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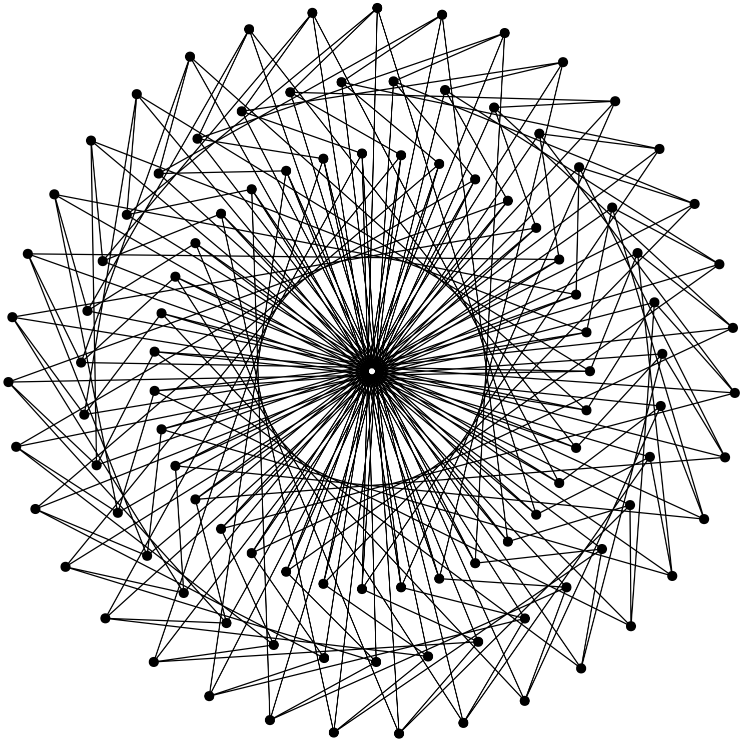
M500 321



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: `m500.org.uk`.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The Revision Weekend is a residential Friday to Sunday event providing revision and examination preparation for both undergraduate and postgraduate students. For details, please go to the Society's website.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, please go to the Society's website.

Editor – *Tony Forbes*

M500 Winter Weekend 2025

The **forty-second M500 Society Winter Weekend** will be held over

Friday 10$^{\text{th}}$ – Sunday 12$^{\text{th}}$ January 2025

at Kents Hill Park Conference Centre, Milton Keynes.

For details, pricing and a booking form, please refer to the M500 website.

`m500.org.uk/the-M500-winter-weekend/`

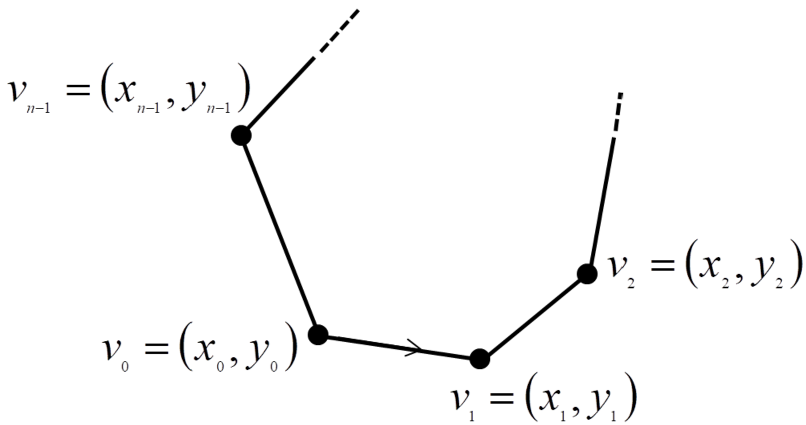
Frederick John Adam Hodgson

We were very saddened to learn that M500 Society member John Hodgson died in May. Employed for many years as an associate lecturer for the East Midlands region, John was a charismatic tutor at Summer Schools held in Durham and Nottingham. He will be remembered by us especially as an enthusiastic tutor who regularly taught at the M500 revision weekend until his retirement. Our deepest condolences go to his family and friends.

Who Invented the Shoelace Formula?

Robin Whitty

The Shoelace Formula allows you to calculate the area of a simple polygon from the coordinates of its vertices, listed anticlockwise.



$$\text{Area} \quad = \quad \frac{1}{2}\left(x_0y_1 - x_1y_0 + x_1y_2 - x_2y_1 + \ldots + x_{n-1}y_0 - x_0y_{n-1}\right) \qquad (1)$$

or, more concisely,

$$\text{area} \quad = \quad \frac{1}{2}\left(v_0v_1 + v_1v_2 + \ldots + v_{n-1}v_0\right), \qquad (2)$$

where v_iv_j is a 'shorthand' for $x_iy_j - x_jy_i$.

It is surprisingly concise, given that it works for non-convex polygons. It takes its name from the 'interlacing' of x and y ordinates. But whom might it be named *after*?

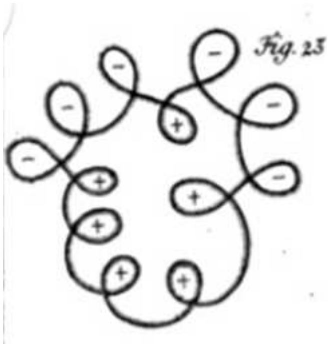
The formula is often referred to as 'Gauss's' but "the formula was certainly not invented by him" according to the German mathematician Burkard Polster in his excellent Mathologer Youtube video "Gauss's magic shoelace area formula and its calculus companion". A rather cursory trawl through those parts of Gauss's collected works that pertain to plane geometry showed me nothing resembling the above formula. But it is also sometimes called 'the surveyor's area formula' and Gauss, as a loyal subject of the Duke of Hanover, undertook the challenge of surveying the duke's territory. Whatever records of that enterprise might exist they have not been examined, even cursorily, by me. In any case Gauss was famously reticent about publishing his discoveries, I suppose to the frustration of even accomplished historians of science.

Wikipedia accounts of Shoelace (as I will call it for short) vary greatly from language to language. The English language page is more assertive than most on history: "The formula was described by Albrecht Ludwig Friedrich Meister (1724–1788) in 1769 and is based on the trapezoid formula which was described by Carl Friedrich Gauss and C. G. J. Jacobi." Since Gauss was born in 1777 and Jacobi in 1804, the phrase 'based on' is not to be taken literally. Nor, in any case, can I find that Gauss wrote down a 'trapezoid formula' in relation to polygon area. But the work of Meister is definitely relevant.

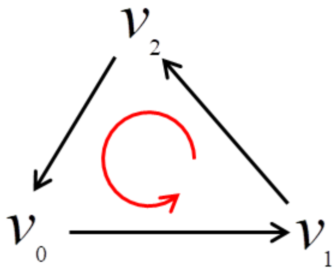
Albrecht Meister wrote a treatise called *Generalia de genesi figurarum planarum et inde pendentibus earum affectionibus* (dated 1770, in fact, not 1769). This may be regarded as the first systematic account of polygons. It addresses how polygons may be specified, classified and manipulated, and has many pages of diagrams. It talks about area in detail. And Gauss's collected correspondence contains a letter of 1825 to his friend and colleague Wilhelm Olbers in which he says (my cursory translation from German):

> I could also have mentioned the theory of the area of figures, which I have been looking at for thirty or more years from a point of view that I have hitherto considered to be new. This, however, is partly a mistake: in fact, only recently did I learn of a treatise by Meister (in my opinion a very brilliant mind) in the first volume of the Novi Commentarii Gotting, in which the matter is viewed in almost exactly the same way and is developed very nicely.

Nevertheless *Generalia de genesi figurarum planarum* definitely does not describe Shoelace, nor indeed any formulae. What was it that so impressed Gauss? I think it was Meister's invention of 'signed area', meaning that area created by a closed curve turning anticlockwise should be taken to be positive while oppositely oriented areas should be negative. He has several diagrams relating to signed area, one of which is reproduced in figure (a) on the next page.



(a)



(b)

One supposes that, for a mathematician of Gauss's stature, this 'rule of signs' was the single, essential insight needed for polygon area; everything else was just bookkeeping. He would hardly have thought it worthwhile recording the fact that the area of a triangle, specified by the direction vectors of its three sides, is half the cross product of any two of these vectors, taken anticlockwise. (The terminology is anachronistic but vector algebra was, in some form, already present in eighteenth century two and three dimensional geometry.) In figure (b), above, v_0, v_1 and v_2 are position vectors (coordinate pairs). The direction vectors of the sides joining v_0 to v_1 and v_1 to v_2 are, respectively, $v_1 - v_0$ and $v_2 - v_1$. So the bookkeeping goes as follows:

$$\begin{aligned}
\text{Area} &= \frac{1}{2}(v_1 - v_0) \times (v_2 - v_1) && \text{(half cross product of edges)} \\
&= \frac{1}{2}(v_1 \times v_2 + v_1 \times -v_1 - v_0 \times v_2 - v_0 \times -v_1) \\
&= \frac{1}{2}(v_1 \times v_2 + 0 - v_0 \times v_2 + v_0 \times v_1) && \text{(since } v_i \times v_i = 0) \\
&= \frac{1}{2}(v_1 \times v_2 + v_2 \times v_0 + v_0 \times v_1) && \text{(since } v_i \times v_j = -v_j \times v_i) \\
&= \frac{1}{2}(v_0 v_1 + v_1 v_2 + v_2 v_0)\,. && \text{(in the notation of equation (2).)}
\end{aligned}$$

And Shoelace has emerged directly from the cross product rule; and indeed equation (2) is simply Shoelace in cross product notation.

It is still something of a leap of imagination to see that clockwise and anticlockwise triangles will combine, in tracing a complicated non-convex polygon, to produce its total area. And although I can believe that this was implicit in what Meister wrote, and explicit in what Gauss thought, still we have yet to see it announced as a formula. This announcement happened, we have it on good authority, in 1810. *Lezioni di statica grafica*, an 1877 textbook by the Italian mathematician and historian of science Antonio Favaro (1847–1922), was translated, as *Leçons de statique graphique*, in two parts by a Monsieur Paul Terrier. I can discover nothing about Monsieur Terrier, except that he studied at the elite École centrale Paris, but he was very thorough in his translation of *Lezioni*, adding copious appendices and footnotes. Chapter 4 of Part 2 opens with a section on "Principe des signes appliqué aux aires". And immediately there is a long footnote in which Terrier traces "Les premières notions sur les signes des aires, spécialement dans les figures aux périmètres croisés" to Meister's 1770 treatise. And he continues "C'est ici le cas de rappeler la célèbre formule de Gauss ..."; and recall the celebrated formula he accordingly does. And he goes on to state (my translation) "This formula was published for the first time in the German translation of *Géométrie de position* de Carnot".

Lazare Carnot's *Géométrie de position* was published in 1803 and was published in German as *Geometrie der Stellung* in 1810. As with Favaro's *Lezioni* the translation provides additional material and this is why Paul Terrier cites it as the Shoelace's place of origin. And on page 362 we find a footnote marked "Anmerkung des Herausgebers", meaning "editor's note", or in this case, I think, "translator's note", which reads

> According to a famous theorem of Gauss, the area of a polygon with n sides, if the coordinates of the vertices are numbered in one direction:
>
> $$\begin{array}{ll} x, & y \\ x', & y' \\ & \cdots \\ x^{n-1}, & y^{n-1} \end{array}$$
>
> is
>
> $$\frac{1}{2}\Big\{x\left(y'-y^{n-1}\right)+x'\left(y^2-y\right)+x''\left(y^3-y'\right)+\ldots \\ +x^{n-1}\left(y-y^{n-2}\right)\Big\}.$$
>
> on which he himself, perhaps, on another occasion, will give us a more complete treatise.

There is no mention of the formula in Carnot's 1803 text. The translator for the German edition was Heinrich Christian Schumacher (1780–1850), a mathematician and astronomer from the German–Norwegian borders. He studied law at Göttingen and Kiel, graduating in 1806 from the former, where he then took up scientific studies under Gauss. In *Monthly Notices of the Royal Astronomical Society*, Vol. 11, February 1851, pp.73–81 we read

> In 1805 he began his translation of Carnot's *Géometri de position* into German, *ad recreationem animi*, as he says himself in the introduction to the work. This translation, with notes by Gauss, was published at Hamburg in two volumes.

There is no doubt that Gauss and Schumacher were, and remained, close colleagues. In the complete correspondence of Gauss, which may be read at the website gauss.adw-goe.de, a sixth of the over eight thousand entries are letters to or from Schumacher. Of those predating the publication of *Geometrie der Stellung* none appear to relate to polygon area. In any case, Schumacher remained at Göttingen until 1810, when he was appointed extraordinary professor of astronomy at Copenhagen. We have to assume that Schumacher's footnote on page 362 of *Geometrie der Stellung* had at least Gauss's blessing, if it was not actually dictated by Gauss. We can imagine Schumacher adding the "on which he himself, perhaps, on another occasion, will give us a more complete treatise" to tease Gauss, the reluctant scribe.

Problem 321.1 – Sums of squares

Tony Forbes

Find all solutions in integers $a > 7$ and $b > 0$ of

$$\sum_{k=7}^{a} k^2 \;=\; b^2.$$

The first few are

$$(a, b) \;=\; (29, 92), (39, 143), (56, 245), (190, 1518), (2215, 60207).$$

Are there any more? More generally, one can replace 7 by k_0. When $k_0 = 1$ it is of course the classic cannon-balls-stacked-in-a-square-pyramid-formation problem, with unique solution $(a, b) = (24, 70)$. Small solutions happen to be unusually common when $k_0 = 7$.

Solution 319.5 – Factorial factorial

How many zeros are at the end of $(n!)!$?
If that's difficult, do $n!$ first.
If that's difficult, try a special case, say 100!.

Jodie Forbes

Clearly brute force calculation doesn't get us very far. The calculator on my PC says:

$(2!)! = 2! = 2$ (0 terminal zeros),
$(3!)! = 6! = 720$ (1 terminal zero),
$(4!)! = 24! = 620448401733239439360000$ (4 terminal zeros),
$(5!)! = 120! \approx 6.6895 \times 10^{198}$ (unknown number of terminal zeros).

What we are looking for is the number of multiples of 10 in the factorial, which is equivalent to looking for the number of times 5 appears in the factorisation (because $10 = 2 \times 5$ and there are many more factors of 2 than 5). If we look first at the simpler case of $n!$, we see the following, noting that we have to be careful to count the additional factors when we reach powers of 5 such as 5^2.

n	$n!$	Number of factors of 5 = number of terminal zeros
5	120	1
10	3628800	2
15	1307674368000	3
20	2432902008176640000	4
25	15511210043330985984000000	6

Using these insights we can give a formula for the number of zeros at the end of $n!$:

$$\sum_{k=1}^{\lfloor \log_5 n \rfloor} \left\lfloor \frac{n}{5^k} \right\rfloor . \tag{1}$$

To extend this to the case of $(n!)!$ we substitute $n!$ in (1), giving the following formula:

$$\sum_{k=1}^{\lfloor \log_5 n! \rfloor} \left\lfloor \frac{n!}{5^k} \right\rfloor . \tag{2}$$

We can further attempt to substitute Stirling's approximation for $n!$ in (2), giving

$$\sum_{k=1}^{\left\lfloor \log_5 \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \right\rfloor} \left\lfloor \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{5^k} \right\rfloor. \tag{3}$$

Because Stirling's approximation sometimes underestimates $n!$, some factors of 5 can be missed and formula (3) will give us an underestimate of the number of terminal zeros. Similarly, in cases where Stirling's approximation is too high, formula (3) may give an overestimate of the number of terminal zeros. Nevertheless, for large n, the use of Stirling's approximation gives a reasonable value. Some examples are shown in the following table, calculated using MATHEMATICA on my PC.

n	$n!$	Stirling approximation for $n!$	Number of terminal zeros in $(n!)!$ according to (2)	Number of terminal zeros in $(n!)!$ according to (3)
2	2	1.919	0	0
3	6	5.836	1	1
4	24	23.502	4	4
5	120	118.02	28	27
6	720	710.08	178	176
7	5040	4980.4	1258	1242
8	40320	39902	10076	9972
9	362880	359536	90717	89881
10	3628800	3598696	907197	899669
12	479001600	475687486	119750395	118921862
100	9.3326×10^{157}	9.3248×10^{157}	2.3332×10^{157}	2.3312×10^{157}
1000	4.0239×10^{2567}	4.0235×10^{2567}	1.0060×10^{2567}	1.0059×10^{2567}
10000	2.8463×10^{35659}	2.8462×10^{35659}	7.1156×10^{35658}	7.1156×10^{35658}

Thanks to Dave Wild for computing the entry in column 4 for 10000.

Solution 319.3 – Sum

Show that

$$\frac{1}{1\cdot 3\cdot 5}+\frac{1}{7\cdot 9\cdot 11}+\frac{1}{13\cdot 15\cdot 17}+\dots \;=\; \frac{\log 3}{16}.$$

The factors in the denominators run through the odd positive integers.

J. M. Selig

I thought I had a good idea how to solve this but when I looked into it I was completely wrong. After a little searching using MATHEMATICA and Wikipedia I figured out how to approach it. The solution uses the digamma function, which I was only vaguely aware of.

Let's name the sum and observe that it can be written as

$$\begin{aligned} S \;&=\; \frac{1}{1\cdot 3\cdot 5}+\frac{1}{7\cdot 9\cdot 11}+\frac{1}{13\cdot 15\cdot 17}+\cdots \\ &=\; \sum_{n=0}^{\infty}\frac{1}{(6n+1)(6n+3)(6n+5)}. \end{aligned}$$

Using partial fractions we can rewrite the sum as

$$S \;=\; \frac{1}{48}\sum_{n=0}^{\infty}\left(\frac{1}{n+1/6}-\frac{2}{n+3/6}+\frac{1}{n+5/6}\right).$$

Now, we cannot split up this sum because the individual terms behave like $1/n$ and summing such terms will not converge.

This is where we introduce the digamma function $\psi(x)$. This is the derivative of the log of the gamma function:

$$\psi(x) \;=\; \frac{d}{dx}\log\big(\Gamma(x)\big).$$

The gamma function satisfies the relation

$$\Gamma(x+1) \;=\; x\Gamma(x);$$

taking the logarithm and differentiating then gives

$$\psi(x+1) \;=\; \frac{1}{x}+\psi(x).$$

This means we can write the reciprocal as a forward difference,

$$\frac{1}{x} \;=\; \psi(x+1) - \psi(x).$$

The partial sum of the first N terms S_N, can then be evaluated by 'telescoping',

$$\begin{aligned} S_N \;=\; \frac{1}{48}\sum_{n=0}^{N-1}\left(\frac{1}{n+1/6} - \frac{2}{n+3/6} + \frac{1}{n+5/6}\right) \;=\; & \\ \frac{1}{48}\big(\psi(N+1/6) - 2\psi(N+3/6) + \psi(N+5/6)\big) & \\ - \frac{1}{48}\big(\psi(1/6) - 2\psi(3/6) + \psi(5/6)\big). & \end{aligned}$$

From Wikipedia [1], we have the asymptotic relation,

$$\log(x - 1/2) \;\leqslant\; \psi(x) \leqslant \log(x)$$

for large x. So, combining

$$\begin{aligned} \log(N + 1/6 - 1/2) \;&\leqslant\; \psi(N+1/6) \;\leqslant\; \log(N+1/6),\\ -2\log(N+3/6) \;&\leqslant\; -2\psi(N+3/6) \;\leqslant -2\log(N),\\ \log(N+5/6-1/2) \;&\leqslant\; \psi(N+5/6) \;\leqslant\; \log(N+5/6) \end{aligned}$$

gives

$$\log\left(\frac{(N^2 - 1/9)}{(N+1/2)^2}\right) \;\leqslant\; X \;\leqslant\; \log\left(\frac{(N+1/6)(N+5/6)}{N^2}\right),$$

where

$$X \;=\; \big(\psi(N+1/6) - 2\psi(N+3/6) + \psi(N+5/6)\big).$$

In the limit $N \to \infty$, the arguments of the logarithmic functions tend to 1. Hence, the expression X is squeezed between functions that tend to zero. Thus we conclude that

$$\lim_{N\to\infty}\big(\psi(N+1/6) - 2\psi(N+3/6) + \psi(N+5/6)\big) \;=\; 0,$$

and so

$$S \;=\; -\frac{1}{48}\big(\psi(1/6) - 2\psi(3/6) + \psi(5/6)\big).$$

The problem has been reduced to the evaluation of the digamma function at three points. Fortunately, there are exact values for the digamma function at these points. Again from the Wikipedia page [1], we have that

$$\psi(3/6) \;=\; \psi(1/2) \;=\; -2\log(2) - \gamma$$

and

$$\psi(1/6) \;=\; -\frac{\pi\sqrt{3}}{2} - 2\log(2) - \frac{3\log(3)}{2} - \gamma,$$

where γ is the Euler–Mascheroni constant. To find the value of the digamma function at $5/6$ we use the reflection formula,

$$\psi(1-x) - \psi(x) \;=\; \pi\cot(\pi x)$$

with $x = 1/6$. The gives

$$\psi(5/6) \;=\; \frac{\pi\sqrt{3}}{2} - 2\log(2) - \frac{3\log(3)}{2} - \gamma.$$

Combining these values we get

$$\big(\psi(1/6) - 2\psi(3/6) + \psi(5/6)\big) \;=\; -3\log(3)$$

and hence

$$S \;=\; -\frac{1}{48}\big(\psi(1/6) - 2\psi(3/6) + \psi(5/6)\big) \;=\; \frac{3\log(3)}{48} \;=\; \frac{\log(3)}{16}.$$

I am not entirely happy with this solution. Using MATHEMATICA to evaluate the original series simply produces the result stated, but I'm confident that it doesn't find it the way I have. There is probably a lot more about the digamma function that I am not aware of and also probably clever tricks that allow one to sum the series without having to use the limiting argument used above. But I am very happy to have learned at least something about the digamma function.

References

[1] Wikipedia contributors, 'Digamma function', Wikipedia, The Free Encyclopedia, `https://en.wikipedia.org/wiki/Digamma_function` (accessed June 17, 2024).

A dog ate my M500 magazine

Dave Wild

When I looked at the few remaining scraps of the magazine I found the following part of a problem

$$\text{Show that } H_1 + H_2 + \ldots + H_n \;=\; n^3.$$

What are the values of the H_i? Putting $n = 1$ we can deduce that $H_1 = 1$. Adding H_{n+1} to both sides of the equation we should expect that

$$n^3 + H_{n+1} \;=\; (n+1)^3\,.$$

As

$$H_{n+1} \;=\; 3n^2 + 3n + 1 \;=\; 3\,(n+1)^2 - 3\,(n+1) + 1$$

then, for $n > 1$,

$$H_n = 3n\,(n-1) + 1.$$

As the value of H_1 is 1, the formula for H_n is valid for all n.

If the sequence was defined by a recurrence relation, then $H_1 = 1$, and subtracting H_{n-1} from H_n gives

$$H_n \;=\; H_{n-1} + 6\,(n-1)\,.$$

The resulting sequence starts

$$1,\ 7,\ 19,\ 37,\ 61,\ 91,\ 127,\ 169,\ \ldots.$$

When I put this into the On-Line Encyclopedia of Integer Sequences I found it is sequence A003215 of Hex(agonal) numbers. So in this case from the formula for the sum of a number of terms of a sequence we have been able to identify the unknown values H_i.

We can replace n^3 in the original problem by another function of n. Using $(n+1)! - 1$, we obtain the formula

$$1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! \;=\; (n+1)! - 1.$$

Why not choose your own function for the sum and see what sequence appears.

Solution 300.1 – Friends

> There are a finite number of people. For any two distinct persons, a and b, there is a unique person c who is friends with both a and b. Show that there exists one person who is friends with every other person.
>
> This is known in the literature on finite projective planes as the Friendship Theorem. What is wanted ideally is a proof that readers of this magazine can readily understand.

Tony Forbes

A *strongly regular graph* $\mathrm{srg}(n, d, \lambda, \mu)$ is a graph with these properties.

(i) The graph is simple (undirected, no loops, no multiple edges) and has n vertices.
(ii) The graph is d-regular (each vertex has d neighbours).
(iii) Each pair of adjacent vertices has λ common neighbours.
(iv) Each pair of non-adjacent vertices has μ common neighbours.

If the graph is complete, we are flexible concerning the parameter μ; you can put anything you like there. Similarly for λ when the graph has no edges.

I am not going to prove the Friendship Theorem. That has been done in [Judith Q. Longyear and T. D. Parsons, The Friendship Theorem, *Indagationes Mathematicae (Proceedings)* **75** (1972), 257–262] by elementary graph theory. In particular, they avoid any mention of finite planes, projective or otherwise. More recently, in the 21st century at the LSBU Maths Study Group, Carrie Rutherford delivered a series of talks based on the Longyear & Parsons paper. After some elementary analysis it becomes clear that when more than three people are involved the Friendship Theorem is equivalent to the non-existence of an $\mathrm{srg}(d^2 - d + 1, d, 1, 1)$ for $d > 2$.

If the number of persons is 3, the Friendship Theorem is either true or false. It depends on whether you are reading the Longyear & Parsons paper or how you interpret 'one' in the statement of Problem 300.1. The corresponding graph is an $\mathrm{srg}(3, 2, 1, 42)$, i.e. a triangle. The theorem is vacuously true when there are fewer than three people.

We aim to show that there is no $\mathrm{srg}(d^2 - d + 1, d, 1, 1)$ for $d > 2$.

Denote the all-1s vector by $\mathbf{j}$, the identity matrix by I and the all-1s square matrix by J, all of whatever dimension is relevant.

Theorem 1 *There is no* $\mathrm{srg}(d^2-d+1,d,1,1)$ *for* $d>2$.

Proof Let $n=d^2-d+1$, $d\geq 2$. Let A be an adjacency matrix of an $\mathrm{srg}(n,d,1,1)$. Then d is even and A^2 is determined since

$$\begin{aligned} [A^2]_{i,i} &= \text{number of 2-step walks from } i \text{ to } i \;=\; d, \\ [A^2]_{i,j,\ j\neq i} &= \text{number of 2-step walks from } i \text{ to } j \;=\; 1. \end{aligned}$$

Hence

$$A^2 \;=\; J+(d-1)I.$$

Multiplying by A and using the formula for A^2 as well as

$$AJ \;=\; dJ \quad\text{and}\quad J^2 \;=\; nJ \;=\; (d^2-d+1)J$$

gives expressions for powers of A in terms of A, I and J:

$$\begin{aligned} A^3 &= (d-1)A+dJ, \\ A^4 &= (d-1)^2I+(d^2+d-1)J, \\ A^5 &= (d-1)^2A+d(d^2+d-1)J, \\ A^6 &= (d-1)^3I+(d^4+d^3-2d+1)J, \\ A^7 &= (d-1)^3A+d(d^4+d^3-2d+1)J, \end{aligned}$$

and so on. Interesting observation: the even powers are independent of A and therefore of the graph's details. We have

$$A^2\equiv J \pmod{d-1} \quad\text{and}\quad AJ \;=\; dJ\equiv J \pmod{d-1}.$$

Hence

$$A^k\equiv J \pmod{d-1} \quad\text{for}\quad k\geq 2.$$

This means that when $k\geq 2$ the number of k-step walks from vertex i to vertex j (not necessarily distinct from i) is congruent to 1 modulo $d-1$. But those who attended Carrie's talks already know this.

Let p be a prime. Since the graph has no loops, the closed p-step walks can be partitioned into sets of size p by the mapping

$$\langle x_1,x_2,\ldots,x_p\sim x_1\rangle \mapsto \langle x_2,x_3,\ldots,x_p,x_1\rangle.$$

But the number of p-step walks is congruent to 1 modulo $d-1$, a contradiction when p divides $d-1$. $\square$

An alternative proof uses linear algebra to show that if the graph exists then it must have an adjacency matrix having an eigenvalue that occurs with a non-integer multiplicity. Whether it counts as elementary graph theory is debateable.

Lemma 1 *Let A be an adjacency matrix of a connected d-regular graph. Then A has eigenvalue d with multiplicity* 1 *and eigenvector* $\mathbf{j}$.

Proof We offer three proofs.

(i) This is well known.

(ii) See [Norman Biggs, *Algebraic Graph Theory*, Cambridge, 1974].

(iii) We give details as follows.

Clearly $A\mathbf{j} = d\mathbf{j}$. Hence d is an eigenvalue of A with eigenvector $\mathbf{j}$.

Assume the graph's vertices are 1, 2, ..., n and denote the neighbours of m by $N(m)$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a vector for which $A\mathbf{x} = d\mathbf{x}$, and suppose x_m is an element of $\mathbf{x}$ having the maximum absolute value. Then

$$[A\mathbf{x}]_m \;=\; \sum_{i \in N(m)} x_i \;=\; dx_m.$$

The sum has d terms. Because $|x_m|$ is maximum it follows that $x_i = x_m$ for all vertices $i \in N(m)$. Since the graph is connected we may repeat the argument with another $x_{m'} = x_m$ until we have shown that $x_i = x_m$ for all vertices i. Therefore $\mathbf{x}$ is a multiple of $\mathbf{j}$. $\square$

Lemma 2 *Let A be an adjacency matrix of a d-regular graph. If A has eigenvalue $-d$, then the graph must be bipartite.*

Proof As in the proof of Lemma 1, assume the graph's vertices are 1, 2, ..., n, let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a vector for which $A\mathbf{x} = -d\mathbf{x}$, and suppose x_m is an element of $\mathbf{x}$ having the maximum absolute value. Then

$$[A\mathbf{x}]_m \;=\; \sum_{i \in N(m)} x_i \;=\; -\,dx_m,$$

and hence $x_i = -x_m$ for all vertices $i \in N(m)$.

We repeat the argument for $\ell \in N(m)$ to deduce that $x_h = -x_\ell = x_m$ for $h \in N(\ell)$. Since the graph is connected, we can continue in like manner until we have proved that for every vertex i, $x_i = -x_j$ if $j \in N(i)$. However, this will lead to a contradiction if the component containing m has a cycle of odd length.

By applying the same argument to each component of the graph, we conclude that there cannot be an odd cycle anywhere. Therefore the graph must be bipartite. □

Theorem 2 *There is no* $\mathrm{srg}(d^2 - d + 1, d, 1, 1)$ *for* $d > 2$.

Proof Let $n = d^2 - d + 1$, $d \geq 2$, and let A be an adjacency matrix of an $\mathrm{srg}(n, d, 1, 1)$. As in Theorem 1 we have

$$A^2 \;=\; J + (d-1)I.$$

By Lemma 1, A has eigenvalue d with multiplicity 1 and eigenvector $\mathbf{j}$. Since the graph contains a triangle it cannot be bipartite. Therefore, by Lemma 2, $-d$ is not an eigenvalue of A.

Let a be an eigenvalue of A corresponding to an eigenvector $\mathbf{w} \neq \mathbf{0}$ that is orthogonal to $\mathbf{j}$. Then

$$A^2\mathbf{w} \;=\; J\mathbf{w} + (d-1)I\mathbf{w};$$

therefore

$$a^2\mathbf{w} \;=\; 0 + (d-1)\mathbf{w},$$

and hence

$$a^2 \;=\; (d-1) \quad\Rightarrow\quad a \;=\; \pm\sqrt{d-1}.$$

If the last paragraph is questionable, [Martin Aigner and Günter M. Ziegler, *Proofs from THE BOOK*, Springer, 1999] does it differently. The eigenvalues of J are n with multiplicity 1 and 0 with multiplicity $n-1$. So A^2 has eigenvalues $d - 1 + n = d^2$ and $d-1$. Since A is symmetric (which implies A has real eigenvalues) and hence diagonalizable (which implies A's eigenvalues are square roots of (A^2)'s eigenvalues), we can conclude that A's eigenvalues are d and $\pm\sqrt{d-1}$.

We can now write down the eigenvalues of A:

$$\begin{array}{rll} d & \text{with multiplicity } 1 & \text{(by Lemmas 1 and 2)},\\ \sqrt{d-1} & \text{with multiplicity } r, & \\ -\sqrt{d-1} & \text{with multiplicity } s. & \end{array}$$

But A has trace 0 and n rows; so

$$d + r\sqrt{d-1} - s\sqrt{d-1} \;=\; 0, \qquad 1 + r + s \;=\; n \;=\; d^2 - d + 1.$$

Solving in the usual way for r and s gives

$$r = \frac{1}{2}\left(n-1-\frac{d}{\sqrt{d-1}}\right), \qquad s = \frac{1}{2}\left(n-1+\frac{d}{\sqrt{d-1}}\right).$$

If $d=2$, then $r=0$, $s=2$ and we have an srg$(3,2,1,1)$, i.e. a triangle, which has eigenvalues 2, -1, -1.

On the other hand, if $d>2$, then r and s are not integers unless possibly $d-1$ is a square that divides d^2. But this cannot happen since $d \equiv 1 \pmod{d-1}$. □

If you happen to be reading M500 **65** (August 1980), you will notice that Problem 65.4 – Friends by Robin Wilson is about something different.

(a) At a gathering of 150 people, why must there be at least two persons with the same number of friends?

(b) They play in a tennis (singles) tournament. In the 1st round there are 75 matches, in the 2nd 37 matches and one bye, ... How many matches?

Both parts were answered by several readers in M500 **67**.

Problem 321.2 – Telephone message

Tony Forbes

I want to send a message to someone using my AI-enhanced smart phone. For each word, I just enter the first letter and then I choose at random one of the two words that the system has suggested. Can you estimate the chances of my message being understood? Assume punctuation is entered correctly.

Here is an example. See if you can decipher it before turning page 18 upside down.

```
If also one time the. People have my done real for
me when it gives him.
```

I suspect the answer is almost certainly, provided the message is sufficiently long or repeated sufficiently often. However, I might be wrong. My device actually makes three suggestions, but one of them is just the letter by itself and is therefore rejected.

Solution 319.1 – Sum

For positive integer r, show that

$$\frac{1}{r^2+(r+1)^2}+\frac{1}{(3r+1)^2+(3r+2)^2}+\frac{1}{(5r+2)^2+(5r+3)^2}+\cdots$$
$$= \frac{\pi}{4r+2}\tanh\frac{\pi}{4r+2}.$$

David Sixsmith

We are asked to show that

$$\sum_{n=1}^{\infty}\frac{1}{((2n-1)r+n-1)^2+((2n-1)r+n)^2} = \frac{\pi}{4r+2}\tanh\frac{\pi}{4r+2}, \quad (1)$$

where r is a positive integer. In fact we will prove the stronger result that (1) holds for all complex r outside an exceptional set which contains no integers.

We begin with the Weierstrass product representation of cosh,

$$\cosh\pi z = \prod_{n=1}^{\infty}\left(1+\frac{z^2}{(n-1/2)^2}\right). \quad (2)$$

Suppose $\cosh\pi z$ is not zero, in other words z is not an odd integer multiple of $i/2$. Then we can take a logarithm of (2), differentiate, and multiply by z, to obtain

$$\pi z\tanh\pi z = \sum_{n=1}^{\infty}\frac{2z^2}{z^2+(n-1/2)^2}. \quad (3)$$

Replacing z with $1/(4r+2)$ gives (1); this can be seen by observing, after a calculation, that

$$((2n-1)r+n-1)^2+((2n-1)r+n)^2 = \frac{1}{2}((n-1/2)^2(4r+2)^2+1).$$

It remains to note the values of r for which the above analysis fails. Firstly, there are the values of r which correspond to the zeros of $\cosh\pi z$, in other words

$$r = \frac{i}{2(2n-1)}-\frac{1}{2}, \quad \text{for } n\in\mathbb{Z}.$$

These are exactly the values of r for which one of the terms in the series in (1) is undefined.

Secondly, there is the value $r = -1/2$, which does not correspond to any finite value of z. When $r = -1/2$, each term in (1) is defined, but the series does not converge.

Finally, we note that dividing both sides of (3) by z^2, and then replacing z with $(r-2)/(2r)$ gives an alternative (and more general) solution to Problem 315.7.

Problem 321.3 – Simplification

Tony Forbes

Simplify $x(-x^k)^{-1/k}$, where x is real and k is a positive integer.

This factor occurs when I ask MATHEMATICA to evaluate the expression at the end of [1]:

$$\int_0^x e^{t^k}\,dt \;=\; e^{x^k}\sum_{n=0}^{\infty}\frac{(-1)^n k^n x^{kn+1}}{\prod_{i=0}^{n}\ (ki+1)}$$
$$= \frac{x(-x^k)^{-1/k}}{k}\left(\Gamma\left(\frac{1}{k}\right)-\Gamma\left(\frac{1}{k},-x^k\right)\right),$$

where $\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}\,dt$, which generalizes the gamma function, which of course generalizes the factorial-minus-one function,

$$(a-1)! \;=\; \Gamma(a) \;=\; \int_0^\infty t^{a-1}e^{-t}\,dt \;=\; \Gamma(a,0).$$

The gamma function is an example of a case where logic and good sense play no part in the development of mathematical notation. Surely we don't need to define $\Gamma(a)$ with that annoying -1 when natural extensions of the factorial function are readily available:

$$a! \;=\; \int_0^\infty t^a e^{-t}\,dt \quad\text{and}\quad (a,z)! \;=\; \int_z^\infty t^a e^{-t}\,dt.$$

[1] Mako Sawin, Solving the integral $\int e^{x^k}\,dx$ by parts, M500 **317**, 18–20.

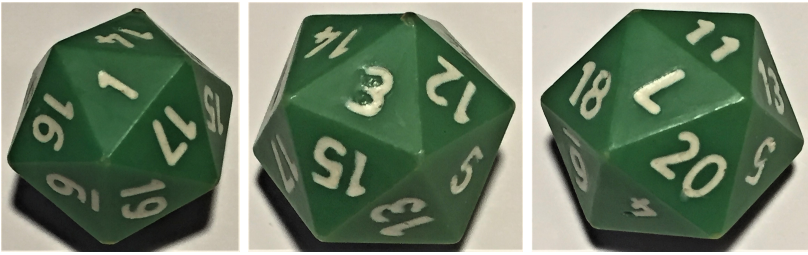
I am on the train. Please have my dinner ready for me when I get home.

An icosahedral die

Tony Forbes

For a vertex v of a given polyhedron with labelled faces, denote by S_v the sum of the labels of the faces that have v as a common point.

The pictures show three views of a typical icosahedral die with the faces labelled 1, 2, ..., 20.



If we list the labels of the faces that meet each vertex v of the die in the pictures, we have {1,16,9,19,17}, {1,17,15,3,14}, {1,14,10,8,16}, {2,4,6,8,10}, {2,10,14,3,12}, {2,12,5,20,4}, {3,12,5,13,15}, {4,6,18,7,20}, {5,13,11,7,20}, {6,18,9,16,8}, {7,11,19,9,18}, {11,13,15,17,19}, and when we calculate the sums S_v we obtain

$$62, 50, 49, 30, 41, 43, 48, 55, 56, 57, 64, 75.$$

In my opinion it would be nice if one could label the die in such a manner that the five faces at each vertex sum to the same constant value, S say; that is, $S_v = S$ for all vertices v. However, a little analysis shows that this is impossible to achieve. The sum of all 20 labels is 210, and each vertex sum must account for five twentieths of this total. That is, $S = 210 \cdot 5/20 = 105/2$, which needs to be integer.

If we cannot have a constant sum, the next best arrangement would be $S_v = 52$ for six vertices and $S_v = 53$ for the other six. But how does one go about constructing such a labelling? Instead of trying to work out a clever way I opted for a sledgehammerish approach.

What I find slightly amazing is that the simple algorithm described on the next page actually works. I was not expecting an instant result, but it did get there eventually. The die is illustrated in Figure 1 as a planar graph, where the label of vertex v is S_v.

(1) Label the faces of the icosahedron with a random permutation of $(1, 2, \ldots, 20)$.

(2) Choose vertices a and b such that S_a takes the maximum value and S_b takes the minimum value.

(3) If $S_a = 53$ and $S_b = 52$, then STOP since the required labelling has been found.

(4) Choose a face F_a with label L_a adjacent to vertex a and a face F_b with label L_b adjacent to vertex b such that $L_a > L_b$.

(5) Swap labels L_a and L_b. That is, label F_a with L_b and F_b with L_a.

(6) If we think we are in a tight loop, go to (1). Otherwise go to (2).

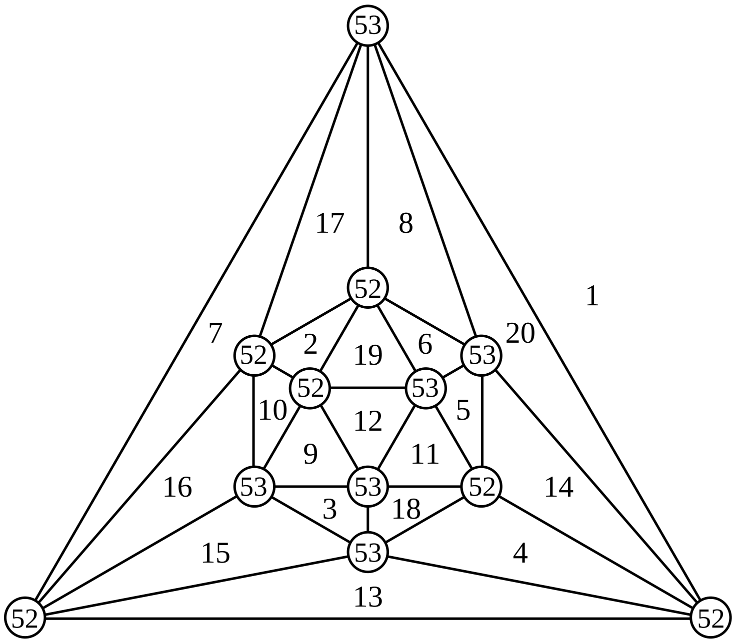


Figure 1: An icosahedral die

Naturally, one wonders if one can do something similar with the other four Platonic solids.

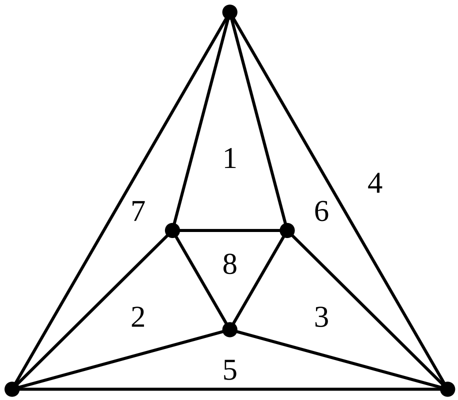


Figure 2: An octahedral die

The tetrahedron won't work because each sum 6, 7, 8, 9 is unavoidable. Nor will the cube, where the sums S_v must be 10 or 11, i.e. $3(1+2+\cdots+6)/6 \pm 0.5$. But that means 1 and 2 must label opposite faces, in which case we cannot avoid $S_v = 1 + 3 + x$ with $x \le 5$.

The octahedron is a pleasant surprise. This time we have a constant sum. We can label the faces with 1, 2, ..., 8 such that $S_v = 18$ for each vertex v. See Figure 2.

For the dodecahedron, S_v must be either 19 or 20. Consider the face with label 1, F_1 say, and the five faces adjacent to it. The only pairs of integers that sum to 18 or 19 are

$$P = \{\{6, 12\}, \{7, 11\}, \{7, 12\}, \{8, 10\}, \{8, 11\}, \{9, 10\}\}.$$

But $\{6, 12\}$ and $\{9, 10\}$ are inadmissible because 6 and 9 each occur only once in P. So the faces adjacent to F_1 must have labels 7, 8, 10, 11, 12 in some order. However, 12 occurs only once in $P \setminus \{\{6, 12\}, \{9, 10\}\}$.

Not long after I had written the last sentence, above, I discovered that the icosahedral die problem had already been solved. Take a look at

`https://mathartfun.com/thedicelab.com/BalancedStdPoly.html`.

Furthermore, their die is much better than the one shown in Figure 1. Opposite labels sum to 21 and the three faces adjacent to a triangle always have labels that sum to 31.5 ± 0.5.

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04.00

Problem 321.4 – Zeros at the end of $(n!)!$

As Dave Wild pointed out (to me (TF), private communication), a glance at the last few entries of the table at the end of Jodie Forbes's Solution 319.5 – Factorial factorial (pages 6–7 of this issue) might lead one to believe that $(n!)!$ ends in approximately $n!/4$ zeros. A problem is suggested.

Prove that for large n, $(n!)!$ terminates in $\dfrac{(n! - n\log n)}{4} + O(n)$ zeros.

Front cover A 4-regular graph with 105 vertices and girth 6.