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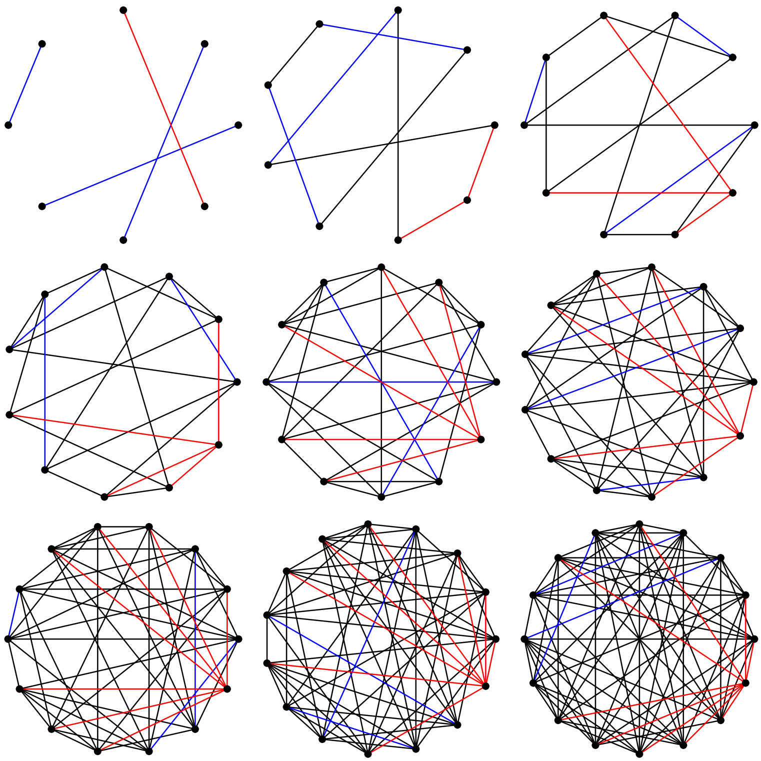
M500 317



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: `m500.org.uk`.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The Revision Weekend is a residential Friday to Sunday event providing revision and examination preparation for both undergraduate and postgraduate students. For details, please go to the Society's website.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, please go to the Society's website.

Editor – *Tony Forbes*

Jeremy Roger Humphries

We are very sorry to have to tell you that Jeremy Humphries has died. He was diagnosed with cancer in 2022, and in December last year he collapsed and had to be taken to hospital. He was able to spend his last days at home, which was what he most wanted. He died peacefully on the 26th of January 2024.

Jeremy was a member of the M500 Society from its early days. He joined the M500 team as Problems Editor in 1977. He was M500 Editor from 1981 to 1998 and remained on the magazine's editorial board until his death. Many Open University mathematics students will remember Jeremy as the organizer of the Revision Weekend, a post which he held in 1987 and from 1996 to 2012. He chaired the M500 Committee from 1982 to 1987 and was Membership Secretary from 2012 to 2021.

He frequently contributed to the M500 magazine, especially delighting its readers with his skilful blending of mathematics and poetry.

Jeremy's passing is a great loss to the M500 Society. He provided a valuable service to the M500 Committee for which he will be remembered with affection. He will be sadly missed. We offer our sympathy to Jeremy's friends and family, especially his widow, Rose.

Packing 1:2:6 bricks into cubes

Kira Bhana and Tony Forbes

Let $M(n)$ denote the maximum number of $1 \times 2 \times 6$ bricks that you can pack into an $n \times n \times n$ cube. In M500 **315** we saw that

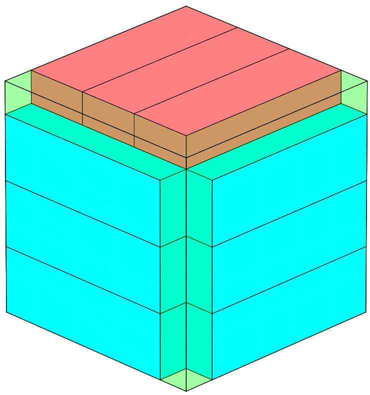
$$M(n) \;=\; \begin{cases} n^3/12 & \text{if } n \equiv 0 \pmod 6, \\ \lfloor n^3/12 \rfloor & \text{if } n \equiv 2 \pmod 6, \\ \lfloor n^3/12 \rfloor - 1 & \text{if } n \equiv 4 \pmod 6) \text{ and } n \geq 10, \end{cases}$$

but all we could manage for odd $n \geq 7$ were non-equal upper and lower bounds.

n	7	9	11	13	15	17	19	21
$\lfloor n^3/12 \rfloor$	28	60	110	183	281	409	571	771
unused	7	9	11	1	3	5	7	9
upper bound	28	60	110	183	280	409	571	771
unused	7	9	11	1	15	5	7	9
lower bound	27	57	108	180	276	405	567	765
unused	19	45	35	37	63	53	55	81

Here we report a small amount of progress. The case $n = 7$ is settled and we have a slight improvement for $n = 12k + 9$.

Perhaps we should make it clear that we are considering only situations where the bricks are orientated so that their faces lie on the grid-planes that divide the $n \times n \times n$ cube into n^3 subcubes. In other words, a brick must occupy exactly 12 of the $1 \times 1 \times 1$ subcubes into which the $n \times n \times n$ cube is naturally divided. Let us call these *regular* packings.



Obviously non-regular packings do exist. For example, there is a certain amount of flexibility with the positioning of some of the 27 bricks in the $7 \times 7 \times 7$ cube shown on the right. However, in all cases we have looked at it is possible to make a non-regular packing regular. Anyway, we shall redefine $M(n)$ to be the maximum possible number of $1 \times 2 \times 6$ bricks in a regular packing of an $n \times n \times n$ cube.

Theorem 1 *If $n \equiv 9 \pmod{12}$, then $M(n) \le \lfloor n^3/12 \rfloor - 1$.*

Proof Suppose $n \equiv 9 \pmod{12}$ and let the cube occupy

$$[0,n] \times [0,n] \times [0,n]$$

in Euclidean 3-dimensional space. Suppose it is packed with $\lfloor n^3/12 \rfloor = (n^3-9)/12$ $1 \times 2 \times 6$ bricks to leave 9 of its subcubes unoccupied.

The proof is the same as that of Theorem 1 on page 3 of M500 **315** except that the ring is different. We represent a point (a,b,c) by the monomial $x^a y^b z^c$. Then the collection of n^3 subcubes is represented by

$$C = \sum_{a=0}^{n-1} \sum_{b=0}^{n-1} \sum_{c=0}^{n-1} x^a y^b z^c,$$

a brick located at (a,b,c) is represented by a polynomial

$$B = x^a y^b z^c \, B_j(x,y,z), \quad j \in \{1,2,\ldots,6\},$$

where

$$\begin{aligned} B_1 &= (1+x+\cdots+x^5)(1+y),\ B_2 = (1+x+\cdots+x^5)(1+z),\\ B_3 &= (1+y+\cdots+y^5)(1+x),\ B_4 = (1+y+\cdots+y^5)(1+z),\\ B_5 &= (1+z+\cdots+z^5)(1+x),\ B_6 = (1+z+\cdots+z^5)(1+y), \end{aligned}$$

and the holes at locations (a_h, b_h, c_h), $h = 1,2,\ldots,9$ are represented by

$$U = \sum_{h=1}^{9} x^{a_h} y^{b_h} z^{c_h}.$$

In the rest of the proof we work in $\mathbb{Z}_{365}$, the ring of integers modulo 365. Put $x = y = z = 9$. The powers of 9 are 1, 9, 81, 364, 356, 284. Also $9^6 = 1$ and

$$1 + 9 + 9^2 + 9^3 + 9^4 + 9^5 = 0.$$

Then

$$C = \left(1 + 9 + 9^2 + \cdots + 9^{n-1}\right)^3 = (1 + 9 + 81)^3 = 211,$$

$$B = 0,$$

and for the packing to exist we must have $C = B + U$. Hence

$$U = 211 \quad \text{for some } a_h, b_h, c_h \in \{0, 1, \ldots, n-1\},\ h = 1, 2, \ldots, 9. \tag{1}$$

But U is a sum of 9 powers of 9, and a straightforward computation shows that no such sum is equal to 211, contradicting (1). □

We don't know if the number $365 = 5 \times 73$ has any special significance. It was found simply by looking for it.

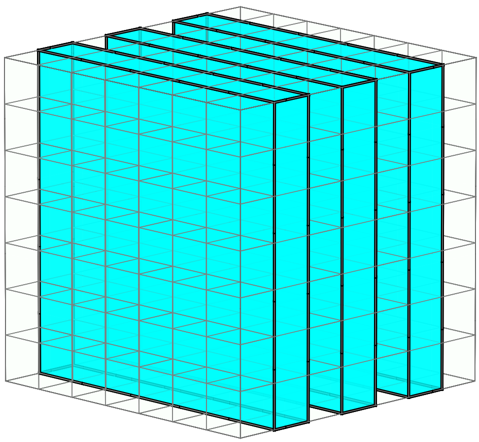
There are 25 numbers modulo 365 that are not representable as sums of 9 powers of 9, namely

$$\begin{gathered} 0, 2, 8, 10, 18, 64, 72, 74, 80, 82, 90, 154, 162, \\ 203, 211, 275, 283, 285, 291, 293, 301, 347, 355, 357, 363, \end{gathered}$$

which occur as 0 and twelve $\pm$ pairs. Fortunately for our proof the list includes 211. One could argue that our probability of success is $25/365 = 0.068$, which is rather small. So we cannot help thinking that something other than coincidence is at work.

We turn now to $n = 7$. A *slice* of the $7 \times 7 \times 7$ cube is a subset in the form of a $1 \times 7 \times 7$ cuboid that may be orientated in any of the three axis directions. A slice is *internal* if neither of its 7×7 faces is adjacent to a cube face. It is clear that there are 21 distinct slices of which 15 are internal.

Lemma 1 *Suppose there is a regular packing of* 28 $1 \times 2 \times 6$ *bricks in a* $7 \times 7 \times 7$ *cube. Then each of the* 15 *internal slices of the cube contains exactly one hole.*



Proof Assume the packing of 28 bricks exists and let T be the union of three distinct parallel internal slices. The picture on page 3 shows a possible arrangement.

A brick laid with its long axis parallel to the slices occupies 12, 6 or 0 subcubes of T. A brick laid orthogonal to the slices spans all three and occupies 6 subcubes of T. These are the only possibilities.

Let b_r denote the number of bricks that occupy r subcubes of T, and let u be the number of holes in T. Since T consists of 147 subcubes we have

$$12\,b_{12} + 6\,b_6 \;=\; 147 - u, \quad 0 \le u \le 7,$$

and therefore $u = 3$.

Denote the slices of T by T_1, T_2, T_3, and let T_4, T_5 be the other two internal slices parallel to those of T.

Suppose one slice, which we may assume is T_1, contains more than one hole. Then the previous argument with T_1, T_4 and T_5 implies $T_4 \cup T_5$ contains at most one hole. But then $T_2 \cup T_4 \cup T_5$ contains at most two holes, a contradiction by the same argument with T_2, T_4 and T_5.

Hence there must be exactly one hole in each of $T_1, T_2, \ldots, T_5$. Similarly for other ten internal slices. $\square$

Theorem 2 *Any regular packing of* $1 \times 2 \times 6$ *bricks in a* $7 \times 7 \times 7$ *cube has at most* 27 *bricks.*

Proof Take the obvious packing of eighteen $1 \times 2 \times 6$ bricks in a $6 \times 6 \times 6$ cube and attach three bricks to each of three mutually orthogonal faces to get a packing of 27 bricks, as illustrated on page 1. Thus $M(7) \ge 27$.

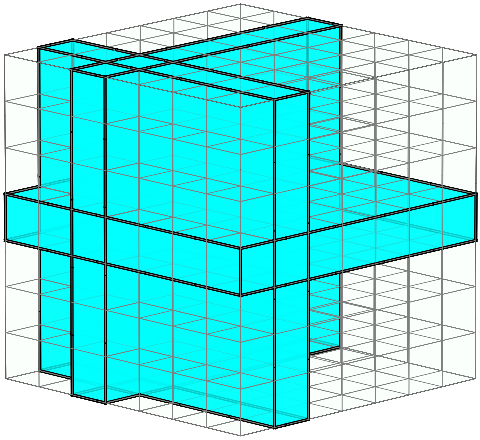
Assume the cube is packed with 28 bricks, and recall that the packing leaves 7 holes. Let U be the union of three mutually orthogonal internal slices. Then U consist of 127 subcubes with a single subcube at the intersection of the three slices. The picture on page 5 shows a possible arrangement.

Depending on how it is laid, a brick must occupy 12, 7 or 2 subcubes of U. Let b_r denote the number of bricks that occupy r subcubes of U, and let u be the number of holes in U. Then

$$\begin{aligned} 12\,b_{12} + 7\,b_7 + 2\,b_2 &= 127 - u, \\ b_{12} + b_7 + b_2 &= 28. \end{aligned}$$

Multiplying the second equality by 7 and subtracting gives

$$-5\,b_{12} + 5\,b_2 \;=\; 69 + u, \text{ which implies } u = 1 \text{ or } 6.$$



But $u \leq 3$ follows from Lemma 1. Therefore $u = 1$ and, again by Lemma 1, the hole must occur at the intersection of the three slices. However, each of the three slices making up U could have been any of five parallel internal slices. Therefore the entire central $5 \times 5 \times 5$ cube is full of holes, a blatant contradiction. $\square$

Problem 317.1 – Product

Tony Forbes

Let $m \geq 2$ be an integer, assume $|q| < 1$, and consider the expansion of the following infinite product into a power series with coefficients $a_m(n)$:

$$P(m) \;=\; \prod_{n=0}^{\infty} \frac{1}{1+q^{m^n}} \;=\; \sum_{n=0}^{\infty} a_m(n) q^n.$$

For example,

$$P(3) \;=\; 1-q+q^2-2q^3+2q^4-2q^5+3q^6-3q^7+3q^8-5q^9+5q^{10}-5q^{11}+\ldots.$$

Show that $a_m(n) \in \{0, -1, 1\}$ when m is even, or find a counter-example. We would also be interested if you can obtain a formula for $a_m(n)$. If that's too difficult, try restricting m to powers of 2. For instance, $P(2) = 1 - q$, which can be proved by multiplying

$$(1+q)(1+q^2)(1+q^4)(1+q^8)\ldots$$

by $1 - q$.

Packing $6 \times 2 \times 1$ bricks into a $7 \times 7 \times 7$ box

Graham Lovegrove

Theorem 1 *The maximum number of* $6 \times 2 \times 1$ *bricks that can be placed in a* $7 \times 7 \times 7$ *box by a regular packing is* 27 *(leaving* 19 *spaces).*

Proof Recall from page 1 that a regular packing of $6 \times 2 \times 1$ bricks is a packing where each brick occupies exactly 12 of the 343 unit cubes into which the $7 \times 7 \times 7$ box is naturally divided.

We divide the space of the whole box into $1 \times 1 \times 1$ cubes, with the lower corner of the box at $(0, 0, 0)$. We represent the cube at coordinates (i, j, k) by the monomial $x^i y^j z^k$; so the whole box is represented by the polynomial

$$\text{Box}(x, y, x) \;=\; \sum_{i,j,k \,=\, 0,1,\ldots,6} x^i y^j z^k \;=\; \left(\sum_{i=0}^{i=6} x^i\right)\left(\sum_{j=0}^{j=6} y^j\right)\left(\sum_{k=0}^{k=6} z^k\right).$$

Now the bricks can be in six different orientations. We represent a generically oriented brick with its lower corner at $(0, 0, 0)$ by the polynomial

$$\text{Brick}_{u,v} \;=\; \left(\sum_{i=0}^{i=5} u^i\right)(1 + v)\,,$$

where the length of the brick is along the u dimension, the width along the v dimension, and the thickness (1) along the remaining dimension.

We want to express $\text{Box}(x, y, z)$ as something of the form

$$\sum_{u,v \,\in\, \{x,y,z\},\; u \neq v} A_{u,v}\text{Brick}_{u,v} + H(x, y, z),$$

where the As are all polynomials in x, y, z where all the coefficients are 1s because no two bricks can occupy the same unit cube. Notice also that $\text{Box}(x, y, z)$ has the same property. We want $H(x, y, z)$ to have the smallest number of terms, so that the bricks occupy the largest possible volume.

This looks difficult in general, but if we just want to know the size of the best packing we can do some substitution to simplify the problem. If we set $y = z = x$, this will preserve the correct number of terms in the polynomial H and the total degree of each term, but not the exact form. So, setting $y = z = x$, we have

$$H(x, x, x) \;\equiv\; \text{Box}(x, x, x) \;(\text{mod Brick}_{x,x}).$$

To simplify the algebra, we can set $S = \left(\sum_{i=0}^{i=5} x^i\right)$; so

$$\mathrm{Brick}_{x,x} \;=\; S(1+x)$$

and

$$\mathrm{Box}(x,x,x) \;=\; \left(S+x^6\right)^3 \;=\; S^3+3x^6S^2+3x^{12}S+x^{18}.$$

Now, note that S has a factor of $(1+x)$, i.e.

$$S \;=\; (1+x^2+x^4)(1+x),$$

so that

$$S^2 \;\equiv\; 0 \pmod{\mathrm{Brick}_{xx}}.$$

Hence the first two terms in the expansion above are composed of complete bricks, leaving only the last two as the remainder:

$$H(x,x,x) \;=\; 3x^{12}S+x^{18}.$$

There are 19 terms in this remainder. We can't extract any more copies of $S(1+x)$ from this without introducing minus signs, so the smallest hole has size 19. □

Since

$$S \;=\; 1+x+x^2+x^3+x^4+x^5,$$

the remainder represents 3 lines of 6 unit cubes plus a singleton. One packing leaves the spaces along three perpendicular edges with the single block in the corner between them.

A remaining question is: do these three lines have to be perpendicular? It is obvious by playing with the bricks that the lines don't all need to be on an outside edge, and that the lines can be broken up by shifting some bricks up and down. I am personally convinced that they have to be perpendicular, but I don't know how to prove it (yet).

Problem 317.2 – Sixes

Given an ordinary die marked 1, 2, …, 6, what's the probability that $6k$ throws will produce at least k sixes? What is the limit of this probability as k tends to infinity?

More generally, given a spinning device that generates equally likely random integers 1, 2, …, n, $n \geq 6$, what's the probability that nk spins will produce at least k sixes? What is the limit of this probability as k tends to infinity?

Covariant derivative on the 3-sphere

Tommy Moorhouse

Introduction The covariant derivative of a tensor field on S^3 will be described in terms of a projection operator derived from the embedding of S^3 in $\mathbb{R}^4$. The projection is from the ambient tangent space $T\mathbb{R}^4$ onto TS^3, the tangent space of the sphere. While the covariant derivative is intrinsic to S^3 (it does not depend on the embedding into $\mathbb{R}^4$) it finds a natural definition in the context of the embedding, which we will use to recover a standard result. The notation is intended to be similar to that used in the old OU set text [O'Neill].

Embedding S^3 in $\mathbb{R}^4$. The unit 3-sphere can be embedded in $\mathbb{R}^4$ as the set of points defined by the equation

$$\sum_{a=1}^{4}(x^a)^2 - 1 \;=\; 0.$$

We will normally use the summation convention to write this as $\eta_{ab}x^a x^b = 1$ with the flat metric $\eta_{ab} = \operatorname{diag}(1,1,1,1)$. The tangent space at a point $x \in S^3$ can be thought of as a subspace of the tangent space to $\mathbb{R}^4$ at the same point. A vector $v_x \in T_x\mathbb{R}^4$ can be projected onto this subspace by setting

$$(Pv)_x \;=\; v_x - (\mathbf{x}\cdot v_x)\mathbf{x}.$$

Here $\mathbf{x}$ is the position vector of the basepoint of v_x, which lies on S^3. It is a straightforward check to confirm that $(Pv)_x$ is tangent to S^3, so orthogonal to $\mathbf{x}$. We will also use the three angular variables χ, ϑ and φ on S^3, analogous to ϑ and φ on S^2. Specifically, we let

$$x \;=\; (\cos\chi, \sin\chi\cos\vartheta, \sin\chi\sin\vartheta\cos\varphi, \sin\chi\sin\vartheta\sin\varphi).$$

Basis vectors We now consider a basis for TS^3. The vector field $\mathbf{x}_\chi$ is defined as

$$\mathbf{x}_\chi \;=\; \frac{\partial}{\partial\chi} \;=\; \frac{\partial x^a}{\partial\chi}\frac{\partial}{\partial x^a}.$$

This expresses $\mathbf{x}_\chi$ both intrinsically as a vector field on S^3 and as a field on $\mathbb{R}^4$. The two other vector fields, $\mathbf{x}_\vartheta$ and $\mathbf{x}_\varphi$ are similarly constructed. These vectors are tangent to S^3 since, for example,

$$0 \;=\; \frac{\partial(x^a x_a)}{\partial\chi} \;=\; 2x^a\frac{\partial x_a}{\partial\chi} \;=\; 2\mathbf{x}\cdot\mathbf{x}_\chi,$$

where the dot denotes the scalar product in $T_x\mathbb{R}^4$ and $\mathbf{x}$ is considered to be the unit radial vector at x. The point of all this is to express any tangent vector at $x \in S^3$ as a sum of the basis vectors:

$$v \;=\; v^\chi \mathbf{x}_\chi + v^\vartheta \mathbf{x}_\vartheta + v^\varphi \mathbf{x}_\varphi.$$

Here the v^μ are the components of v in the basis consisting of the vectors x_μ. Thus the v^μ are real numbers at $x \in S^3$, and functions of the base point in a neighbourhood of x.

The covariant derivative We denote the coordinate derivatives in $\mathbb{R}^4$ by ∂_a. We are interested in expressions such as $\nabla_V W = P(V \cdot \partial W)$, the covariant derivative of W along V, where both V and W are tangent to S^3. This effectively means that we should look at derivatives of the form

$$\nabla_\mu \mathbf{x}_\nu.$$

Here we have used the subscript μ to denote x_μ to avoid double subscripts. These derivatives are easily constructed, since $\mathbf{x}_\mu \cdot \partial = \partial_\mu$ by definition. We can do the differentiation, and the projection simply involves dropping any terms proportional to $\mathbf{x}$. As a couple of examples let us consider

$$\begin{aligned}
\nabla_\chi \mathbf{x}_\chi &= P(\mathbf{x}_\chi \cdot \partial \mathbf{x}_\chi) \\
&= P(\partial_\chi \mathbf{x}_\chi) \;=\; P(-\mathbf{x}) \;=\; 0, \\
\nabla_\varphi \mathbf{x}_\varphi &= P(\partial_\varphi \mathbf{x}_\varphi) \\
&= P((0, 0, -\sin\chi \sin\vartheta \cos\varphi, -\sin\chi \sin\vartheta \sin\varphi)) \\
&= P(-\sin^2\chi \sin^2\vartheta \mathbf{x} - \sin\chi \cos\chi \sin^2\vartheta \mathbf{x}_\chi - \sin\vartheta \cos\vartheta \mathbf{x}_\vartheta) \\
&= -\sin\chi \cos\chi \sin^2\vartheta \mathbf{x}_\chi - \sin\vartheta \cos\vartheta \mathbf{x}_\vartheta.
\end{aligned}$$

These expressions lead to the Christoffel symbols usually used to write down covariant derivatives. Take v as above. Then define the Christoffel symbols as

$$\nabla_\mu \mathbf{x}_\nu \;=\; \Gamma^\sigma_{\mu\nu} \mathbf{x}_\sigma.$$

Note the sum over the index σ. The usual expression for the covariant derivative is then recovered:

$$\begin{aligned}
\nabla_\mu v &= P(\partial_\mu v^\nu \mathbf{x}_\nu + v^\nu \nabla_\mu \mathbf{x}_\nu) \\
&= P(\partial_\mu v^\nu \mathbf{x}_\nu + v^\nu \Gamma^\sigma_{\mu\nu} \mathbf{x}_\sigma) \\
&= (\partial_\mu v^\nu + v^\sigma \Gamma^\nu_{\mu\sigma}) \mathbf{x}_\nu \\
&\equiv (\nabla_\mu v)^\nu \mathbf{x}_\nu.
\end{aligned}$$

Here I have relabelled dummy indices (repeated upper and lower indices) to group the terms into the covariant derivative.

Thus the covariant derivative of a tangent vector along a second tangent vector is the projection of the 'flat' derivative in $\mathbb{R}^4$ onto TS^3.

Projection operator and induced metric The projection operator P may be written in index form as

$$P^a{}_b \;=\; \delta^a{}_b - \mathbf{x}^a \mathbf{x}_b.$$

We can rewrite this in terms of the basis vectors, since it projects any vector onto a sum of the basis vectors:

$$P^a{}_b \;=\; \frac{1}{|\mathbf{x}_\chi|^2}\mathbf{x}^a_\chi \mathbf{x}_{\chi b} + \frac{1}{|\mathbf{x}_\vartheta|^2}\mathbf{x}^a_\vartheta \mathbf{x}_{\vartheta b} + \frac{1}{|\mathbf{x}_\varphi|^2}\mathbf{x}^a_\varphi \mathbf{x}_{\varphi b}.$$

This can be checked directly using the defining properties $P\mathbf{x}_\mu = \mathbf{x}_\mu, P\mathbf{x} = 0$. We can also see that the induced metric on S^3 is

$$g_{\mu\nu} \;=\; \mathbf{x}_\mu \cdot \mathbf{x}_\nu.$$

A defining property of the covariant derivative is that it preserves the metric. We can derive this from our definition. Since $g_{\mu\nu}$ is just a function for fixed μ and ν we have $\nabla_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu}$. But in terms of the expression for $g_{\mu\nu}$ as a scalar product of basis vectors

$$\begin{aligned} \nabla_\sigma g_{\mu\nu} &= \nabla_\sigma(\mathbf{x}_\mu)\cdot \mathbf{x}_\nu + \mathbf{x}_\mu \cdot \nabla_\sigma \mathbf{x}_\nu \\ &= \Gamma^\rho_{\sigma\mu}\mathbf{x}_\rho \cdot \mathbf{x}_\nu + \mathbf{x}_\mu \cdot \mathbf{x}_\rho \Gamma^\rho_{\sigma\nu} \\ &= g_{\rho\nu}\Gamma^\rho_{\sigma\mu} + g_{\mu\rho}\Gamma^\rho_{\sigma\nu}. \end{aligned}$$

This tells us that

$$\partial_\sigma g_{\mu\nu} - g_{\rho\nu}\Gamma^\rho_{\sigma\mu} - g_{\mu\rho}\Gamma^\rho_{\sigma\nu} \;=\; 0.$$

This is the definition of the covariant derivative of the metric tensor found in standard texts. It is the starting point for the derivation of an expression for $\Gamma^\rho_{\mu\nu}$ in terms of the induced metric: form the three equations that result from cyclic permutations on σ, μ and ν, subtract the third from the sum of the first two and contract with $g^{\mu\lambda}$.

Reference

[O'Neill] Barrett O'Neil, *Elementary Differential Geometry*, Academic Press, 1966.

Solution 314.10 – Limit

Show that $n/(n!)^{1/n} \to e$ as $n \to \infty$.

Henry Ricardo

Proof 1. Let $x_n = n/(n!)^{1/n}$. Then

$$\ln x_n = \ln n - \frac{1}{n}\ln n! = \frac{1}{n}(n\ln n - \ln n!) = -\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}\right),$$

which is a right Riemann sum for $-\int_0^1 \ln x\,dx = 1$. It follows that $\lim_{n\to\infty} x_n = e$.

Proof 2. It is a known result in analysis (the Cauchy–d'Alembert criterion) that $\lim_{n\to\infty} x_{n+1}/x_n = x$ with $x_n > 0$ for all n implies $\lim_{n\to\infty} \sqrt[n]{x_n} = x$. Letting $x_n = n^n/n!$, we see that

$$\frac{x_{n+1}}{x_n} = \frac{\dfrac{(n+1)^{n+1}}{(n+1)!}}{\dfrac{n^n}{n!}} = \left(1+\frac{1}{n}\right)^n \to e.$$

Therefore $\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} n/\sqrt[n]{n!} = e$.

Proof 3. It is a known result in analysis that $\lim_{n\to\infty} x_n = x$ with $x_n > 0$ for all n implies

$$\lim_{n\to\infty} \sqrt[n]{x_1x_2\cdots x_n} = x. \tag{1}$$

Letting $x_n = (1+1/n)^n > 0$ for all n, we have by (1)

$$\begin{aligned}
e &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\sqrt[n]{x_1x_2\cdots x_n} \\
&= \lim_{n\to\infty}\sqrt[n]{\left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\cdots\left(\frac{n+1}{n}\right)^n} \\
&= \lim_{n\to\infty}\sqrt[n]{\frac{(n+1)^n}{n!}} \\
&= \lim_{n\to\infty}\frac{n+1}{\sqrt[n]{n!}} = \lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}} + \lim_{n\to\infty}\frac{1}{\sqrt[n]{n!}} = \lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}.
\end{aligned}$$

(See Proof 6 for the fact that $(n!)^{1/n} \to \infty$ as $n \to \infty$.)

Proof 4. Stirling's formula gives us

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}} \left(\frac{e}{n}\right)^n = 1.$$

Consequently,

$$\lim_{n\to\infty} \frac{n}{(n!)^{1/n}} = \lim_{n\to\infty} \frac{e}{(2\pi n)^{1/2n}} = e \cdot \lim_{n\to\infty} \frac{1}{(2\pi)^{1/2n} \cdot n^{1/2n}} = e$$

since $n^{1/2n} \to 1$ is easily proved by the AGM inequality and the squeeze theorem:

$$\begin{aligned} 1 < n^{1/2n} = (n^{n/2})^{1/n^2} &< \frac{\overbrace{1 + 1 + \cdots + 1}^{n^2 - n \text{ terms}} + \overbrace{\sqrt{n} + \cdots + \sqrt{n}}^{n \text{ terms}}}{n^2} \\ &= \frac{n^2 - n + n\sqrt{n}}{n^2} = 1 - \frac{1}{n} + \frac{1}{\sqrt{n}}. \end{aligned}$$

Proof 5. If $x_n \geq 0$, then the following inequalities hold:

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \leq \liminf_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

Letting $x_n = n^n/n!$, it follows that

$$\frac{x_{n+1}}{x_n} = \frac{\dfrac{(n+1)^{n+1}}{(n+1)!}}{\dfrac{n^n}{n!}} = \left(1 + \frac{1}{n}\right)^n \to e$$

and so

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} = \limsup_{n\to\infty} \frac{x_{n+1}}{x_n} = e \quad \text{and} \quad \sqrt[n]{x_n} \to e.$$

Proof 6. First we establish that $(n!)^{1/n} \to \infty$ as $n \to \infty$:

$$\begin{aligned} (n!)^{1/n} = \exp\left(\frac{1}{n} \ln n!\right) &= \exp\left(\frac{1}{n} \sum_{k=1}^{n} \ln k\right) \\ &\geq \exp\left(\frac{1}{n} \int_1^n \ln x \, dx\right) \end{aligned}$$

$$= \exp\left(\frac{n \ln n - n + 1}{n}\right) = \frac{n}{e} \cdot \sqrt[n]{e}.$$

Therefore $(n!)^{1/n} \to \infty$. Now let $a_n = n$ and $b_n = (n!)^{1/n}$. It is easy to see that $\{b_n\}$ is a positive increasing sequence and

$$(a_{n+1} - a_n)/(b_{n+1} - b_n) = 1/(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}).$$

But it is known (Traian Lalescu, 1900) that

$$\lim_{n\to\infty} \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} = 1/e.$$

Thus we may apply the Cesàro–Stolz lemma to conclude that

$$\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} n/\sqrt[n]{n!} = 1/(1/e) = e.$$

Problem 317.3 – Eight triangles

Tony Forbes

Denote the area of a triangle with vertices X, Y, Z by $\triangle(X, Y, Z)$.

(i) A circle has the six points A, B, C, D, E, F in that order on its circumference. Show that

$$\begin{aligned} &\triangle(A,B,C)\triangle(D,E,F) - \triangle(A,B,D)\triangle(C,E,F) \\ &+ \triangle(A,C,D)\triangle(B,E,F) - \triangle(B,C,D)\triangle(A,E,F) = 0. \end{aligned}$$

Here we have a curious expression involving six points, eight triangles and four 4-dimensional objects formed from multiplying pairs of areas. It is actually a special case of a more general theorem, where the circle is replaced by an arbitrary convex shape. However we think that the stated equality should be much easier to prove for points on a circle. Thanks to Robin Whitty for the idea behind this problem.

(ii) The problem can also be generalized in a manner that does not depend on the ordering of the points. Choose any six points in the plane, A, B, C, D, E, F. Show that

$$\prod_{\epsilon_1, \epsilon_2, \epsilon_3 = \pm 1} \Big(\triangle(A,B,C)\triangle(D,E,F) + \epsilon_1 \triangle(A,B,D)\triangle(C,E,F) + \epsilon_2 \triangle(A,C,D)\triangle(B,E,F) + \epsilon_3 \triangle(B,C,D)\triangle(A,E,F)\Big) = 0,$$

or find a counter-example.

Solution 314.7 – Exponential function equation

Find all solutions of

$$e^{\pi x} = x^2.$$

Peter Fletcher

We take the square root of both sides of the given equation

$$\mp e^{\pi x/2} = x,$$

move the LHS to the RHS

$$\mp 1 = x e^{-\pi x/2}$$

and multiply both sides by $-\pi/2$

$$\pm \frac{\pi}{2} = -\frac{\pi x}{2} e^{-\pi x/2}.$$

We can now take the Lambert W function of both sides

$$\begin{aligned} W\left(\pm\frac{\pi}{2}\right) &= W\left(-\frac{\pi x}{2} e^{-\pi x/2}\right) \\ &= -\frac{\pi x}{2}. \end{aligned}$$

Therefore

$$x = -\frac{2W_k(\pi/2)}{\pi} \quad \text{or} \quad x = -\frac{2W_k(-\pi/2)}{\pi}$$

for $k \in \mathbb{Z}$. In particular, the principal values are

$$x = -\frac{2W_0(\pi/2)}{\pi} \approx -0.474541 \quad \text{per Wolfram Alpha}$$

and

$$\begin{aligned} x &= -\frac{2W_0(-\pi/2)}{\pi} \\ &= -\frac{2(\mathrm{i}\pi/2)}{\pi} \quad \text{also per Wolfram Alpha} \\ &= -\mathrm{i}, \end{aligned}$$

which might have been spotted just by looking at the original equation.

I assume most readers will not have heard of Lambert's W function, which I only learnt about recently. Wikipedia and Wolfram Alpha have pages on it and there are other references online and on YouTube. I used YouTube's `blackpenredpen`. The basic equation, which I used above, is

$$W\left(\text{🐈‍⬛} \exp\left(\text{🐈‍⬛}\right)\right) = \text{🐈‍⬛},$$

where 🐈‍⬛ is any mathematical expression.

TF—Solutions computed by MATHEMATICA's ProductLog[k,z] function.

k	$-2W_k(\pi/2)/\pi$	$-2W_k(-\pi/2)/\pi$
-20	$2.78189 + 78.9776\,i$	$2.76557 + 76.9771\,i$
-19	$2.74883 + 74.9767\,i$	$2.73163 + 72.9762\,i$
-18	$2.71395 + 70.9757\,i$	$2.69577 + 68.9751\,i$
-17	$2.67706 + 66.9746\,i$	$2.65777 + 64.974\,i$
-16	$2.63789 + 62.9733\,i$	$2.61736 + 60.9727\,i$
-15	$2.59615 + 58.972\,i$	$2.57421 + 56.9713\,i$
-14	$2.55149 + 54.9705\,i$	$2.52793 + 52.9696\,i$
-13	$2.50345 + 50.9688\,i$	$2.47801 + 48.9678\,i$
-12	$2.4515 + 46.9668\,i$	$2.42383 + 44.9657\,i$
-11	$2.39492 + 42.9646\,i$	$2.36462 + 40.9633\,i$
-10	$2.33281 + 38.9619\,i$	$2.29933 + 36.9604\,i$
-9	$2.26399 + 34.9588\,i$	$2.22657 + 32.9571\,i$
-8	$2.18681 + 30.9551\,i$	$2.14439 + 28.9529\,i$
-7	$2.09895 + 26.9505\,i$	$2.05001 + 24.9478\,i$
-6	$1.99699 + 22.9447\,i$	$1.93914 + 20.9412\,i$
-5	$1.87549 + 18.9372\,i$	$1.80474 + 16.9324\,i$
-4	$1.72511 + 14.9267\,i$	$1.63401 + 12.9199\,i$
-3	$1.52758 + 10.9115\,i$	$1.39951 + 8.90071\,i$
-2	$1.23855 + 6.88672\,i$	$1.02132 + 4.86835\,i$
-1	$0.684583 + 2.84992\,i$	i
0	-0.474541	$-i$
1	$0.684583 - 2.84992\,i$	$1.02132 - 4.86835\,i$
2	$1.23855 - 6.88672\,i$	$1.39951 - 8.90071\,i$
3	$1.52758 - 10.9115\,i$	$1.63401 - 12.9199\,i$
4	$1.72511 - 14.9267\,i$	$1.80474 - 16.9324\,i$
5	$1.87549 - 18.9372\,i$	$1.93914 - 20.9412\,i$

Ted Gore

Let $x = a + bi$ so that

$$x^2 = (a^2 - b^2) + 2abi$$

and

$$e^{\pi x} = e^{\pi a}[\cos(\pi b) \pm i \sin(\pi b)].$$

The following results were obtained by using the online DESMOS graph calculator to get approximate solutions that were then made more accurate using a binary section.

When $b = 0$ we have $e^{\pi a} = a^2$, which gives

$$a = -0.47454051971435546.$$

When $a = 0$ we have $\cos(\pi b) = -b^2$, which gives $b = \pm 0.63$ or $b = \pm 1$. When a and b are both non zero we have

$$(a^2 - b^2) = e^{\pi a}\cos(\pi b) \quad \text{and} \quad 2ab = \pm e^{\pi a}\sin(\pi b).$$

Thus

$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2 = e^{2\pi a}[\cos^2(\pi b) + \sin^2(\pi b)] = e^{2\pi a}$$

from which we have $e^{\pi a} = (a^2 + b^2)$ since $e^{\pi a} > 0$. Then

$$b^2 = e^{\pi a} - a^2 \quad \text{and} \quad b = \pm\sqrt{e^{\pi a} - a^2}.$$

So

$$e^{\pi x} = e^{\pi a}[\cos(\pm\pi\sqrt{e^{\pi a} - a^2} + i\sin(\pm\pi\sqrt{e^{\pi a} - a^2})]$$

from which we have

$$e^{\pi a}\cos(\pi\sqrt{e^{\pi a} - a^2}) = \pm(a^2 - b^2) = \pm(2a^2 - e^{\pi a}).$$

From

$$e^{\pi a}\cos(\pi\sqrt{e^{\pi a} - a^2}) = (2a^2 - e^{\pi a})$$

we get

$$a \in \{-0.38576351165771483, 0.6845831298828124, 0.7475990295410156\}.$$

From

$$e^{\pi a}\cos(\pi\sqrt{e^{\pi a} - a^2}) = -(2a^2 - e^{\pi a})$$

we get

$$a \in \{-0.38576351165771483, 0.41080026626586924, 0.5033500671386719\}.$$

The results from using all these values of a are listed in the table below. Note that in this table I have truncated the solutions given above in order to fit them easily on a page.

The sign of $\sin(\pi b)$ depends on whether we use plus or minus $\sqrt{e^{\pi a} - a^2}$ for b and in which quadrant this places πb. The table should clarify this.

I have carried out calculations using both the positive and negative square roots of b^2 and have used b_+ and b_- in the column headings to distinguish between them.

The row for 0.685 gives a solution of the form $e^{\pi a}[\cos(\pi b) + i\sin(\pi b)]$.

The row for 0.748 gives a solution of the form $e^{\pi a}[\cos(\pi b) - i\sin(\pi b)]$.

The row for 0.411 gives a solution of the form $-e^{\pi a}[\cos(\pi b) - i\sin(\pi b)]$.

The row for 0.503 gives a solution of the form $-e^{\pi a}[\cos(\pi b) + i\sin(\pi b)]$.

There is a problem with the value -0.38576351165771483 for a that gives ± 0.3857644868435072 for b. This might at first seem to be a valid solution to the problem where $a^2 = b^2$ but the numbers in the table do not support that interpretation. In fact there can be no such solution.

If $(a^2 - b^2) = 0$, then $e^{\pi a}\cos(\pi b) = 0$ so that $b = \pm 0.5$ and $a = \pm b$. Then $2ab = \pm 0.5$ while $e^{\pi a}\sin(\pi b) = \pm e^{\pi a}$. However, if $e^{\pi a} = 0.5$, then

$$a \;=\; \frac{\ln(0.5)}{\pi} \;=\; -\,0.2206356001,$$

which is a contradiction. Therefore there is no solution such that a and b are both non-zero and $a^2 = b^2$.

a	b	$a^2 - b^2$	$e^{\pi a}\cos\pi b$	$2ab_+$	$e^{\pi a}\sin\pi b_+$	$2ab_-$	$e^{\pi a}\sin\pi b_-$
-0.475	0	0.226	0.226	0	0	0	0
0	±0.63	-0.397	-0.397	0	0	0	0
0	±1	-1	-1	0	0	0	0
0.685	±2.85	-7.653	-7.653	3.902	3.902	-3.902	-3.902
0.748	±3.15	-9.3536	-9.3536	4.7075	-4.7075	-4.7075	4.7075
0.411	±1.86	-3.3	3.3	1.53	-1.53	-1.53	1.53
0.503	±2.145	-4.35	4.36	2.16	2.16	-2.16	-2.16
-0.386	±0.386	0.000585	0.1	-0.297	0.278	0.297	-0.278

Solving the integral $\int e^{x^k} dx$ by parts

Mako Sawin

The integral

$$\int e^{x^2} dx$$

is a classic example and is not expressible in terms of elementary functions such as polynomials, exponentials, trigonometric, or logarithmic functions. Therefore it cannot be solved using elementary methods like algebraic manipulations or simple substitutions. Instead, it is typically solved using more advanced techniques from calculus and mathematical analysis. The solution involves expressing the integral as a special function called the *error function* (erf). The result is as follows:

$$\int_0^x e^{t^2} dt = \frac{\sqrt{\pi}\, i}{2} \operatorname{erf}(-ix),$$

where $\operatorname{erf}(x)$ is the Gaussian error function, defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Nevertheless, we can evaluate this integral through the application of multiple iterations of integration by parts.

Let

$$u = e^{x^2} \Rightarrow du = 2xe^{x^2} dx \quad \text{and} \quad dx = dv \Rightarrow v = x.$$

Then

$$\int u dv = uv - \int v du.$$

Thus

$$\begin{aligned}
\int e^{x^2} dx &= xe^{x^2} - \int \left(2xe^{x^2}\right) x dx \\
&= xe^{x^2} - 2 \int x^2 e^{x^2} dx.
\end{aligned}$$

By taking a second integration by parts,

$$u = e^{x^2} \Rightarrow du = 2xe^{x^2} dx \quad \text{and} \quad x^2 dx = dv \Rightarrow v = \frac{1}{3}x^3,$$

$$\begin{aligned}\int e^{x^2}dx &= xe^{x^2} - 2\left(\frac{1}{3}x^3e^{x^2} - \int \left(2xe^{x^2}\right)\frac{1}{3}x^3dx\right)\\ &= xe^{x^2} - \frac{2}{3}x^3e^{x^2} + \frac{(2)(2)}{3}\int x^4e^{x^2}dx.\end{aligned}$$

To solve the integral

$$\int x^4e^{x^2}dx$$

by parts, let

$$u = e^{x^2} \Rightarrow du = 2xe^{x^2}dx \quad \text{and} \quad x^4dx = dv \Rightarrow v = \frac{1}{5}x^5.$$

Then

$$\begin{aligned}\int e^{x^2}dx &= xe^{x^2} - \frac{2}{3}x^3e^{x^2} + \frac{(2)(2)}{3}\left(\frac{1}{5}x^5e^{x^2} - \frac{1}{5}\int \left(2xe^{x^2}\right)x^5dx\right)\\ &= xe^{x^2} - \frac{2}{3}x^3e^{x^2} + \frac{(2)(2)}{(3)(5)}x^5e^{x^2} - \frac{(2)(2)(2)}{(3)(5)}\int x^6e^{x^2}dx.\end{aligned}$$

In the process of employing continued integration by parts, we will derive a series pattern:

$$\begin{aligned}\int e^{x^2}dx = {} & xe^{x^2} - \frac{2}{3}x^3e^{x^2} + \frac{(2)(2)}{(3)(5)}x^5e^{x^2} - \frac{(2)(2)(2)}{(3)(5)(7)}x^7e^{x^2}\\ &+ \frac{(2)(2)(2)(2)}{(3)(5)(7)(9)}x^9e^{x^2} - \frac{(2)(2)(2)(2)(2)}{(3)(5)(7)(9)}\int x^{10}e^{x^2}dx. \ \ldots\end{aligned}$$

Therefore

$$\int e^{x^2}dx = e^{x^2}\sum_{n=0}^{\infty}\frac{(-1)^n2^nx^{2n+1}}{\prod_{i=0}^{n}(2i+1)}.$$

Based on this approach, the integral

$$\int e^{x^3}dx$$

yields the following result:

$$\begin{aligned}\int e^{x^3}dx = {} & xe^{x^3} - \frac{3}{4}x^4e^{x^3} + \frac{(3)(3)}{(4)(7)}x^7e^{x^3} - \frac{(3)(3)(3)}{(4)(7)(10)}x^{10}e^{x^3}\\ &+ \frac{(3)(3)(3)(3)}{(4)(7)(10)}\int x^{12}e^{x^3}dx. \ \ldots\end{aligned}$$

Therefore we obtain

$$\int e^{x^3}\,dx \;=\; e^{x^3}\sum_{n=0}^{\infty}\frac{(-1)^n 3^n x^{3n+1}}{\prod_{i=0}^{n}\;(3i+1)}.$$

For $k = 1, 2, 3, \ldots$, the integral

$$\int e^{x^k}\,dx$$

can be expressed as follows:

$$\int e^{x^k}\,dx \;=\; e^{x^k}\sum_{n=0}^{\infty}\frac{(-1)^n k^n x^{kn+1}}{\prod_{i=0}^{n}\;(ki+1)}.$$

Problem 317.4 – International chess

David Sixsmith

This is a true story.

Every player on the website `http://chess.com/` selects their country when they sign up. There are 241 countries. Whenever you play a game, the site notes if this is a player from a country you have not played before, and keeps a log. So, for example, after 20 games you might have played people from 17 different countries; in other words, 3 games were duplicates.

I have played 22,000 games, all at random. Using plausible statistical assumptions, estimate how many of the 241 countries I have played a player from.

Problem 317.5 – Approximation

Tony Forbes

Show that for small x,

$$\exp(\tan x) \;=\; \sqrt{\frac{1+x}{1-x}} + O(x^5).$$

Thus, for example, $e^{2\tan(1/1000)} \approx 1.002002002002002$.

Problem 317.6 – Floored square roots

Tony Forbes

For $x \geq 0$ and non-negative integer n, define the function $q_n(x)$ by

$$q_0(x) \;=\; x, \qquad q_n(x) \;=\; \lfloor \sqrt{q_{n-1}(x)} \rfloor \text{ for } n \geq 1.$$

(i) Show that for $x \geq 0$ and positive integer n,

$$q_n(x) \;=\; \left\lfloor \sqrt{\left\lfloor \sqrt{\cdots \left\lfloor \sqrt{\lfloor \sqrt{x} \rfloor} \right\rfloor \cdots} \right\rfloor} \right\rfloor \;=\; \left\lfloor \sqrt{\sqrt{\cdots \sqrt{\sqrt{x}}}} \right\rfloor.$$

That is, repeatedly applying the function $x \mapsto \lfloor \sqrt{x} \rfloor$ is the same as repeatedly square-rooting and rounding down only at the end.

(ii) Compute

$$q_{80}\,(((4!)!)!) \;=\; \left\lfloor (((4!)!)!)^{2^{-80}} \right\rfloor.$$

That is, take the number 4, apply the factorial function 3 times followed by the square root function 80 times and round down.

Problem 317.7 – Nine cards

Colin Aldridge

This puzzle was presented at the M500 Winter Weekend in January 2024. The event was oversubscribed and we had 40 people attend. The presenter Mel Starkings said it was the most annoying maths problem he knows.

Consider nine cards with numbers written on each card arranged in two rows as follows.

1	2	3	6	9
4	5	7	8	

The task is to move one and only one card so that the two rows add up to the same number.

When set it was assumed there was only one solution but Mel asserted that there were now 9 known solutions. The only hint Mel gave was that the task was originally designed as a puzzle for primary schoolchildren, and this is why mathematicians find it so annoying. I know of 7 valid answers.

Contents

04.00

Problem 317.8 – Sum

Show that

$$\frac{1}{1 \cdot 2} + \frac{1}{4 \cdot 5} + \frac{1}{7 \cdot 8} + \ldots \;=\; \frac{\sqrt{3}\,\pi}{9}.$$

Front cover Regular graphs; See Problem 316.6.