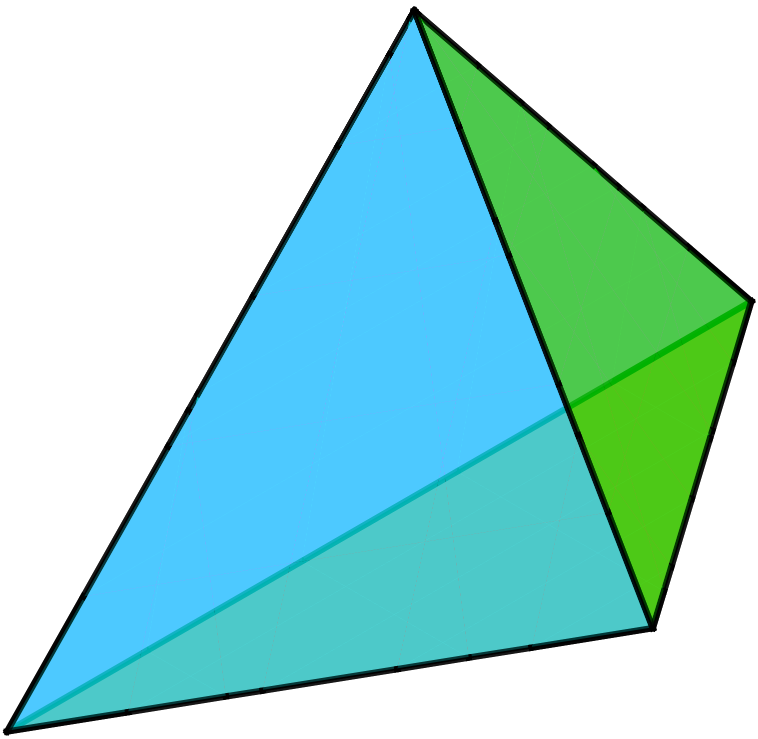

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Lines in $\mathbb{R}^3$ and vector fields on S^2: postscript

Tommy Moorhouse

Introduction In M500 **313** I described vector fields on S^2 derived from sets of lines in $\mathbb{R}^3$. These fell into two broad classes, which I described, by analogy, as electric and magnetic (although electromagnetism on S^2 actually takes a different form). I speculated on the existence of a duality between the sets of lines leading to each type of field. Here the link is explained in another way.

The link between the electric and magnetic fields Readers may have noticed, or may wish to check, that rotating the electric field on S^2 at each point through $\pi/2$ radians anticlockwise leads to a field looking very much like the magnetic field. In fact if we represent a tangent vector to the sphere by the pair $(\vec{u}, \vec{v}) \in S^2 \times \mathbb{R}^3$ we can quite naturally produce another tangent vector $(\vec{u}, J(\vec{v})) \equiv (\vec{u}, \vec{u} \times \vec{v})$ at right angles to the first. Here, of course, we consider the sphere to be embedded in $\mathbb{R}^3$ and the cross product is the usual one. This construction defines an 'almost complex structure' on the tangent space to S^2, with $J(J(\vec{v})) = -\vec{v}$, and it makes sense to ask what effect the operation of J has on the allowed set of lines giving rise to, say, the electric field. We can show that the effect on the set of lines passing through a point (the allowed set for the electric field) is to map those lines to the odd-looking allowed set for the magnetic field presented previously, as can be shown by a simple calculation.

Generate the electric field by taking the allowed set of lines through $\vec{q} = (0, 0, z_0) \in \mathbb{R}^3$. These lines are parametrised by $t \mapsto \vec{q} + t\vec{u}$ where $\vec{u}$ is a unit vector. Taking the base point for the vector $\vec{v}$ to be $\vec{p} = (0, 0, 0)$ our construction gives rise to the vector

$$\vec{v} \;=\; (-u_1 u_3 z_0, -u_2 u_3 z_0, z_0(1 - u_3^2))$$

on S^2 at $\vec{u}$, an electric field pointing from the north to the south of the sphere. Now take the cross product to find $J(\vec{v}) = \vec{u} \times \vec{v} = (u_2 z_0, -u_1 z_0, 0)$. This gives a new vector field $(\hat{\vec{u}}, J(\vec{v}))$ derived from a new allowed set of lines. Taking the base point again to be $(0, 0, 0)$ we have $J(\vec{v})$ ending at $(x, y, 0) = (u_2 z_0, -u_1 z_0, 0)$. Using the fact that $|\hat{\vec{u}}|^2 = 1$, we calculate $\hat{\vec{u}}$ for the allowed set of lines as

$$\hat{\vec{u}} \;=\; \frac{1}{z_0}\left(-y, x, \pm\sqrt{z_0^2 - (x^2 + y^2)}\right).$$

This is just the set of unit tangent vectors to the allowed set of lines giving rise to the magnetic field.

Geodesics on the 2-sphere

Tommy Moorhouse

Introduction The geodesics on a 2-sphere, the set of points

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

are easy to visualise but not quite so easy to describe by explicit equations. Here I will set out an alternative description not involving the solution of differential equations except as a check. First a variational principle is used to obtain the conventional geodesic equations, then the alternative description is introduced.

Deriving the geodesic equations from a variational principle The metric on the unit 2-sphere can be written in terms of the angular coordinates ϑ and φ as

$$ds^2 \;=\; d\vartheta^2 + \sin^2\vartheta d\varphi^2.$$

If we consider the distance function to be the action of the system in terms of the dynamical variables ϑ and φ we can write

$$L \;=\; \dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2.$$

We write φ and ϑ for x^μ in turn in the variational equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} \;=\; 0,$$

finding

$$\begin{aligned}\frac{d}{dt}\left(\dot{\varphi}\sin^2\vartheta\right) &= 0,\\ \ddot{\vartheta} - \sin\vartheta\cos\vartheta\dot{\varphi}^2 &= 0.\end{aligned}$$

Expanding the derivatives if required we can read off the Christoffel coefficients from the geodesic equation

$$\ddot{x^\mu} + \Gamma^\mu_{\nu\rho}\dot{x^\nu}\dot{x^\rho} \;=\; 0.$$

For example

$$\Gamma^\vartheta_{\varphi\varphi} \;=\; -\sin\vartheta\cos\vartheta.$$

While it is possible to obtain expressions for the geodesics in this way there is a different, more intuitive approach. We know that geodesics take the shortest path between two given points. On the sphere these paths are the great circles, which are the intersections of the sphere centred on the origin with oriented planes through the origin.

Parametrised geodesics We will describe a geodesic using the intersection of any plane through the origin with the sphere. Choose two orthonormal unit 3-vectors $\vec{u}$ and $\vec{v}$ spanning the plane and form the curve

$$\cos\omega(t)\vec{u} + \sin\omega(t)\vec{v}.$$

This is a curve following the path of the geodesic described by the plane. It can thus be described by the usual coordinates on the 2-sphere $(\cos\vartheta, \sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi)$. This means that

$$\begin{aligned}\cos\vartheta &= u^1\cos\omega + v^1\sin\omega,\\ \sin\vartheta\cos\varphi &= u^2\cos\omega + v^2\sin\omega,\\ \sin\vartheta\sin\varphi &= u^3\cos\omega + v^3\sin\omega.\end{aligned}$$

Readers may wish to check that these expressions are consistent (e.g. that $\cos^2\theta + \sin^2\theta = 1$) using the orthonormality of the set $\{\vec{u}, \vec{v}\}$. From this we have

$$\begin{aligned}\vartheta &= \arccos(u^1\cos\omega + v^1\sin\omega),\\ \varphi &= \arctan\left(\frac{u^3\cos\omega + v^3\sin\omega}{u^2\cos\omega + v^2\sin\omega}\right).\end{aligned}$$

By differentiating these expressions and substituting into the geodesic equations it is possible to show that the curve is a geodesic provided that $\ddot{\omega} = 0$. This follows from the equation for φ, with that for θ then being satisfied identically. It is an interesting (dare I say fun?) exercise to work through the calculation. Thus, to specify a geodesic, choose a constant a and set $\omega(t) = at$, choose two orthonormal 3-vectors based at the origin and substitute into the equations for ϑ and φ.

Conclusion This description of the solutions to the geodesic equations on S^2 in terms of two orthogonal unit vectors hopefully provides a satisfying explanation of the slightly complicated expressions for ϑ and φ normally encountered. This approach may generalise to geodesics on other manifolds such as S^3, which at face value are much more complicated to describe.

Geodesics on the 3-sphere

Tommy Moorhouse

Introduction The 3-sphere S^3 can be embedded in $\mathbb{R}^4$ as the set of points (x, y, z, w) satisfying $x^2 + y^2 + z^2 + w^2 = 1$. It is three dimensional and so any point can be specified using three angular variables analogous to the two used on the 2-sphere S^2. By analogy with the 2-sphere we can describe the path of a geodesic in terms of two independent 4-vectors $\vec{u}$ and $\vec{v}$, to be discussed below. An orientation can be defined on $\mathbb{R}^4$ so that geodesics can be consistently assigned a direction, but we will not make explicit use of this here. A geodesic path is the set of points that the geodesic passes through.

The geodesic equations Here the points on S^3 are parametrised as

$$(p_1, p_2, p_3, p_4) \;=\; (\cos\chi, \sin\chi\cos\vartheta, \sin\chi\sin\vartheta\cos\varphi, \sin\chi\sin\vartheta\sin\varphi).$$

The geodesic equations following from the metric

$$ds^2 \;=\; d\chi^2 + \sin^2\chi\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)$$

are

$$\begin{aligned} \frac{d}{dt}(\dot{\varphi}\sin^2\chi\sin^2\vartheta) &= 0, \\ \frac{d}{dt}\left(\dot{\vartheta}\sin^2\chi\right) - \dot{\varphi}^2\sin^2\chi\sin\vartheta\cos\vartheta &= 0, \\ \ddot{\chi} - \sin\chi\cos\chi\left(\dot{\vartheta}^2 + \dot{\varphi}^2\sin^2\vartheta\right) &= 0. \end{aligned}$$

It is possible, with some effort, to find a general expression for the geodesics. However, intuition tells us that the geodesics are one-dimensional paths on the surface of a 4-dimensional ball, and the intersection of planes with the 3-sphere should lead us to a solution.

The parametrised geodesics In $\mathbb{R}^4$ the natural analogue of a plane is a hyperplane through the origin, which is three dimensional. This meets S^3 in a two dimensional surface, which is not what we need. We can, however, take a two dimensional plane subspace of $\mathbb{R}^4$ and consider the curve given by

$$\gamma(t) \;=\; \vec{u}\cos\omega(t) + \vec{v}\sin\omega(t)$$

where the orthogonal unit 4-vectors $\vec{u}$ and $\vec{v}$ span the plane. We can check that $\gamma(t)$ lies on S^3 so that it is a candidate geodesic. The coordinates of points on the geodesic can be found from

$$\begin{aligned}
\cos\chi &= u^1\cos\omega + v^1\sin\omega,\\
\sin\chi\cos\vartheta &= u^2\cos\omega + v^2\sin\omega,\\
\sin\chi\sin\vartheta\cos\varphi &= u^3\cos\omega + v^3\sin\omega,\\
\sin\chi\sin\vartheta\sin\varphi &= u^4\cos\omega + v^4\sin\omega.
\end{aligned}$$

We can readily deduce expressions for each angular variable in terms of inverse trigonometric functions of the components of $\vec{u}$ and $\vec{v}$. Substituting into the geodesic equations we find that provided $\ddot{\omega}(t) = 0$ these curves are geodesics. We can take $\omega(t) = t$, for example. Readers may object that we have not demonstrated that all the geodesics arise in this way, and this will be touched on below. We should also remark that the choice of the orthonormal pair $\{\vec{u}, \vec{v}\}$ is not unique. A different choice with the same orientation, however, simply corresponds to a reparametrisation of the geodesic by $t \mapsto t + t_0$.

Some pairs of geodesics never meet In contrast to the geodesics on S^2 we can find pairs of geodesics that do not meet. For example the plane spanned by $\vec{u} = (0,0,0,1)$ and $\vec{v} = (0,0,1,0)$ only meets the plane spanned by $\vec{w} = (1,0,0,0)$ and $\vec{z} = (0,1,0,0)$ in a single point, so geodesics in these planes do not meet. In fact every plane defined by a pair of vectors in $\mathbb{R}^4$ has a 'dual' plane which it meets in a single point, so every geodesic has a dual geodesic which it does not meet.

Geodesics through a point meet again The geodesic paths through a point p of S^3 spread out through a 2-sphere surrounding the point and refocus at $-p$. To establish this we use the parametrisation $\gamma(t) = \vec{p}\cos t + \vec{v}\sin t$. We may as well take the point $p = (1,0,0,0) \in S^3$ using the symmetries of S^3. We require that $\vec{p}\cdot\vec{v} = 0$, and a general expression for $\vec{v}$ is therefore a parametrisation of the 2-sphere such as $(0, \cos\theta, \sin\theta\sin\phi, \sin\theta\cos\phi)$. Each point of this 2-sphere defines a geodesic path through p, and it is clear that these paths meet again when $t = \pi$. One could perhaps adapt this argument using the manifold structure (i.e. the fact that S^3 looks like $\mathbb{R}^3$ close to p) to show that the geodesic paths are all of the form we have found.

Conclusion Visualising geodesic paths on S^3 poses challenges, especially if we start from the Euler-Lagrange equations. However, a bit of intuition has allowed us to find and characterise them and even start to explore some of their properties. A useful tool is projection onto $\mathbb{R}^3$ from a point of S^3 and interested readers might enjoy looking into this.

Solution 313.5 – Integers

For which x is $\sqrt{24x+1}$ an integer?

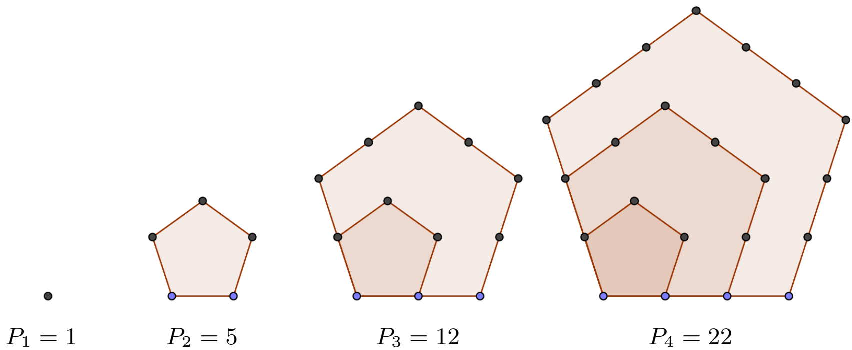
Peter Fletcher

If we try $x = 0, 1, 2, \ldots$ in the given expression, we soon get a list of values for which $\sqrt{24x+1}$ is an integer:

$$x = 0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, 57, 70, 77, 92, 100, 117, \ldots.$$

It's not obvious what sort of formula would generate this list, but if we put these numbers in the search box of the The On-Line Encyclopedia of Integer Sequences (`https://oeis.org/`), we soon find that the x's are generalized pentagonal numbers.

Recall that 'small' pentagonal numbers are very simply calculated thus (image from Wikipedia):



The dots that comprise P_q above may be split into a triangle and a rectangle, which leads to

$$P_q = \frac{q(q+1)}{2} + q(q-1)$$

for positive q.

If, however, we include negative q in the equation for P_q in the order

$$q = 0, +1, -1, +2, -2, +3, -3, \ldots$$

we get generalised pentagonal numbers x in ascending order, as we calculated above.

Ken Greatrix

At last, a problem that my age-weary bones can manage!

Following the procedure we had at summer school for M101 (the maths foundation course that was current then), I first tried a few integers in the relationship to see what resulted. I assumed that x should take an integer-value and the results were:

> If x took values of 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, ... this results in integer-values of 5, 7, 11, 13, 17,19, 23, 25, 29, 31, ... respectively.

I then noticed a pattern was emerging; the results seemed to be following $6n \pm 1$ and I also noticed that values of x came in pairs separated by n.

Values of $6n + 3$ did not appear in this list; so to investigate further I worked the relationship backwards from these values of n. At this stage, other than the relationship between pairs of x noted above I didn't know which values of x would fit the problem.

Let z represent the function then

$$z = \sqrt{24x + 1}.$$

By rearranging this for x we get

$$x = \frac{(z^2 - 1)}{24}.$$

So if z takes a value of $6n \pm 1$ then

$$x = \frac{(36n^2 \pm 12n)}{24}.$$

When a value of $z = 6n \pm 1$ is considered for even values of n, then this expression can be rewritten as

$$x = 72\frac{n^2}{2} \pm 24\frac{n}{2},$$

which can be divided by 24 to give a pair of integer-values for x.

But if we consider odd values of n, the expression can now be rewritten as

$$x = 72\frac{(n^2 - 1)}{2} + 36 \times 1 \pm \left(24\frac{(n-1)}{2} + 12 \times 1\right),$$

where $n-1$ and n^2-1 are even numbers. This is similar to the above expression with the addition of 36 ± 12, which again is divisible by 24 to give further x-values as integers.

The difference between pairs of x-values associated with each n-value is

$$x = \frac{(36n^2 \pm 12n)}{24},$$

which can be rewritten as

$$x = \frac{3n^2 \pm n}{2},$$

where the difference between the two values of x is n.

But when z takes a value of $6n+3$, then x does not have an integer solution. i.e.

$$x = \frac{(36n^2 + 36n) + 8}{24},$$

which is not an integer for odd or even values of n.

This problem has shown that a solution is obtained by considering the inverse of the function, but the values of x could be found by finite differencing. However it is likely that anyone trying this technique is more likely to spot the easier method of finding z (my designation).

Another feature of the solution is obtained when n takes a value of 0. x now becomes 0 in both cases—which is two values of 0 separated by 0 or n in this case.

If negative values of n are considered, x takes positive values again, since in the inverse function, z is squared. In which my statement 'two values separated by n' must be considered in the reverse direction: e.g. if $n = -2$ then the two x-values become 7 and 5 rather than 5 and 7.

I wonder if anyone would like to consider looking for a solution in complex numbers?

Problem 316.1 – Irrational series

Let k be a positive integer. Suppose a_1, a_2, ... are non-negative integers of which an infinite number are positive and only a finite number satisfy $a_n > n^k$. Show that

$$\frac{a_1}{1!} + \frac{a_2}{2!} + \dots$$

is irrational, or find a counter-example.

Problem 316.2 – Integer tetrahedron

Tony Forbes

A tetrahedron has edge lengths which are 6 consecutive integers. If its volume is an integer, show that it must be 48. Or find another value.

Writing the edges in the order (AB, AC, AD, BC, BD, CD) for a tetrahedron with vertices A, B, C, D, we have these two useful formulæ for the volume:

$$\frac{1}{6}\left|\det\begin{bmatrix} A_x & B_x & C_x & D_x \\ A_y & B_y & C_y & D_y \\ A_z & B_z & C_z & D_z \\ 1 & 1 & 1 & 1 \end{bmatrix}\right|$$

in terms of the coordinates of the vertices, and

$$\left(\frac{1}{288}\det\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & |AB|^2 & |AC|^2 & |AD|^2 \\ 1 & |AB|^2 & 0 & |BC|^2 & |BD|^2 \\ 1 & |AC|^2 & |BC|^2 & 0 & |CD|^2 \\ 1 & |AD|^2 & |BD|^2 & |CD|^2 & 0 \end{bmatrix}\right)^{1/2}. \tag{1}$$

Using them (or otherwise) one can confirm that volume 48 is obtained for some orderings of $\{6, 7, \ldots, 11\}$. In particular we have this example:

$$(|AB|, |AC|, |AD|, |BC|, |BD|, |CD|) \ = \ (6, 7, 8, 11, 10, 9) \ \to \ \text{volume } 48.$$

The volume of a tetrahedron must be real. The alternative ordering of edge lengths $(6, 7, 10, 11, 8, 9)$ when substituted into (1) gives $12i$, and one can confirm by trying to build it out of cardboard that the corresponding tetrahedron does not exist.

Problem 316.3 – Two integrals

Tony Forbes

Show that

$$\int_0^1 \frac{\arctan t}{\sqrt{t}}\,dt \ = \ \sqrt{2}\,\mathrm{arccoth}\sqrt{2} - \frac{\pi}{2}\left(\sqrt{2} - 1\right) \ = \ 0.595805$$

and

$$\int_0^1 \frac{\arctan t}{t^{1/3}}\,dt \ = \ \frac{3}{8}\left(2\sqrt{3}\log\left(2 + \sqrt{3}\right) - \pi\right) \ = \ 0.532681.$$

Evaluation of series via the Laplace transform

Henry Ricardo

The Laplace transform is the first integral transform an undergraduate STEM student encounters, usually in a differential equations course. This transformation is defined by

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

for a function f whose rate of growth yields a convergent improper integral. An important property of this transform is its linearity: $\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha\mathcal{L}[f(t)] + \beta\mathcal{L}[g(t)]$ for constants α and β. This characteristic allows one to convert a linear differential equation to an algebraic equation and a system of linear differential equations to a system of algebraic equations. Lists of Laplace transforms can be found in many textbooks and books of maths tables, and the function is embedded in many computer algebra systems.

What students usually don't see is that this transformation can be used to evaluate certain infinite series. The method is effective when you have an infinite series $\sum_{n=1}^\infty a_n$ whose summand a_n can be represented by a Laplace transform integral: $a_n = \int_0^\infty e^{-nt} f(t)\, dt$ for an appropriate function f. Let's look at some examples to see how this technique works.

Example 1. Evaluate $\displaystyle\sum_{n=1}^\infty \frac{(-1)^{n+1}}{2n+1}$.

We have

$$\frac{1}{2n+1} = \frac{1}{2(n+1/2)} \text{ and } \frac{(-1)^{n+1}}{2n+1} = \frac{(-1)^{n+1}}{2(n+\frac{1}{2})} = \frac{(-1)^{n+1}}{2}\mathcal{L}[e^{-t/2}].$$

Therefore, recognizing a geometric series in the calculation,

$$\begin{aligned}
\sum_{n=1}^\infty \frac{(-1)^{n+1}}{2n+1} &= \sum_{n=1}^\infty \frac{(-1)^{n+1}}{2} \cdot \mathcal{L}[e^{-t/2}] \\
&= \sum_{n=1}^\infty \frac{(-1)^{n+1}}{2} \cdot \int_0^\infty e^{-nt} e^{-t/2}\, dt \\
&= \frac{1}{2}\int_0^\infty e^{-t/2} \left(\sum_{n=1}^\infty (-1)^{n+1} e^{-nt}\right) dt \\
&= \frac{1}{2}\int_0^\infty e^{-t/2} \left(\frac{e^{-t}}{1+e^{-t}}\right) dt
\end{aligned}$$

$$= \frac{1}{2}\int_0^\infty \frac{e^{-t/2}}{e^t+1}\,dt$$
$$\overset{u=e^{t/2}}{=} \int_1^\infty \frac{1}{u(u^2+1)}\,du = 1 - \frac{\pi}{4}.$$

A crucial observation is that in the third line of this derivation, we have interchanged integration and summation. Generally, if we have a convergent infinite series, $\sum\int \neq \int\sum$, so that we must be able to justify this swap by one of the standard criteria found in most analysis texts: uniform convergence, monotone convergence, dominated convergence, or bounded convergence of the series in the integrand. In this example, the swap can be justified by dominated convergence: $\left|\sum_{n=1}^{N}(-1)^{n+1}e^{-nt}\right| \leq \dfrac{2}{1+e^t} = g(t)$ and $\int_0^\infty g(t)\,dt$ converges.

If the interchange of summation and integration can be justified, then we have

$$\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty \int_0^\infty e^{-nt}f(t)\,dt = \int_0^\infty f(t)\sum_{n=1}^\infty e^{-nt}\,dt = \int_0^\infty f(t)\left(\frac{e^{-t}}{1-e^{-t}}\right)dt.$$

Then we must evaluate this integral to get the sum of the original series. However, the availability of such resources as *Tables of Integrals, Series, and Products* by I. S. Gradshteyn and I. M. Ryzhik makes this part of the process relatively easy.

Our next example is a problem (310.8) whose solution appeared in issue 313 of this journal.

Example 2. Show that $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+2n^3+n^4} = \frac{\pi^2}{3} - 3.$

As in the original solution, the first step is a partial fraction decomposition:

$$\frac{1}{n^2+2n^3+n^4} = -\frac{2}{n} + \frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{2}{n+1}.$$

Then we find that

$$-\frac{2}{n} = -2\mathcal{L}[1], \quad \frac{1}{n^2} = \mathcal{L}[t], \quad \frac{1}{(n+1)^2} = \mathcal{L}[te^{-t}], \quad \frac{2}{n+1} = 2\mathcal{L}[e^{-t}].$$

Therefore,

$$\frac{1}{n^2+2n^3+n^4} = --2\mathcal{L}[1] + \mathcal{L}[t] + \mathcal{L}[te^{-t}] + 2\mathcal{L}[e^{-t}]$$

and

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n^3 + n^4} &= \sum_{n=1}^{\infty} \mathcal{L}[-2 + t + te^{-t} + 2e^{-t}] \\
&= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-nt}(-2 + t + te^{-t} + 2e^{-t})\, dt \\
&= \int_0^{\infty} (-2 + t + te^{-t} + 2e^{-t}) \cdot \left(\frac{e^{-t}}{1 - e^{-t}}\right) dt \\
&= -2 \int_0^{\infty} \frac{e^{-t}}{1 - e^{-t}}\, dt + \int_0^{\infty} \frac{te^{-t}}{1 - e^{-t}}\, dt \\
&\quad + \int_0^{\infty} \frac{te^{-2t}}{1 - e^{-t}}\, dt + 2 \int_0^{\infty} \frac{e^{-2t}}{1 - e^{-t}}\, dt. \\
&= -2 \int_0^{\infty} e^{-t}\, dt + \int_0^{\infty} te^{-t} \cdot \frac{e^t + 1}{e^t - 1}\, dt \\
&= -2 + \left(-1 + \frac{\pi^2}{3}\right) = \frac{\pi^2}{3} - 3.
\end{aligned}$$

The last integral involves *polylogarithms* and can be found in a good table of integrals.

Two solutions of the next problem (310.3) appear in M500 313, and we provide a third possibility.

Example 3. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{3 + 10n + 10n^2 - 6n^4}{(n^2 + n)^4}$.

We have

$$\begin{aligned}
\frac{3 + 10n + 10n^2 - 6n^4}{(n^2 + n)^4} &= \frac{2}{(n+1)^3} - \frac{2}{n^3} - \frac{3}{(n+1)^4} + \frac{3}{n^4} \\
&= \mathcal{L}[t^2 e^{-t}] - \mathcal{L}[t^2] - \frac{1}{2}\mathcal{L}[t^3 e^{-t}] + \frac{1}{2}\mathcal{L}[t^3] \\
&= \mathcal{L}[t^2 e^{-t} - t^2 - \frac{1}{2}t^3 e^{-t} + \frac{1}{2}t^3].
\end{aligned}$$

It follows that

$$\sum_{n=1}^{\infty} \frac{3 + 10n + 10n^2 - 6n^4}{(n^2 + n)^4} = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-nt}(t^2 e^{-t} - t^2 - \frac{1}{2}t^3 e^{-t} + \frac{1}{2}t^3)\, dt$$

$$\begin{aligned}
&= \int_0^\infty (t^2e^{-t} - t^2 - \frac{1}{2}t^3e^{-t} + \frac{1}{2}t^3)\left(\frac{e^{-t}}{1-e^{-t}}\right)dt \\
&= \int_0^\infty \frac{t^2e^{-2t}}{1-e^{-t}}\,dt - \int_0^\infty \frac{t^2e^{-t}}{1-e^{-t}}\,dt - \frac{1}{2}\int_0^\infty \frac{t^3e^{-2t}}{1-e^{-t}}\,dt \\
&\quad + \frac{1}{2}\int_0^\infty \frac{t^3e^{-t}}{1-e^{-t}}\,dt.
\end{aligned}$$

Now the first two integrals can be combined and the third and fourth integrals can be combined. Then simple factorizations leave us with

$$-\int_0^\infty t^2e^{-t}\,dt \;+\; \frac{1}{2}\int_0^\infty t^3e^{-t}\,dt,$$

which, after an appropriate number of integrations by parts, yields 1.

Problem 310.1 in this Journal, with two solutions appearing in M500 312, asks the reader to show that $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n} = \sum_{n=1}^{\infty}\frac{\sin^2 n}{n^2} = \frac{\pi-1}{2}$. This can be done using the Laplace transform (cf. [3]), but with some complexification (pun intended). The general term $\dfrac{\sin n}{n}$ should be written as a product $\sin n \cdot \dfrac{1}{n}$, and only the second factor regarded as a Laplace transform: $\dfrac{1}{n} = \mathcal{L}[1]$. See [3] for further information.

Not every infinite series has terms that correspond to Laplace transforms, but when you can make this connection, the evaluation of such a series reduces to the evaluation of integrals in a straightforward way. Originally the values of the series in Examples 2-4 were attained in a fairly elementary manner (partial fractions and telescoping series), but these solution strategies involved a certain amount of intuition and luck and would not have been applicable, say, to Example 1.

References

[1] Efthimiou, C. J., Finding Exact Values For Infinite Sums, *Mathematics Magazine*, **72**, 45–51.

[2] Lesko, J. P. and Smith, W. D., A Laplace Transform Technique for Evaluating Infinite Series, *Mathematics Magazine*, **76**, 394–398.

[3] Efthimiou, C. J., Trigonometric Series via Laplace Transforms, *Mathematics Magazine*, **79**, 376–379.

Solution 313.3 – Tetrahedron

A tetrahedron has vertices A, B, C, D, and

$$\angle BAD \;=\; \angle BAC \;=\; \angle CAD \;=\; 90^\circ.$$

Show that the face areas $\triangle BCD$, $\triangle BAC$, $\triangle CAD$ and $\triangle DAB$ satisfy

$$(\triangle BCD)^2 \;=\; (\triangle BAD)^2 + (\triangle BAC)^2 + (\triangle CAD)^2.$$

Tommy Moorhouse

Let the vertices of the tetrahedron be at $A = (0,0,0)$, $B = (b,0,0)$, $C = (0,c,0)$ and $D = (0,0,d)$. This arrangement clearly satisfies the conditions set out in the problem. Each of the angles of $\triangle BCD$ is acute and so we can use the following lemma to develop the solution.

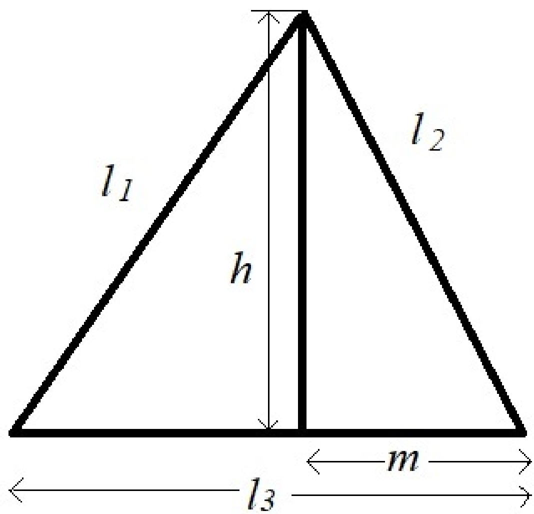


Figure 1: Triangle construction

Lemma The area of a plane triangle with sides l_1, l_2 and l_3 is

$$A_T \;=\; \frac{1}{4}\sqrt{2l_1^2l_2^2 + 2l_1^2l_3^2 + 2l_2^2l_3^2 - l_1^4 - l_2^4 - l_3^4}.$$

Proof From Figure 1 we see that

$$\begin{aligned} m^2 + h^2 &= l_2^2, \\ (l_3 - m)^2 + h^2 &= l_1^2. \end{aligned}$$

Solving for m and h we find

$$\begin{aligned} m &= \frac{l_3^2 + l_2^2 - l_1^2}{2l_3}, \\ h^2 &= \frac{2l_1^2 l_3^2 + 2l_2^2 l_3^2 + 2l_1^1 l_2^2 - l_1^4 - l_2^4 - l_3^4}{4l_3^2}. \end{aligned}$$

The area of the triangle is $hl_3/2$, which is A_T as required. □

Now the triangle areas are

$$\triangle ABC = \frac{bc}{2}, \quad \triangle ADC = \frac{dc}{2} \quad \text{and} \quad \triangle ABD = \frac{bd}{2}.$$

We use Pythagoras' Theorem to set

$$l_1^2 = b^2 + c^2, \quad l_2^2 = d^2 + c^2 \quad \text{and} \quad l_3^2 = b^2 + d^2,$$

where l_1, l_2 and l_3 have been chosen in no particular order. With this choice $A_T = \triangle BCD$. Expanding the terms of $(\triangle BCD)^2$ in terms of b, c and d we find

$$\begin{aligned} 2l_1^2 l_2^2 &= 2(b^4 + b^2 d^2 + c^2 b^2 + c^2 d^2), \\ 2l_1^2 l_3^2 &= 2(b^2 c^2 + b^2 d^2 + c^4 + c^2 d^2), \\ 2l_2^2 l_3^2 &= 2(b^2 c^2 + b^2 d^2 + c^2 d^2 + d^4), \\ l_1^4 &= b^4 + 2b^2 c^2 + c^2, \\ l_2^4 &= b^4 + 2b^2 d^2 + d^4, \\ l_3^4 &= c^4 + 2c^2 d^2 + d^4. \end{aligned}$$

This gives

$$\begin{aligned} (\triangle BCD)^2 &= \frac{1}{16}(2l_1^2 l_2^2 + 2l_1^2 l_3^2 + 2l_2^2 l_3^2 - l_1^4 - l_2^4 - l_3^4) \\ &= \frac{1}{4}(b^2 d^2 + b^2 c^2 + c^2 d^2) \\ &= (\triangle ABC)^2 + (\triangle ABD)^2 + (\triangle ADC)^2, \end{aligned}$$

proving the result.

Problem 316.4 – Goat

Tony Forbes

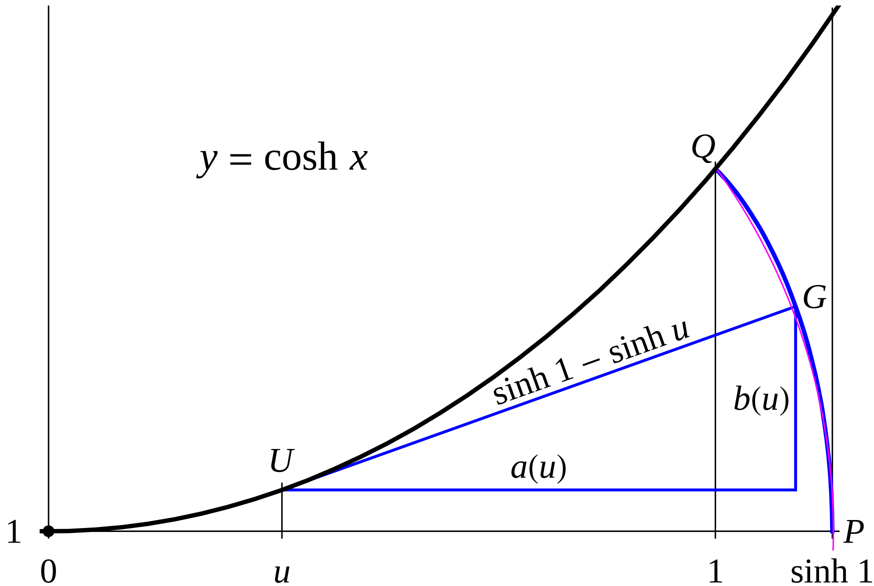
The recreational mathematical literature (M500 included) is sprinkled with problems that involve a goat, a field and the grazing of the one by the other. Sometimes the goat is a horse but always the problem is to determine the area that the goat can access when it is tethered to a fixed point in the field by a rope of constant length. To make things less trivial, there has to be some obstruction within range of the tether, possibly a boundary fence, or a barn of some simple shape, elliptical, rectangular, whatever.

In this problem there is a boundary in the shape of the function $\cosh x$. The goat is tethered to $(0, 1)$ by a rope of length $\sinh 1$ and the said animal is free to graze south of the curve. What area can he access?

A diagram will make this clearer. Assume the goat is north of the line $y = 1$. With the rope taut we see that it hugs the curve from $(0, 1)$ to $U = (u, \cosh u)$ for a length of $\sinh u$ and then leaves U and heads straight for

$$G = (a(u) + u,\ b(u) + \cosh u),$$

where $a(u)$, $b(u)$ and $\sinh 1 - \sinh u$ are the sides of the right-angled triangle shown. Observe that $b(u)/a(u) = \cosh'(u) = \sinh u$. As u goes from 0 to 1, G traces the thick blue curve from P to Q.



I offer this problem to M500 readers because it is doable for an exact answer involving only elementary functions. The cosh function has the interesting property that lengths are readily computable. Indeed, the length along the curve from $(0,1)$ to $(u, \cosh u)$ is

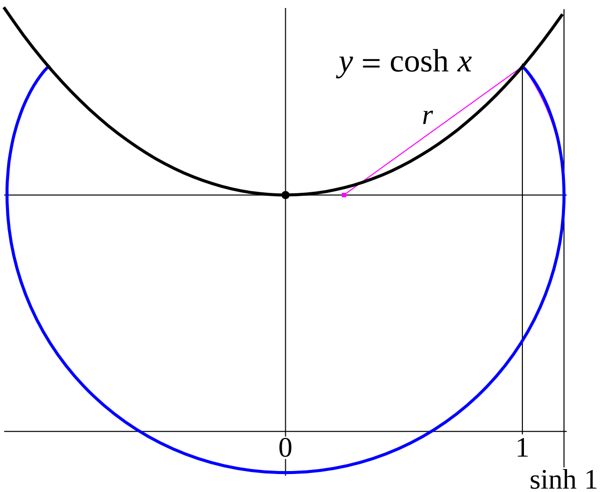
$$\int_0^u \sqrt{1+\left(\frac{d\cosh(x)}{dx}\right)^2}\,dx \;=\; \int_0^u \sqrt{1+\sinh^2(x)}\,dx \;=\; \int_0^u \cosh(x)\,dx \;=\; \sinh u.$$

As a check, you can verify that the area of the part in the region $x \geq 0$, $y \geq 1$ is slightly greater than the area enclosed by the circle passing through P and Q that meets the line $y = 1$ at 90 degrees. This is the feint magenta curve in the picture. The radius of the circle is

$$r \;=\; \frac{2-2e+\cosh 2}{2\,(\sinh 1 - 1)} \;=\; 0.929309,$$

its centre is at $(\sinh(1) - r,\ 0)$, and the area bounded by the circle, the line $y = 1$ and the curve $y = \cosh x$, $0 \leq x \leq 1$ is

$$\int_0^1 (\cosh u - 1)du + \arcsin\left(\frac{\cosh 1 - 1}{r}\right)\frac{r^2}{2} - \frac{(\cosh 1 - 1)(r - \sinh 1 + 1)}{2}$$
$$= \; 0.239935.$$



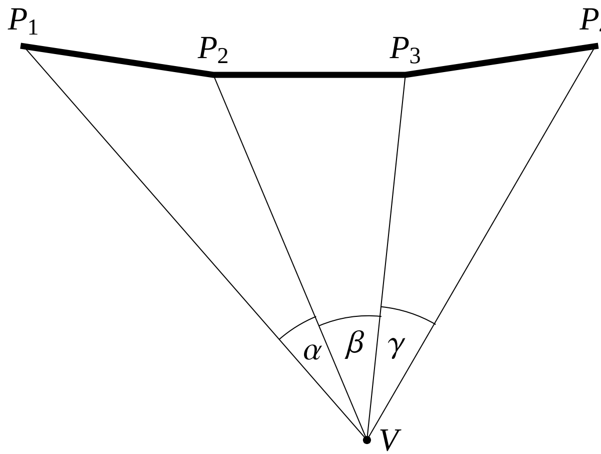
Problem 316.5 – Prism

Tony Forbes

This is like Problem 315.6 – Prism in that you are to determine from limited observations the n for a regular n-sided prism. Suppose you are standing on the ground at point V outside the prism such that four consecutive corners P_1, P_2, P_3 and P_4 are visible. Let

$$\alpha \;=\; \angle P_1VP_2, \quad \beta \;=\; \angle P_2VP_3, \quad \gamma \;=\; \angle P_3VP_4.$$

Compute n as a function of α, β and γ.



For instance, if $\alpha = 18.29°$, $\beta = 28.84°$ and $\gamma = 24.11°$, then n must be 42.

Ideally we want an exact expression for n. However, in the example the angles are given to only two decimal places and yet the precision is sufficient to determine n exactly, as you can see if you set up the relevant equations and solve them numerically.

The problem as stated looks fiendish. Perhaps one can begin by working a special case—where the viewpoint V is on the perpendicular bisector of P_2P_3. Then $\gamma = \alpha$ and, assuming the prism side length $|P_1P_2|$ is equal to 2, we have

$$|P_2V| \;=\; \operatorname{cosec}\frac{\beta}{2} \qquad \text{and} \qquad \angle VP_2P_1 \;=\; \frac{2\pi}{n} + \frac{\pi}{2} + \frac{\beta}{2}.$$

Hence, by the sine rule,

$$\frac{\sin\alpha}{2} = \frac{\sin(\angle VP_1P_2)}{|P_2V|} = \frac{\sin\left(\dfrac{\pi}{2} - \alpha - \dfrac{\beta}{2} - \dfrac{2\pi}{n}\right)}{\operatorname{cosec}\dfrac{\beta}{2}},$$

which can be solved to obtain

$$n = \frac{2\pi}{\dfrac{\pi}{2} - \alpha - \dfrac{\beta}{2} - \arcsin\left(\dfrac{\sin\alpha}{2\sin(\beta/2)}\right)}.$$

This might not work when V is at infinity.

Problem 316.6 – 6-Regular graph construction

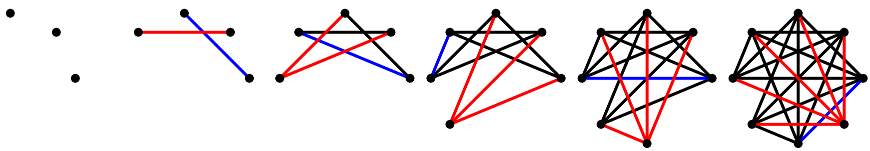
Tony Forbes

Given a 5-regular graph on 12 vertices $\{1, 2, \ldots, 12\}$, one can construct a 6-regular graph on 13 vertices as follows. Take the 5-regular graph and find three pairs of non-adjacent vertices, $\{v_1, v_2\}$, $\{v_3, v_4\}$, $\{v_5, v_6\}$, such that v_1, v_2, …, v_6 are distinct. Add them as edges. Then add edges $\{w, 13\}$ for the remaining vertices $w \in \{1, 2, \ldots, 12\} \setminus \{v_1, v_2, \ldots, v_6\}$.

Can all 13-vertex, 6-regular graphs be obtained in this way? Alternatively, given a 13-vertex, 6-regular graph, is it always possible to remove a vertex v as well as three edges that span six vertices other than v?

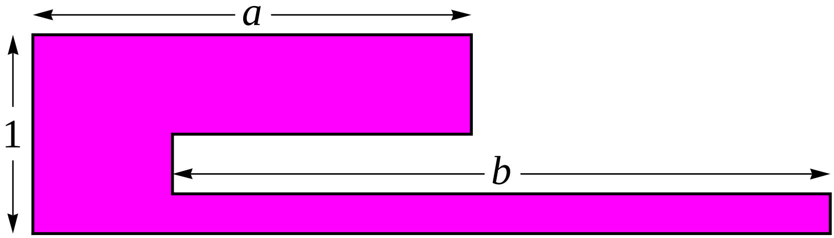
A general construction suggests itself. Take a d-regular graph with $2m + d + 1$ vertices. Add m edges that span distinct vertices, v_1, v_2, …, v_{2m}. Add a new vertex w and a further $d+1$ edges $\{w, v_{2m+1}\}$, $\{w, v_{2m+2}\}$, …, $\{w, v_{2m+d+1}\}$. The result is a $(d + 1)$-regular graph with $2m + d + 2$ vertices.

It would be interesting to see what is created when you start the construction process with an empty graph. For example, here are the first six stages of a sequence with $m = 1$. An available edge chosen at random is coloured blue. The edges incident with the new vertex are shown in red.



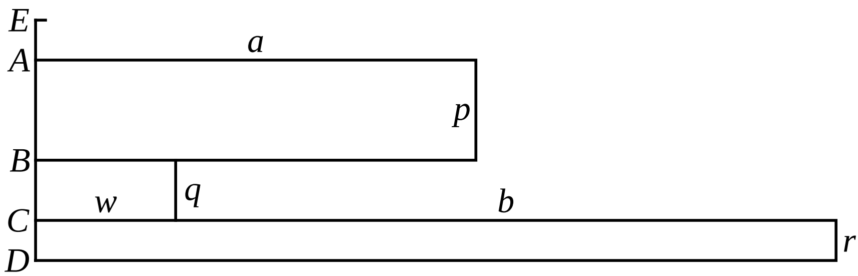
Solution 312.3 – Area and perimeter

What's the area of the shape? What about the perimeter?



Ted Gore

The following sketch is a reasonably accurate copy of that in the question and is marked up to make clear the steps presented below.



To determine the area of the figure we need to find p, q, r, w, a and b. This can be done using a compass and straight edge. The following steps are only meaningful when used on an accurate scale copy of the figure.

Step 1. We know that $AD = 1$. With the compass needle on B draw a circle through A. This also passes through D so that $p = AB = 1/2$ and $q + r = BD = 1/2$.

Step 2. Draw a horizontal straight line and mark a point X roughly in the middle of the line. With the compass width set to r mark a point Y a distance of $3r$ to the left of X. With the compass width set to q mark a point Z at a distance $2q$ to the right of X. With the compass needle on X draw a circle that passes through Y. It also passes through Z so that $3r = 2q$. Since we know that $q + r = 1/2$ we calculate that $3r/2 + r = 1/2$ so that $r = 1/5$ and $q = 3/10$.

Step 3. Set the compass width to AD and with the needle on C mark the point E. Then $AE = r$. Move the needle to B and set the compass width to w. A circle drawn with this radius passes through E so that $w = p + r = 7/10$.

Step 4. We have that $AD = 1$ and we can bisect p to give a length of 1/4. With these and the other lengths that we have found we can measure $a = 2.25$ and $b = 3.4$.

Result. The area of the figure is

$$ap + wq + (b + w)r \;=\; \frac{a}{2} + \frac{21}{100} + \frac{b}{5} + \frac{7}{50} \;=\; \frac{a}{2} + \frac{b}{5} + \frac{35}{100} \;=\; 2.155.$$

The perimeter is $1 + (p + q + r)$ for the vertical lines and $a + (a - w) + b + (b + w)$ for the horizontal giving $2 + 2a + 2b = 13.3$.

Note that the figure can be generated from r alone using valid straight edge and compass operations and that the actual size of the figure depends only on the initial choice of r.

Tony Forbes

I spent several days on and off thinking about this problem until it occurred to me (or somebody told me—I can't remember) that the area of the shape does not have a value that depends only on a and b. All I can say is that

$$\text{area} \;=\; ap + br + w\,(1 - p),$$

where the parameters w, p and r can take any sensible values.

After I had succeeded in 'solving' the area problem I was only a little bit astonished to discover that the shape's perimeter *does* have a calculable value. We make the assumption that for some $\epsilon > 0$, a disc of radius ϵ can move freely around inside the entire shape as depicted. Using the notation established on page 20 we have $w > 0$ and

$$\text{perimeter} \;=\; 1 + a + p + (a - w) + (1 - p - r) + b + r + (b + w).$$

Then all those extra parameters vanish to give

$$\text{perimeter} \;=\; 2(1 + a + b).$$

The degenerate case ($w = 0$) is left to the reader.

Contents

04.00

Problem 316.7 – Cubes

A $5 \times 5 \times 5$ cube is partitioned into 125 $1 \times 1 \times 1$ subcubes of which five are coloured red. The other 120 are coloured something other than red. Each $1 \times 5 \times 5$ slice of the cube contains exactly one red subcube. In how many ways can this be done?

Front cover A tetrahedron with transparent faces, the example on page 9.