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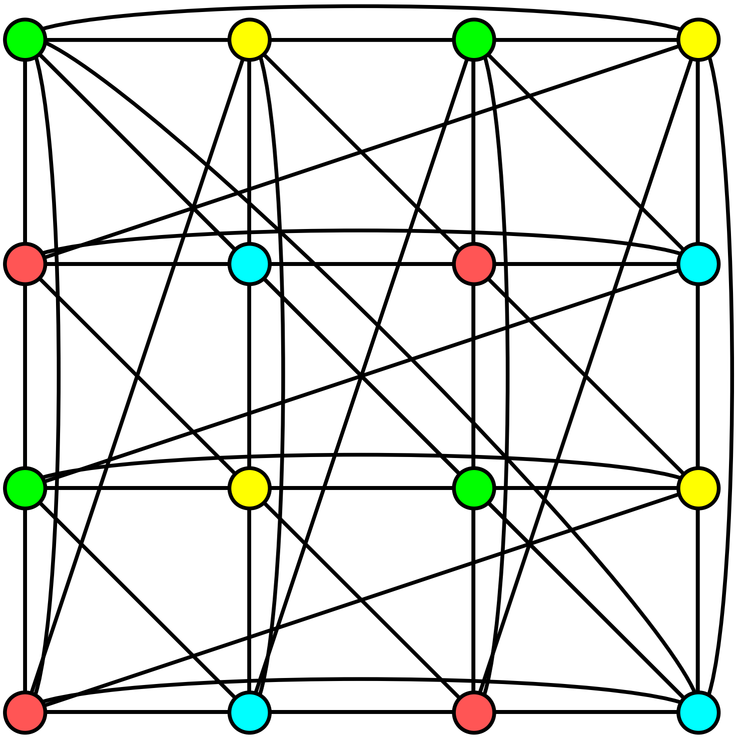
M500 325



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Editor – *Tony Forbes*

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Alan Borthwick, BEM

We would like to congratulate Alan Borthwick, Artistic Director, Edinburgh Gilbert and Sullivan Society, on his receipt of the British Empire Medal. The award is for services to Music and was included in the King's Birthday Honours, 2025. M500 Society members will of course know of Alan as an outstanding part-time tutor at the Open University and at our M500 Revision Weekends.

A general result for the scaling of metrics

Tommy Moorhouse

Introduction In an earlier investigation (M500 **322**, p. 19) we considered the effect on the integral $\int_M d^2x\sqrt{g}R$ of a rescaling of a metric on a surface, namely

$$\hat{g}_{ab} \;=\; \exp(2f)g_{ab}.$$

The general result for D-dimensional manifolds M is

$$\sqrt{\hat{g}}\hat{R} \;=\; e^{(D-2)f}\sqrt{g}\left(R - 2(D-1)\nabla^2 f - (D-1)(D-2)g(\nabla f, \nabla f)\right).$$

When $D = 2$ this gives our previous result. Here we find a proof of the general result using differential forms on M. The proof requires some care and the notation for arbitrary D can seem cluttered, so I have set out some results for $D = 3$ which the interested reader might like to use to get a concrete idea of the steps. We use the Einstein summation convention for repeated upper and lower indices so that

$$A_aB^a \;=\; A_1B^1 + \cdots A_DB^D.$$

We also use the Kronecker delta symbol δ^a_b which is 1 if $a = b$ and zero otherwise. We concentrate on metrics with signature $(+, +, \cdots, +)$ but the results carry over to other signatures with the appropriate modifications.

Proof using forms This proof uses properties of differential forms on manifolds with a metric. The metric allows us to define the dual of a p-form, which is a $(D - p)$-form (e.g. if $D = 3$ the dual of a 1-form is a 2-form), and this in turn allows us to find an expression for the change in the curvature 2-form $\rho^a{}_b$ due to a scaling of the metric, leading to the general result.

Properties of forms There are many good accounts of the properties of forms (such as [1]), and this brief description is just for quick reference. I have used some of the notation from [2]. A p-form ϕ_x defined on a manifold M maps p tangent vectors at $x \in M$ to a real number. The map is linear in each argument so that

$$\phi_x(av_1 + bw_1, v_2, \cdots, v_p) \;=\; a\phi_x(v_1, v_2, \cdots, v_p) + b\phi_x(w_1, v_2, \cdots, v_p)$$

and so on, with each tangent vector defined at x. It is skew when any two arguments are interchanged, so that

$$\phi_x(v_1, v_2, \cdots, v_p) \;=\; -\,\phi_x(v_2, v_1, \cdots, v_p)$$

etc. Here ϕ varies smoothly with x and we will generally omit the x in our notation. Both the tangent and cotangent spaces at any point of M are $\mathbb{R}^D$ so we can define a basis of 1-forms $\{e^a | a \in \{1, \cdots, D\}\}$. The index a labels the forms, which we could expand in a local coordinate basis as $e^a = e^a_\mu dx^\mu$. In fact we can take the forms to be such that

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu.$$

Here $g_{\mu\nu}$ is the manifold metric and η_{ab} gives the dot product in the cotangent space, so this definition is like that for an induced metric. Define a complementary basis $\{X_a\}$ for the vectors at x by the rule $e^a(X_b) = \delta^a_b$. We consider a 0-form (a function on M) f. In coordinates we define the 1-form $df = (\partial_\mu f) dx^\mu$. Then it is not hard to show that

$$df = X_a(f) e^a,$$

an expression that will prove very useful. Here $X_a(f)$ is the directional derivative of f along the vector X_a, which is $X^\mu_a \partial_\mu f$. (Note that some authors use the same kernel symbol for the 1-forms e^a and their complementary vectors e_b).

Curvature The curvature 2-form can be expressed in terms of the basis 1-forms. First we define the spin connection with components (all 1-forms) ${\omega^a}_b$ in terms of the exterior derivative of the basis forms:

$$de^a = -{\omega^a}_b \wedge e^b.$$

Essentially this is an expansion of the exterior derivative in terms of the basis forms. The curvature 2-form is then given by

$${\rho^a}_b = d{\omega^a}_b + {\omega^a}_c \wedge {\omega^c}_b.$$

This can be related to the Riemann and Ricci curvature tensors, as we shall see.

Dual forms The dual of a p-form

$$\psi = \psi_{a_1 a_2 \cdots a_p} e^{a_1} \wedge e^{a_2} \wedge \cdots \wedge e^{a_p}$$

is the $(D-p)$-form

$$(*\psi)_{b_1 b_2 \cdots b_{D-p}} = \frac{1}{(D-p)!} \psi_{a_1 a_2 \cdots a_p} {\varepsilon^{a_1 a_2 \cdots a_p}}_{b_1 b_2 \cdots b_{D-p}} e^{b_1} \wedge e^{b_2} \wedge \cdots \wedge e^{b_{(D-p)}}.$$

As this expression is no beauty (I wonder what it would score according to Jodie Forbes's scheme in M500 **323**) I will set out the expressions for $D = 3$:

$$\begin{aligned} *e_1 &= e^2 \wedge e^3, \\ *e_2 &= e^3 \wedge e^1, \\ *e_3 &= e^1 \wedge e^2, \\ *(e_1 \wedge e_2) &= e^3, \\ *(e_2 \wedge e_3) &= e^1, \\ *(e_3 \wedge e_1) &= e^2, \end{aligned}$$

with the others determined by antisymmetry:

$$e^a \wedge e^b \;=\; -\, e^b \wedge e^a.$$

The indices are raised and lowered with the 'flat' metric η_{ab}, but for us e_a and e^a both represent forms. The volume form is denoted

$$*1 \;=\; ed^D x \;=\; e^1 \wedge e^2 \wedge \cdots \wedge e^D,$$

where $e = \sqrt{g}$, $g = \det(g_{\mu\nu})$.

An expression like $D * 1$ means D times the volume form. We will need some identities involving forms and their duals:

$$\begin{aligned} e^c \wedge *e_b &= \delta^c_b * 1; \\ e^a \wedge *e_a &= \delta^a_a * 1 = \delta^1_1 + \delta^2_2 + \cdots + \delta^D_D = D * 1; \\ (e^c \wedge e^d) \wedge *(e_a \wedge e_b) &= (\delta^c_a \delta^d_b - \delta^c_b \delta^d_a) * 1; \\ (e^c \wedge e^a) \wedge *(e_a \wedge e_b) &= -(D-1)\delta^c_b * 1; \\ (e^a \wedge e^b) \wedge *(e_a \wedge e_b) &= D(D-1) * 1. \end{aligned}$$

Note the order of terms in the last identity. The reader might like to check these for the case $D = 3$, remembering the summation convention (e.g. $\delta^a_a = D$). It can be shown that

$$* * \psi \;=\; (-1)^{p(D-1)} \psi$$

for any p-form ψ.

The coderivative and the Laplacian operator We will need an expression for the Laplacian operator acting on a 0-form. Given p-forms ϕ and ψ the expression

$$(\phi, \psi) \;=\; \int_M \phi \wedge *\psi$$

is a well defined scalar product given by the integral over M of a D-form. For example $(e^a, e_b) = \delta^a_b$ (in units of the manifold volume) using the first of our identities above. It is easily checked that $(\phi, \psi) = (\psi, \phi)$. We can also define an operator related to the exterior derivative called the coderivative δ. This acts on p-forms to give $(p-1)$-forms and satisfies $(\phi, d\omega) = (\delta\phi, \omega)$. In each scalar product the forms on the left and right must have the same weight (i.e. they must both be p-forms for some p).

We can find an expression for the coderivative as follows, with ϕ a $(q+1)$-form and ψ a q-form:

$$\begin{aligned}
(\delta\phi, \psi) &= \int_M \phi \wedge *d\psi \\
&= \int_M d\psi \wedge *\phi \\
&= \int_M (d(\psi \wedge *\phi - (-1)^q \psi \wedge d * \phi) \quad \text{using partial integration} \\
&= \int_M (-1)^{q+1+(D-q)(D-1)} \psi \wedge * * d * \phi, \quad (d * \phi \text{ is a } (D-q)\text{-form}) \\
&= \int_M (-1)^{(D-q)(D-1)+q+1} * d * \phi \wedge *\psi.
\end{aligned}$$

Thus

$$\delta = (-1)^{(D-q)(D-1)+q+1} * d*$$

as an operator on the $(q+1)$-form ϕ. This equates to $\delta = (-1)^{D(p+1)+1} * d*$ acting on p-forms (you can check this by trying D and $p = q-1$ odd and even). In the partial integration step we have discarded the total derivative, assuming that there is no boundary or that other suitable boundary conditions hold. The Laplacian acts on 0-forms as

$$\nabla^2 f = -\delta df = *d * df.$$

In detail

$$\begin{aligned}
\nabla^2 f &= *d * df \\
&= *d * (X_a(f) e^a) \\
&= *d(X_a(f) * e^a) \\
&= *(X_b(X_a(f))e^b \wedge *e^a - X_a(f) d * e^a) \\
&= X_a(X^a(f)) + X_a(f)\omega_b^{ab}.
\end{aligned}$$

Here we have expanded ω^{ab} in the local 1-form basis as $\omega^{ab}_c e^c$. As always repeated upper and lower indices are summed over. The reader may like to work out the case $D = 3$ to see exactly what is happening. Note, for example, that in a term such as $\omega^a{}_b \wedge e^1 \wedge e^2$ only the part of $\omega^a{}_b$ proportional to e^3 (i.e $\omega^a_{3\,b}$) contributes.

Putting the forms to work After all this preparation we begin to put everything together. If $\hat{e}^a = \exp(f) e^a$ then

$$\begin{aligned} d\hat{e}^a &= df \wedge \hat{e}^a - \omega^a{}_b \wedge \hat{e}^b \\ &= X_b(f) e^b \wedge \hat{e}^a - \omega^a{}_b \wedge \hat{e}^b \\ &= -(X_b(f) e^a + \omega^a{}_b) \wedge \hat{e}^b. \end{aligned}$$

We should allow for a term proportional to e^b, which would have a zero wedge product with e^b. To find this term we calculate $d\hat{e}^b$ and get

$$\hat{\omega^a}_b = \omega^a{}_b + X_b(f) e^a - X^a(f) e_b.$$

Observe that the right hand side involves only unhatted forms. The scaled curvature 2-form is

$$\hat{\rho^a}_b = d\hat{\omega^a}_b + \hat{\omega^a}_c \wedge \hat{\omega^c}_b.$$

The reader might like to show that the volume integral expressed in terms of forms is

$$\int_M d^D x \sqrt{g} R = \int_M \rho^{ab} \wedge *(e_a \wedge e_b)$$

using the standard definition (in the notation of [2])

$$\rho^{ab} = \frac{1}{2} R_{\mu\nu}{}^{ab} dx^\mu \wedge dx^\nu = \frac{1}{2} R_{cd}{}^{ab} e^c \wedge e^d$$

with the identities above together with $R_{ab}{}^{ab} = R$, and $*1 = \sqrt{g} d^D x$.

In the scaled, hatted version, since we are dealing with the scaled metric, the dualisation (denoted $\hat{*}$) needs to include the scaling factors. In detail

$$\begin{aligned} \hat{*}(\hat{e}_a \wedge \hat{e}_b) &= \frac{1}{(D-2)!} \varepsilon_{abc\cdots d} \hat{e}^c \wedge \cdots \wedge \hat{e}^d \\ &= e^{(D-2)f} \frac{1}{(D-2)!} \varepsilon_{abc\cdots d} e^c \wedge \cdots \wedge e^d \\ &= *(e_a \wedge e_b) e^{(D-2)f}. \end{aligned}$$

We saw above that $\hat{\rho}^{ab}$ does not carry any scaling factors. This results in an overall factor of $\exp((D-2)f)$ in the final expression below. We can simply carry this forward to the final result as no other operators are applied to this factor.

Working through the details we find some cancellations between terms, and get:

$$\begin{aligned}\hat{\rho}^{ab} &= d\hat{\omega}^{ab} + \hat{\omega^a}_c \wedge \hat{\omega}^{cb} \\ &= \rho^{ab} + X_c(X^b(f))e^c \wedge e^a - X_c(X^a(f))e^c \wedge e^b - X^c(f){\omega^a}_c \wedge e^b \\ &\quad + X_c(f)\omega^{bc} \wedge e^a + X_c(f)X^b(f)e^a \wedge e^c - X_c(f)X^c(f)e^a \wedge e^b \\ &\quad + X^a(f)X_c(f)e^c \wedge e^b.\end{aligned}$$

Finally we use the wedge product identities to expand $\hat{\rho}^{ab} \wedge \hat{*}(\hat{e}_a \wedge \hat{e}_b)$. We find that the rescaled $\sqrt{\hat{g}}\hat{R}$ is

$$e^{(D-2)f}\sqrt{g}\,\big(R - 2(D-1)\nabla^2 f - (D-1)(D-2)X_c(f)X^c(f)\big) * 1.$$

The reader may like to show that

$$X_c(f)X^c(f) \;=\; g^{\mu\nu}(\partial_\mu f)(\partial_\nu f) \;=\; g(\nabla f, \nabla f),$$

from which the general result for a D-dimensional manifold follows.

References

[1] M. Crampin and F. A. E. Pirani, *Applicable Differential Geometry*, CUP, 1986.

[2] D. Z. Freedman & A. Van Proeyen, *Supergravity*, CUP, 2012, Chapter 7.

Problem 325.1 – A sieve for squares

Write down a sequence of infinitely many 1s. Then:

for $k = 2$ to ∞:
  for $j = 1$ to ∞:
    add 1 mod 2 to the (jk)-th number in the sequence.

Show that the nth number in the sequence is 1 if and only if n is an integer square—or find a counter-example.

Solution 323.1 – Step partitions

Given any positive integer N we define a step partition to be an expression for N as a sum of consecutive positive integers. For example, there are two step partitions of 5, namely 5 and $2+3$.

To get a feel for these partitions consider the step partition $30 = 4+5+6+7+8$. Since $8 = 4 \times 2$ we can immediately find another step partition by 'sharing out' the 8 between the four smaller elements, adding 2 to each element: $30 = 6+7+8+9$. This conversion can be done whenever the smallest or largest element of a partition is divisible by the number of remaining elements. Looking at it another way, since $4+5 = 9$ we can remove 4 and 5 from the partition $30 = 4+5+6+7+8$ and add in 9 to get, once again, $30 = 6+7+8+9$. 'Sharing out' the 6 gives the new partition $30 = 9+10+11$.

The number of step partitions $P_s(N)$ can be found by trial and error for small integers: $P_s(5) = 2$, $P_s(30) = 4$, for example, but can you find a simple formula for the number of step partitions of an integer N?

David Sixsmith

Suppose N is a natural number. Problem 323.1 asks us to calculate in how many ways we can write N as a 'step partition', i.e. a sum of consecutive positive integers. This is denoted by $P_S(N)$. (The problem does not state 'positive' explicitly, but I am assuming this is implicit.)

Since any sum of consecutive positive integers can be written as a difference of two triangular numbers (allowing 0 to be a triangular number), the question is asking how many ways we can choose non-negative integers n, m such that

$$N = \frac{1}{2}n(n+1) - \frac{1}{2}m(m+1).$$

Rearranging gives

$$2N = (n+m+1)(n-m).$$

So each choice of n, m corresponds to a factorization of $2N$. Now suppose we have such a factorization, so that $2N = AB$, for integers A, B. Solving

$$n+m+1 = A, \quad \text{and} \quad n-m = B,$$

gives

$$n = \frac{A+B-1}{2}, \quad \text{and} \quad m = \frac{A-B-1}{2}.$$

Hence each choice of n, m correspond to exactly one choice of factors such that one is odd and the other is even, and vice versa. So $P_S(N)$ is equal to the number of ways of factorizing $2N$ with one factor odd and the other even.

If N has prime factorization

$$N = 2^\alpha p_1^{k_1} p_2^{k_2} \dots p_t^{k_t},$$

with the p_j all odd primes and $\alpha \geq 0$, then there are exactly $(1+k_1)(1+k_2)\dots(1+k_t)$ ways to choose the even factor, which must include all powers of 2, and this then fixes the odd factor. Hence

$$P_S(N) = \prod_{j=1}^{t}(1+k_j), \quad \text{where} \quad N = 2^\alpha p_1^{k_1} p_2^{k_2} \dots p_t^{k_t}.$$

To give an example, $P_S(15) = 4$. This is because $2 \times 15 = 30$ factorizes as $30 \times 1 = 15 \times 2 = 10 \times 3 = 6 \times 5$, which gives the step partitions

$$15 = 15 = 7+8 = 4+5+6 = 1+2+3+4+5.$$

Note that another way to calculate $P_S(N)$ is first to divide out all powers of 2 that factor N, and then apply the τ function to the result.

Reinhardt Messerschmidt

We will call a positive divisor d of a positive integer N a *small* divisor if $d \leq N/d$, and an *opposite parity* divisor if d and N/d have opposite parities. Let $A(N)$ be the set of small opposite parity divisors of $2N$.

If a is a real number and ℓ is a positive integer such that

$$a + (a+1) + \cdots + (a+\ell-1) = N,$$

then it follows from the formula

$$a + (a+1) + \cdots + (a+\ell-1) = (2a+\ell-1)\ell/2$$

that

$$a = \frac{(2N/\ell) - (\ell-1)}{2}.$$

This implies that

$$\begin{aligned} &\text{There exists a step partition of } N \text{ with a length of } \ell \\ \iff\ & \frac{(2N/\ell)-(\ell-1)}{2} \text{ is a positive integer} \\ \iff\ & (2N/\ell)-(\ell-1) \text{ is a positive even integer} \\ \iff\ & (2N/\ell)-\ell \text{ is a nonnegative odd integer} \\ \iff\ & \ell \in A(N). \end{aligned}$$

If a step partition of N with a length of ℓ exists, then it is unique, because any such step partition has $((2N/\ell)-(\ell-1))/2$ as its initial term. It follows that

$$P_s(N) \;=\; |A(N)|.$$

For example, the small divisors of $2 \cdot 210 = 420$ are

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 10, \quad 12, \quad 14, \quad 15, \quad 20;$$

and the opposite parity divisors among these are

$$1, \quad 3, \quad 4, \quad 5, \quad 7, \quad 12, \quad 15, \quad 20; \tag{1}$$

therefore 210 has 8 step partitions; with lengths as in (1); and with corresponding initial terms

$$210, \quad 69, \quad 51, \quad 40, \quad 27, \quad 12, \quad 7, \quad 1.$$

Problem 325.2 – Two functions

There are two functions, A and B. Each maps $\{1,2\}$ into $\{1,2\}$. They satisfy

$$\begin{aligned} A(1) &\;=\; 1, \\ A(2)+B(2) &\;=\; 3, \\ A(x)+B(y) &\;\equiv\; 0 \pmod 2 \quad \text{otherwise.} \end{aligned}$$

Determine A and B.

Re: Problem 324.7 – Recursion

On the assumption that $f(n)$ is a function that maps non-negative integers to integers, define the sums $S_{f(n)}(i,j)$ and $S_{f(n)}(j)$ by

$$\begin{aligned} S_{f(n)}(i,0) &= 1, \\ S_{f(n)}(i,j) &= \sum_{k=0}^{i+f(j)} S_{f(n)}(k,j-1), \qquad & j \geq 1, \\ S_{f(n)}(j) &= S_{f(n)}(0,j), & j \geq 0. \end{aligned}$$

(i) Show that $S_{n\mapsto 0}(j) = 1$ for $j = 0,1,2,\ldots$.

(ii) Show that $S_{n\mapsto 1}(j)$ generates the Catalan numbers slightly displaced,

$$1,2,5,14,42,132,429,1430,\ldots = \frac{1}{j+2}\binom{2j+2}{j+1}, \quad j = 0,1,2,\ldots.$$

(iii) Obtain a nice formula for $S_{n\mapsto 2}(j)$.

(iv) More generally, obtain nice formulæ for $S_{n\mapsto k}(j)$, $k = 0,1,\ldots$.

(v) Even more generally, get a closed formula for $S_{f(n)}(j)$ in terms of the function $f(n)$ and the parameter j. If that's too difficult, just do it for your favourite nonconstant function $f(n)$. For example,

$$S_{n\mapsto(n \bmod 2)}(j) = 2,2,7,7,30,30,143,143,728,728,\ldots.$$

Robin Whitty

I will make life easy for myself by

1. not even attempting to think about general integer functions $f(n)$ (the constant function is hard enough for me!) and
2. allowing the value of i to take non-negative values, ignoring the last of the three defining functions.

Thus, denoting the constant function $n \mapsto t$ by t,

$$S_t(i,j) = \begin{cases} 1 & j = 0 \\ \sum_{k=0}^{i+t} S_t(k,j-1) & j > 0. \end{cases} \tag{1}$$

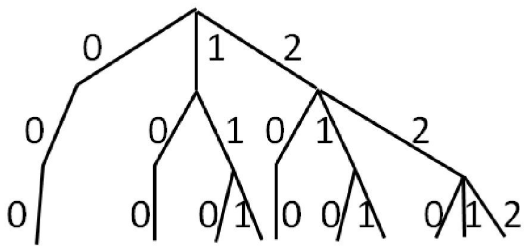
This version of S is a recursive implementation of a nested summation:

$$S_t(i,j) = \sum_{k_1=0}^{i+t} \sum_{k_2=0}^{k_1+t} \cdots \sum_{k_j=0}^{k_{j-1}+t} 1. \qquad (2)$$

I find it easier to think about nested sums than recursion but of course the latter is much more convenient to program and to write about in neat sentences.

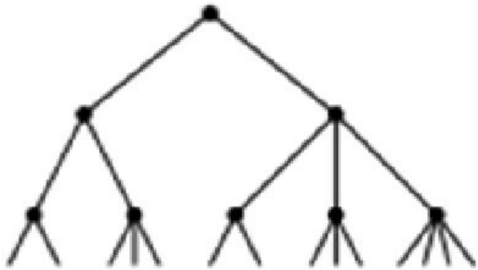
The nested sum in equation (2) with value $t = 0$ is studied by Butler and Karasik, "A Note on Nested Sums", *Journal of Integer Sequences*, Vol. 13, 2010, article 10.4.4. They start by remarking that $S_0(i,j) = \binom{i+j}{j}$. This follows by showing that the summation is equivalent to placing i identical balls into $j+1$ bins, which has a textbook binomial answer (you can check the details, the article is open access at cs.uwaterloo.ca/journals/JIS/). And it immediately answers Question (i): $S_0(0,j) = \binom{0+j}{j} = 1$. Butler and Karasik generalise $S_0(i,j)$ very interestingly by replacing the summand in equation (2) by things more complicated than just 1. We are being asked to generalise in a different direction by letting t take values more complicated than just 0.

Another way to think about $S_t(i,j)$ is as a tree. Each level of nesting of the summation in equation (2) is a level in the tree. The root has $i+t+1$ children; at each level the k-th eldest child (from left) of a vertex has $k+t$ children. Here is the tree for $S_0(2,3) = \binom{5}{3}$:



Now Question (ii) also has a textbook answer, the textbook being Richard Stanley's *Enumerative Combinatorics, Vol. 2*. Exercise 6.19 famously presents 66 different combinatorial objects which are counted by the Catalan numbers. The list continues on to Stanley's webpage math.mit.edu/~rstan/ec/, where a 96-page last-ever addendum to exercise 6.19 takes the number of objects to 207. On page 7 of this addendum is exercise 6.19.m[4], reproduced below:

(m4) Vertices of height $n-1$ of tree T defined by the property that the root has degree 2, and if the vertex x has degree k, then the children of x have degrees 2, 3, …, $k+1$.



You will see that the tree corresponds to $S_1(0,3)$ and the number of its leaves (vertices at height 3) is equal to the value of C_4, the 5th Catalan number (starting from zero). In general $S_1(0,n)=C_{n+1}$.

We can say more by evaluating $S_1(i,j)$ for other values of i. The Catalan numbers are the diagonal entries of a Pascal-type triangle (the Catalan triangle) in which entry n of row m (counting from zero) is the number of sequences of m Xs and n Ys in which the number of Ys, starting from left, never exceeds the number of Xs. We may refer to these as 'good' sequences. The number of good sequences may be denoted $C_{m,n}$ and it has value

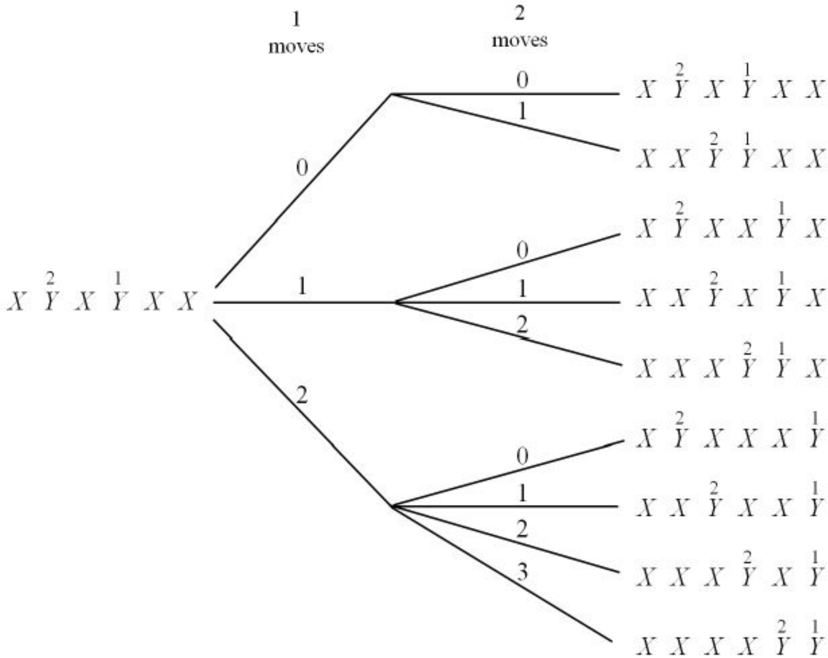
$$\frac{m+1-n}{m+1}\binom{m+n}{n}.$$

And we have

$$S_1(i,j)=C_{i+j+1,j}=\frac{i+2}{i+2+j}\binom{i+1+2j}{j}. \tag{3}$$

If we check this against our answer for $S_1(0,j)$ we get $C_{j+1,j}$ which is not a diagonal entry of the Catalan triangle. But in fact the diagonal entries are identical to those in the first subdiagonal. This is because every good sequence of m Xs and m Ys counted on the diagonal must end in a Y and so are in one-to-one correspondence with good sequences of m Xs and $m-1$ Ys.

We can confirm equation (3) by another appeal to the correspondence between nested sums and trees. In the picture below all good sequences of 4 Xs and 2 Ys are generated from a sequence (on the left of the picture) that is 'minimally' good in the sense that a shift of either Y to the left will result in a bad sequence.



Lemma *Let* $\sigma_{m,n}$ *be the sequence*

$$(XY)^n X^{m-n}.$$

Then the good sequences of m X*s and* n Y*s are generated uniquely from* $\sigma_{m,n}$ *by all possible repeated exchanges of a* Y *with an* X *immediately to its right.*

Proof. Notice that $\sigma_{m,n}$ is good; and moving any Y to the right will preserve this property. Any good distribution of the m Ys can be produced by repeated right-shifts of Ys. Suppose the Y are labelled, right-to-left, $1, 2, \ldots, m$. Then right-shifts of Y must maintain this ordering and therefore cannot produce duplicate good sequences. □

Applying the Lemma produces a tree in which the number of leaves is $C_{m,n}$. This tree corresponds to the evaluation of the nested sum in equation (2), with $t = 1$ and suitable values of i and j. For example, the tree shown above corresponds to the value $C_{4,2} = 9$. As a nested sum this will have two Σs, so $j = 2$. And the initial sum runs from $k_1 = 0$ to $k_1 = 2$. And indeed, with $i+j+1 = 4$ in equation (3) we have $i = 3-j = 3-2 = 1$, as expected.

Is there some sequence of combinatorial triangles that starts with Pascal and Catalan? This might supply an answer to Questions (iii) and (iv). The book by Thomas Koshy, *Triangular Arrays with Applications*, Oxford University Press, 2011, might be good place to look. Koshy also has a book on Catalan numbers and, not having had an opportunity to check, for all I know the above exploration may already reside there; there is not much that is new to say about Catalan numbers!

Question (v) seems to me of a different order of difficulty altogether. In fact, I feel inclined to suggest the following as a preemptive strike: show that it is undecidable that a given integer sequence can be represented by the general form nested sum.

Representation of integers as sums of odd squares

Dave Wild

If you have studied the OU course Number theory and Mathematical Logic (M381) then you will have come across the problem of representing integers as the sum of squares. When I studied the course the number theory part was based on readings from *Elementary Number Theory* by David M. Burton. I did not find the proof that any positive integer could be written as the sum of at most four non-zero squares to be elementary. Here I look at how many squares are required if we decide to use only the squares of odd integers.

If we look at the square of an odd integer $2n + 1$ we find that

$$(2n+1)^2 \;=\; 4n^2+4n+1 \;=\; 4n\,(n+1)+1.$$

Since either n or $n + 1$ is even then it means that all of the squares are of the form $8m + 1$. So if we want to find a representation for an integer n then we immediately know how many terms, modulo 8, are required in the sum. So, for example, any sum of odd squares which totals 2025 can only contain either 1, 9, 17, . . . , or 2025 terms. In this case the representation we are interested in is $2025 = 45^2$.

I wrote a program to determine how many squares were required to write all the integers up to a million.The results showed that if $n \equiv 1 \pmod 8$ then either 1 or 9 squares were required. For example $17 = 3^2 + 8 \times 1^2$, where the multiplication sign indicates that one-squared is repeated 8 times.

If $n \equiv 2 \pmod 8$ then either 2 or 10 squares were required. The smallest example which required 10 squares is

$$42 = 5^2 + 3^2 + 8 \times 1^2.$$

The following theorem from Burton explains why some of the numbers require more than 2 squares.

> Let the positive integer n be written as $n = N^2 m$, where m is square-free. Then n can be represented as the sum of two squares if and only if m contains no prime factor of the form $4k + 3$.

See the 'Sum of two squares theorem' article in Wikipedia for an equivalent theorem.

All the other integers can be expressed using the minimum possible number of squares according to their value modulo 8; i.e. 3, 4, 5, 6, 7, or 8. So it looks as if every integer could be expressed using a maximum of 10 squares.

If we assume that every number of the form $n \equiv 3 \pmod 8$ can always be expressed as the sum of 3 squares then we can also conclude the maximum number of squares required for any integer is 10. For example, if we represent 3 as a sum of 3 squares then we can find a representation of the integers 4 to 10 in the form of a sum of 10 or fewer squares simply by repeatedly adding one-squared. The next integer, 11, is of the form $8n+3$. By our assumption this can be expressed as a sum of three squares and so we can repeat the above process. The procedure described above does not necessarily produce the shortest representation. For example $10 = 10 \times 1^2$ could have been expressed as $3^2 + 1^2$.

We now have to show that any positive integer of the form $4n + 3$ can be expressed as the sum of 3 squares. Even if we allow both odd and even squares then we require 3 odd squares in the representation plus, possibly, one even square. Wikipedia's article on 'Legendre's three square theorem' comes to the rescue. This states that any integer can be represented as the sum of three squares unless it is of the form $4^a\,(8b + 7)$ where a and b are non-negative integers. Therefore all numbers of the form $8n + 3$ can be expressed as the sum of 3 odd squares.

Our task is finished. Any positive integer can be expressed as the sum of 10 or fewer odd squares.

On 1 + 2 = 3

Ted Gore

In M500 **323** Dilwyn Edwards posed several questions in his article 'On 1 + 2 = 3'.

> Write down a finite string of sequential natural numbers such as 123456 and insert a 'break point' | so that the left hand and right hand sums are as close as possible. In this case 1234|56 giving sums that differ by 1. For 123456789 the best we can do is 123456|789 giving a difference of 3.
>
> 1. Am I [DE] right in thinking that the two sums can never be equal except for the simple case 12|3?
> 2. In the case of 12345678 there are two equally good break points 123456|78 and 12345|678, both giving a difference of 6. Are there other similar cases?
> 3. The ratio of the number of numbers on the left to those on the right seems to converge to about 2:1 as we increase the length of the sequence. What is the correct limit (if the ratio does converge)?

Let L be the sum of integers to the left of the break point and R be the sum of the integers to the right. If p is the number of integers to the left and k the number to the right, then $L = p(p+1)/2$ and $R = k(k+1)/2 + pk$. The difference is $R - L$.

When $R - L = 0$,

$$k = \frac{-(2p+1) \pm \sqrt{8p^2 + 8p + 1}}{2}.$$

We want a positive value for k; so

$$k = \sqrt{2p^2 + 2p + 1/4} - (p + 1/2).$$

We can use this result to find the smallest difference for a particular value of p.

For $p = 6$, $k = 2.67878$. Being non-integer this gives us a choice of nearby integers

$$k_0 = \lfloor k \rfloor \quad \text{and} \quad k_1 = \lceil k \rceil.$$

For $k_0 = 2$, $R - L = -6$ and $p/k_0 = 3$. For $k_1 = 3$, $R - L = 3$ and $p/k_1 = 2$.

Thus $k_0 = 2$ provides the difference for 123456|78. I have chosen to use the geometric mean of p/k_0 and p/k_1 as a measure of the convergence of p/k. For $p = 6$, the geometric mean of p/k_0 and p/k_1 is 2.44949.

For $p = 5$, $k = 2.26209$. For $k_0 = 2$, $R - L = -2$ and $p/k_0 = 2.5$. For $k_1 = 3$, $R - L = 6$ and $p/k_1 = 1.66666667$.

Thus $k_1 = 3$ provides the difference for 12345|678. For $p = 5$, the geometric mean of p/k_0 and p/k_1 is 2.04124.

The convergence of p/k as p increases is of interest.

For $p = 100000000$, $k = 41421356.44442$.

For $k_0 = 41421356$, $R - L = -62849954$ and $p/k_0 = 2.41421358$.

For $k_1 = 41421357$, $R - L = 78571403$ and $p/k_1 = 2.41421352$

The geometric mean of p/k_0 and p/k_1 is $2.41421355 \approx 1 + \sqrt{2}$.

I should mention that this calculation is based on numbers of integers rather than numbers of characters. For example, I count 204 as 1 rather than 3. To do otherwise would be complicated and might depend on which number base you used.

For R to equal L we need $\left(-(2p+1) + \sqrt{8p^2 + 8p + 1}\right)/2$ to be an integer. Values of $p \in [2, 1000]$ that ensure this are 2, 14, 84, 492 for which $k = 1$, 6, 35, 204 and for which $R = L = 3$, 105, 3570, 121278.

For a constant value of $p + k$, when do we get two equally good break points?

Let $R_0 - L_0$ be the difference for one break point and let $R_1 - L_1$ be that for the break point moved one place to the right.

We want to find p and k such that

$$L_0 = p(p+1)/2, \quad R_0 = k(k+1)/2 + pk,$$

$$L_1 = (p+1)(p+2)/2 \quad \text{and} \quad R_1 = (k-1)k/2 + (p+1)(k-1).$$

Setting $R_0 - L_0 = R_1 - L_1$ results in $p = -1$, which is unproductive. Setting $R_0 - L_0 = L_1 - R_1$ gives

$$k = \sqrt{2p^2 + 4p + 9/4} - (p + 1/2)$$

and we require that this is an integer. Values of $p \in [2, 1000]$ that ensure this are 5, 34, 203 for which $k = 3$, 15, 85 and for which the differences are 6, 35, 204.

DE gives the example for a difference of 6.

On $1 + 2 = 3$

Reinhardt Messerschmidt

In M500 **323** Dilwyn Edwards asked several questions concerning the splitting of a finite string of sequential natural numbers such as 123456 by a 'break point' | so that the left hand and right hand sums are as close as possible. [See page 16 for the details.]

For every positive integer n and $b \in \{0, \ldots, n\}$, let

$$\begin{aligned} S(b) &= 1 + \cdots + b = b(b+1)/2 \\ T_n(b) &= (b+1) + \cdots + n = (n+b+1)(n-b)/2 \\ f_n(b) &= S(b) - T_n(b) = b^2 + b - n(n+1)/2. \end{aligned}$$

Question 1

This question is asking for solutions (n, b) to the equation $f_n(b) = 0$. From the identity

$$-4f_n(b) = n^2 + (n+1)^2 - (2b+1)^2,$$

we have that $f_n(b) = 0$ is equivalent to

$$n^2 + (n+1)^2 = (2b+1)^2.$$

In other words, solving $f_n(b) = 0$ is equivalent to finding *almost isosceles Pythagorean triples* (AI-PTs), which are triples (r, s, t) of positive integers such that $r^2 + s^2 = t^2$ and $|r - s| = 1$. We will construct an infinite number of such triples.

Let $(P_0, P_1, \ldots)$ be the sequence of integers defined by

$$P_0 = 0, \qquad P_1 = 1, \qquad P_k = 2P_{k-1} + P_{k-2}, \qquad k \geq 2.$$

This recurrence relation resembles the Fibonacci recurrence relation, the only difference being the coefficient of 2. The numbers in the sequence $(P_0, P_1, \ldots)$ are the *Pell numbers*. The first few Pell numbers are

$$0, \quad 1, \quad 2, \quad 5, \quad 12, \quad 29, \quad 70, \quad 169, \quad 408, \quad \ldots$$

We have

$${P_1}^2 - {P_0}^2 - 2P_1P_0 = 1$$

and

$${P_k}^2 - {P_{k-1}}^2 - 2P_kP_{k-1} = -\big({P_{k-1}}^2 - {P_{k-2}}^2 - 2P_{k-1}P_{k-2}\big);$$

therefore

$$\left|{P_k}^2 - {P_{k-1}}^2 - 2P_kP_{k-1}\right| \;=\; 1$$

for every $k \geq 1$. Furthermore, for any integers u and v we have

$$(u^2 - v^2)^2 + (2uv)^2 \;=\; (u^2 + v^2)^2.$$

It follows that the triple

$$\left({P_k}^2 - {P_{k-1}}^2, \quad 2P_kP_{k-1}, \quad {P_k}^2 + {P_{k-1}}^2\right)$$

is an AI-PT for every $k \geq 2$.

The first few AI-PTs in this sequence are

$$(3,\ 4,\ 5), \quad (21,\ 20,\ 29), \quad (119,\ 120,\ 169), \quad (697,\ 696,\ 985), \quad \ldots \tag{1}$$

The corresponding solutions (n, b) to $f_n(b) = 0$ are

$$(3,\ 2), \qquad (20,\ 14), \qquad (119,\ 84), \qquad (696,\ 492), \qquad \ldots$$

Question 2

Since f_n satisfies the functional equation

$$f_n(b+1) \;=\; f_n(b) + 2(b+1),$$

this question is asking for solutions (n, b) to the equation $f_n(b) = -(b+1)$. Such a solution also satisfies $f_n(b+1) = b+1$. The equation $f_n(b) = -(b+1)$ is equivalent to

$$n(n+1) \;=\; 2(b+1)^2. \tag{2}$$

We will construct two solutions to (2) for each AI-PT. Since we constructed an infinite number of AI-PTs earlier, this will give an infinite number of solutions to (2).

Suppose $(r,\ r+1,\ t)$ is an AI-PT. From $r^2 + (r+1)^2 = t^2$, we have

$$(2r+1)^2 + 1 \;=\; 2t^2. \tag{3}$$

Let

$$(x,\ y) \;=\; (2t - 2r - 1,\ 2r + 1 - t);$$

then

$$2y + x \;=\; 2r + 1, \qquad x + y \;=\; t.$$

It follows from (3) that

$$(2y + x)^2 + 1 \;=\; 2(x + y)^2;$$

therefore

$$2y^2 + 1 \;=\; x^2. \tag{4}$$

This is an instance of *Pell's equation.* Let

$$(n,\ b) \;=\; (2y^2,\ xy - 1);$$

then it follows from (4) that (n, b) satisfies (2).

The AI-PTs in (1) give the following solutions (x, y) to (4):

$$(3,\ 2), \qquad (17,\ 12), \qquad (99,\ 70), \qquad (577,\ 408), \qquad \ldots$$

The corresponding solutions (n, b) to (2) are

$$(8,\ 5), \qquad (288,\ 203), \qquad (9800,\ 6929), \qquad (332928,\ 235415), \qquad \ldots$$

To construct another solution to (2) from $(r,\ r + 1,\ t)$, let

$$(x,\ y) \;=\; (2r + 1,\ t).$$

It follows from (3) that

$$x^2 + 1 \;=\; 2y^2. \tag{5}$$

This is an instance of the *negative Pell's equation.* Let

$$(n,\ b) \;=\; (x^2,\ xy - 1);$$

then it follows from (5) that (n, b) satisfies (2).

The AI-PTs in (1) give the following solutions (x, y) to (5):

$$(7,\ 5), \qquad (41,\ 29), \qquad (239,\ 169), \qquad (1393,\ 985), \qquad \ldots$$

The corresponding solutions (n, b) to (2) are

$$(49,\ 34), \quad (1681,\ 1188), \quad (57121,\ 40390), \quad (1940449,\ 1372104), \quad \ldots$$

Question 3

If b is allowed to have real values, then f_n is strictly increasing on $[0, n]$ and has a zero in this interval at

$$b_0 \;=\; \frac{-1+\sqrt{2n^2+2n+1}}{2}.$$

This implies that, for $b \in \{0, \ldots, n\}$, the quantity $|f_n(b)|$ achieves a minimum at $\lfloor b_0 \rfloor$ or $\lceil b_0 \rceil$. Let L_n be $\lfloor b_0 \rfloor$ or $\lceil b_0 \rceil$ (whichever minimizes $|f_n(b)|$), and let $R_n = n - L_n$. We have

$$L_n \;=\; x_n + b_0 \;=\; \frac{2x_n - 1 + \sqrt{2n^2+2n+1}}{2}$$

for some real number $x_n \in (-1, 1)$; therefore

$$L_n/n \;=\; \frac{2x_n - 1}{2n} + \frac{\sqrt{2+2/n+1/n^2}}{2} \;\longrightarrow\; \sqrt{2}/2;$$

therefore

$$L_n/R_n \;=\; \frac{L_n/n}{1-L_n/n} \;\longrightarrow\; \frac{\sqrt{2}/2}{1-\sqrt{2}/2} \;=\; \sqrt{2}+1.$$

The number $\sqrt{2}+1$ is known as the *silver ratio*, because it has properties that are similar to the properties of the golden ratio. For example, the golden ratio is the limit of the ratio of consecutive Fibonacci numbers, and the silver ratio is the limit of the ratio of consecutive Pell numbers.

Problem 325.3 – Factors

Tony Forbes

Let

$$L \;=\; \{1, 3, 9, 5, 7, 11, 13, 17, 19, 23\},$$

the odd prime powers less than 24. Suppose n divides 334639305, the lcm of the numbers in L. Suppose also that n has a factorization $n = f_1 f_2 f_3$ where each factor f_i, $i = 1, 2, 3$, is the product of elements of L that sum to at most 24.

How many n are there?

For example, $n = 255255 = (3\cdot 17)(5\cdot 13)(7\cdot 11)$ is included in the count, but not $81719 = 11 \cdot 17 \cdot 19 \cdot 23$.

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04.00

Problem 325.4 – Two pairs

Tony Forbes

Let $n \geq 4$ be an integer. How many structures

$$\{\{a, b\}, \{c, d\}\}$$

are there that satisfy all of the following:

(i) $1 \leq a, b, c, d \leq n$;
(ii) a, b, c, d are distinct;
(iii) either $a + c = b + d = n + 1$, or $a + d = b + c = n + 1$.

Front cover The Shrikhande graph. It is strongly regular with parameters srg$(16, 6, 2, 2)$, it is 4-chromatic and its complement is 6-chromatic. It is named after Sharad–Chandra S. Shrikhande (1917–2020), who showed that it is the only graph with the stated properties.