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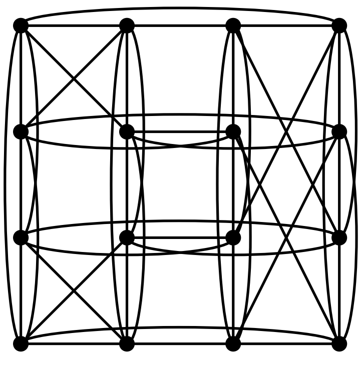

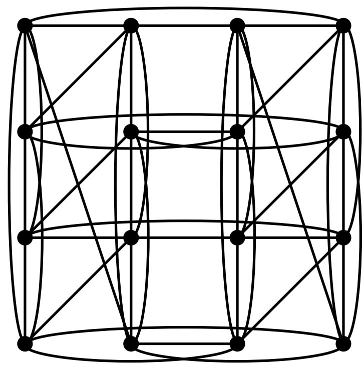

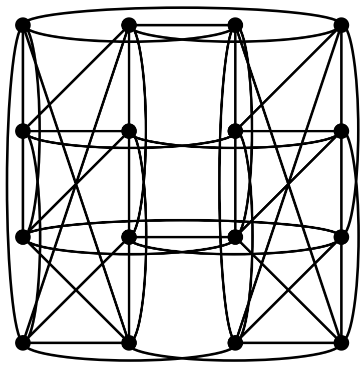

The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The Revision Weekend is a residential Friday to Sunday event providing revision and examination preparation for both undergraduate and postgraduate students. For details, please go to the Society's website.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, please go to the Society's website.

Editor – *Tony Forbes*

M500 Winter Weekend 2026

The **forty-third M500 Society Winter Weekend** will be held over

Friday 9th – Sunday 11th January 2026
at Kents Hill Park Conference Centre, Milton Keynes.

For details, pricing and a booking form, please refer to the M500 web site.

m500.org.uk/the-M500-winter-weekend/

Stuart Walmsley

Sadly, we have to report that Stuart Walmsley died in June 2025, aged 90. We are told that he really enjoyed his membership of the M500 Society—it was an excellent way to help keep him mentally agile to the end. He provided many contributions to the magazine M500 over the years.

Mirages

J. M. Selig

In an article in M500 **320** I promised to look at how refraction in the atmosphere affects the position of the horizon. I'm afraid this article does not make good on that promise. Rather, it is a step in that direction but doesn't address the question directly. Look at it more as a "warm-up" exercise. The purpose of this article is to say something about atmospheric refraction and look at some simple but, I hope, interesting examples—mirages.

According to Einstein, the speed of light in a vacuum is a physical constant, $c = 299792458\,\mathrm{ms}^{-1}$, and gives an upper limit for the speed of particles and the transmission of information. In a medium such as air, water or glass light slows down, and the ratio of c to the speed of light in the medium is called the refractive index of the medium, usually denoted n. When light passes from a medium with refractive index n_1 to another medium with refractive index n_2 it obeys Snell's law,

$$n_1 \sin\phi_1 \;=\; n_2 \sin\phi_2.$$

Here, ϕ_1 is the angle of incidence and ϕ_2 is the angle of refraction; see Figure 1.

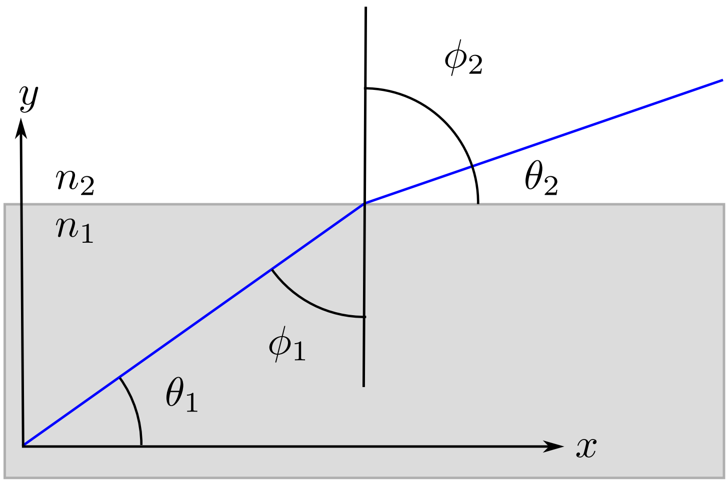

Figure 1: Snell's law.

Suppose that we are considering a reasonably short distance over which we can approximate the Earth to be flat. Further, assume that the refractive index of the air varies only with height, $n = n(y)$. The refractive index of air depends on several factors but mainly temperature, pressure and humidity. I don't want to get lost in the details here, so below we will assume simple models for how the refractive index varies with height above the ground.

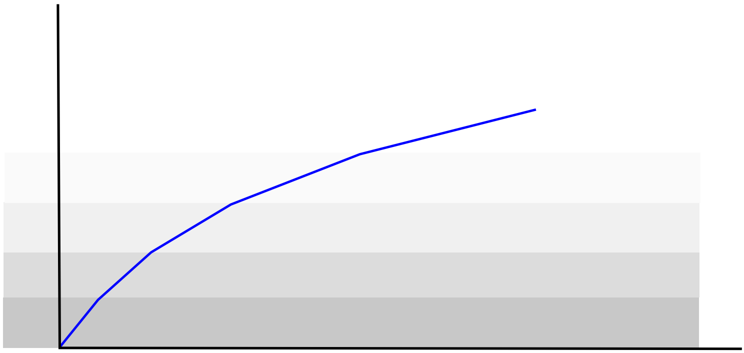

Figure 2: Slabs of atmosphere with constant refractive index.

We can model the atmosphere here as a sequence of thin, constant thickness slabs stacked vertically, each with constant refractive index. But each slab has a slightly different refractive index from its neighbours. The path of a light ray will be a piecewise linear curve, which can be found using Snell's law at each interface between slabs, see Figure 2. Then we take the limit as the thickness of the slabs tends to zero. The path of a light ray will now be a smooth curve in a vertical plane. In terms of Cartesian coordinates Snell's law becomes

$$n_1 \cos\theta_1 \;=\; n_2 \cos\theta_2,$$

where θ_i is the angle the light rays make with the x-axis. If we assume that the refractive index varies smoothly with height, then we can write

$$n(y)\cos\theta \;=\; k,$$

where k is a constant. In general, $(dy/dx) = \tan\theta$, so we have a differential equation for the curve,

$$\frac{dy}{dx} \;=\; \pm\sqrt{\left(\frac{n(y)}{k}\right)^2 - 1}, \tag{1}$$

since $\sec\theta = n(y)/k$ and $\tan^2\theta = \sec^2\theta - 1$.

Notice, by the way, that the light rays are reversible; so it doesn't matter if our light ray begins in the eye of the observer and travels out from there. Let's choose coordinates so that the observer's eye is located at the origin. This means that we can set $k = n_0\cos\theta_0$, where $n_0 = n(0)$ and θ_0 is the initial (or final) angle the light ray makes with the horizontal.

We also need an explicit model for the refractive index. For simplicity, let's assume that it varies linearly with height,

$$n(y) \;=\; n_0(ay+1),$$

where a is a constant. This formula ensures that when $y = 0$ the value of the refractive index is n_0. More complicated models could be approximated by such a linear model.

We can resolve the sign ambiguity by looking at the curve at the origin. At that point the angle to the horizontal is θ_0 and so $dy/dx = \tan\theta_0$. The square root on the right-hand side of the differential equation reduces to $\sqrt{\sec^2\theta_0 - 1}$. So we have to limit the angle to $-\pi/2 < \theta_0 < \pi/2$. We must also take the positive square root if $0 \leqslant \theta_0 < \pi/2$ and the negative square root when $-\pi/2 < \theta_0 < 0$, to agree with the sign of $\tan\theta_0$.

The differential equation above can now be solved for the curve by integration:

$$x \;=\; \pm\int_0^y \left(\left(\frac{n(y')}{k}\right)^2 - 1\right)^{-1/2} dy'.$$

This can be transformed into a standard integral using the substitution

$$u \;=\; \frac{n(y)}{k} \;=\; \frac{(ay+1)}{\cos\theta_0} \qquad \text{with} \qquad \frac{du}{dy} \;=\; \frac{a}{\cos\theta_0}.$$

This gives

$$x \;=\; \pm\frac{\cos\theta_0}{a}\int_{\sec\theta_0}^{(ay+1)\sec\theta_0} \frac{1}{\sqrt{u^2-1}}\,du \;=\; \pm\frac{\cos\theta_0}{a}\Big[\operatorname{arccosh}(u)\Big]_{\sec\theta_0}^{(ay+1)\sec\theta_0}.$$

So that

$$y \;=\; \frac{1}{\alpha}\cosh(\alpha x + \beta) - \frac{1}{a} \quad \text{when} \quad 0 \leqslant \theta_0 < \pi/2$$

and

$$y \;=\; \frac{1}{\alpha}\cosh(-\alpha x + \beta) - \frac{1}{a} \quad \text{when} \quad -\pi/2 < \theta_0 < 0,$$

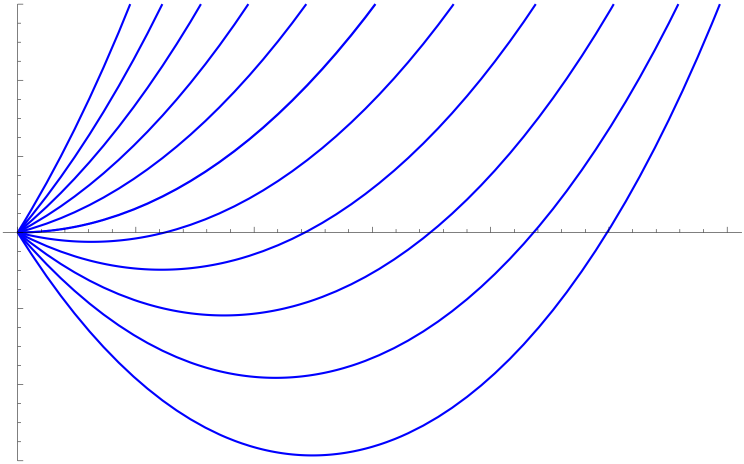

Figure 3: Light Rays for an Inferior Mirage.

where

$$\alpha = a \sec \theta_0 \quad \text{and} \quad \beta = \operatorname{arccosh}(\sec \theta_0).$$

The light rays follow catenaries. If the coefficient a is positive, the refractive index is increasing with height; this can happen over a hot surface, like the sand of a desert or a road on a hot day. The air near the hot surface has a low refractive index but as you get further above the surface the air cools and the refractive index increases; see Figure 3. Notice that some of the light rays from the sky will enter the eye from below. So looking down you will see the sky. This can lead to people confusing the light from the sky with water on the ground. This phenomenon is known as an inferior mirage.

It is also possible to get a superior mirage, also sometimes given the wonderful name "Fata Morgana" after Morgan le Fay, the legendary King Arthur's legendary half-sister, [1]. This can occur over ice or snow if the temperature gradient in the air is increasing and hence the refractive index is decreasing, the case where a is negative. Here, light from the land appears in the sky giving the illusion of "floating islands". Notice that the solution paths for the light rays can cross each other, meaning that a distant object will appear upside-down; see Figure 4.

Before finishing I would like to just say something more about Snell's

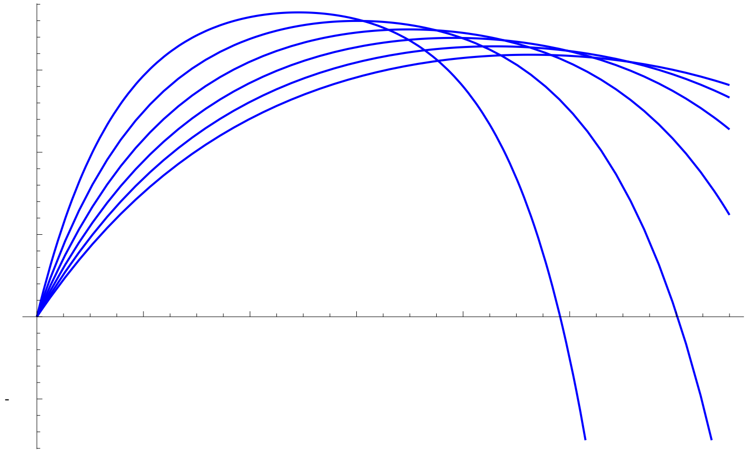

Figure 4: Light Rays for a Superior Mirage.

law. We were lucky with the analysis above because we could approximate the atmosphere by thin slabs with constant vertical height, so we could use Snell's law. What if the refractive index had varied in a more complicated way? And anyway, where does Snell's law come from? According to the Wikipedia article [2], Fermat's principle is the link between ray optics and wave optics. The principle states that light travels along the path that minimises the optical path length, where the optical path length is the Euclidean distance d along the path times the refractive index n. That is, $n \times d = cd/v$, with v the speed of light in the medium. Hence the optical path length is c times the time taken by the light.

This principle can be used to give a simple derivation of Snell's law. Consider Figure 5. The optical path length λ of the path shown from A to B is clearly

$$\lambda \;=\; n_1\sqrt{x^2 + y_1^2} + n_2\sqrt{(l-x)^2 + y_2^2}.$$

Treating x as the variable here, we can differentiate and set the result to zero to find the minimum,

$$\frac{d\lambda}{dx} \;=\; n_1\frac{x}{\sqrt{x^2 + y_1^2}} - n_2\frac{(l-x)}{\sqrt{(l-x)^2 + y_2^2}} \;=\; 0.$$

Recognising that $x/\sqrt{x^2 + y_1^2} = \sin\phi_1$ and $(l-x)/\sqrt{(l-x)^2 + y_2^2} = \sin\phi_2$

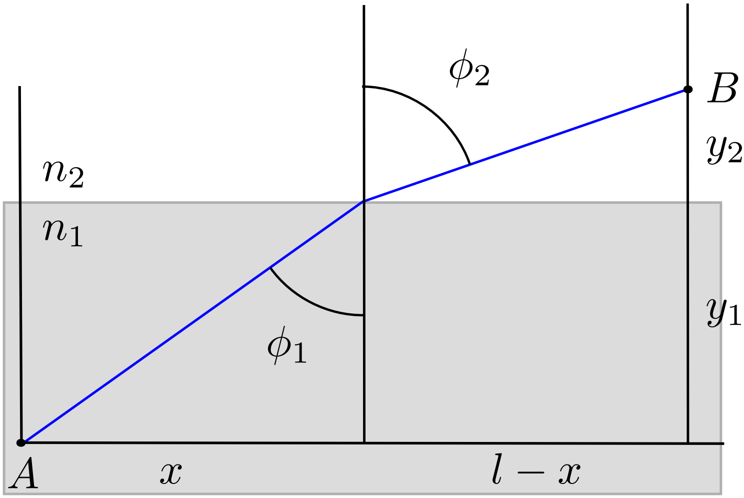

Figure 5: Deriving Snell's law.

the equation above can be rearranged to give Snell's law,

$$n_1 \sin \phi_1 \;=\; n_2 \sin \phi_2.$$

There is also a way to show this without using any calculus. See [3].

By the way, the above solves another problem: Suppose you are at A on a sandy beach and see someone drowning in the sea at B. Your running speed on the sand is $1/n_1$ and you can swim at a speed of $1/n_2$, what is the quickest route to get to the casualty? Of course, in such a situation computing the value of x might delay the rescue by more than just guessing.

If the refractive index is a function of the position in the medium then the optical path length is given by the integral

$$\int n \, ds,$$

where n is the refractive index of the medium and ds is the element of Euclidean length along a curve. This turns the problem of finding the path that light takes into a calculus of variations problem. For the problem of mirages solved above we would have to find the function $y(x)$ that minimises the integral

$$\lambda \;=\; \int n(y)\sqrt{dx^2+dy^2} \;=\; \int n(y)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Given a model for the variation of the refractive index with height, we could derive the Euler–Lagrange equation for this functional. This turns out to have a fairly simple first integral and after some simplification we arrive back at equation (1). This is something of a battle, however it is a battle we will have to fight to look at refraction on a spherical Earth!

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[1] Wikipedia contributors, “Fata Morgana (mirage),” Wikipedia, The Free Encyclopedia, `https://en.wikipedia.org/wiki/Fata_Morgana_(mirage)` (accessed June 11, 2025).

[2] Wikipedia contributors, “Fermat’s principle,” Wikipedia, The Free Encyclopedia, `https://en.wikipedia.org/wiki/Fermat’s_principle` (accessed June 11, 2025).

[3] Grant Sanderson, (3Blue1Brown), (2016), “Snell’s law proof using springs”, `https://www.youtube.com/watch?v=Iq1a_KJTWJ8`

Problem 326.1 – Binomial coefficient products

Tony Forbes

Let $P(n)$ be the product of the elements of row n of Pascal’s triangle; i.e.

$$P(n) \;=\; \prod_{m=0}^{n}\binom{n}{m}.$$

Show that for given integer k,

$$\frac{P(n-k)P(n+k)}{P(n)^2} \;\to\; \exp(k^2) \qquad \text{as } n\to\infty.$$

For large k the convergence can be quite slow. For example, with $k = 10$ we have

$$\frac{P(90)\;P(110)}{P(100)^2\;e^{100}} \;=\; 0.608047, \qquad \frac{P(990)\;P(1010)}{P(1000)^2\;e^{100}} \;=\; 0.95126.$$

The derivative of a simple graph

Tommy Moorhouse

Introduction

In M500 **294** I described a type of graph multiplication, the smash product. It works on simple graphs, i.e. sets of vertices with edges possibly connecting some of them. There are no multiple edges, and edges must start and finish at distinct vertices.

Here we investigate a derivative acting on connected graphs and its relationship with the smash product. $G+H$ denotes the disconnected graph consisting of G and H and we will say that G is strongly connected if it cannot be disconnected by deleting a single vertex (together with any edges connected to it). We take the empty graph, the graph consisting of a single vertex, and K_2, the connected graph with two vertices, to be strongly connected.

The smash product and the derivative

The smash product $G \vee H$ of two connected graphs G and H is constructed by drawing the graphs side by side and connecting every vertex of G to every vertex of H, making no other changes. The empty graph E_0 is the smash product unit, meaning $E_0 \vee G = G$. The smash product is extended to disconnected graphs by the rule

$$(G_1 + G_2) \vee H \;=\; G_1 \vee H + G_2 \vee H.$$

Perhaps surprisingly we can define a derivative ∇ acting on connected graphs. We consider a graph G. If G is disconnected, we break it down:

$$\begin{aligned} G &= G^{(1)} + G^{(2)} + \cdots + G^{(k)}; \\ \nabla G &= \nabla G^{(1)} + \nabla G^{(2)} + \cdots + \nabla G^{(k)}. \end{aligned}$$

We now define ∇ acting on a connected simple graph G with n vertices. Make n copies of G, which we call $G_1, G_2, \ldots, G_n$, each with the vertices labelled in the same way, i.e. as $v_1, v_2, \ldots, v_n$. From G_1 delete v_1 together with all the edges connected to it. From G_2 delete v_2 and all its edges. Continue until vertex v_i and all the edges connected to it have been removed from G_i for $i = 1, 2, \ldots, n$. Now we have a disconnected graph consisting of n modified copies of G each with $n - 1$ vertices, and this disconnected graph is what we call ∇G. The altered copies need not be isomorphic. The procedure is shown schematically in Figure 1.

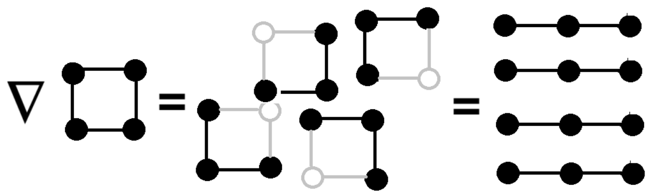


Figure 1: $\nabla C_4 = 4l_3$

Examples

Readers can check the following examples using the recipe above.

$$\begin{aligned} \nabla E_n &= nE_0, \\ \nabla C_n &= nl_{n-1}, \\ \nabla K_n &= nK_{n-1}. \end{aligned}$$

Here E_n is the graph consisting of n vertices and no edges, C_n is the cyclic graph with n vertices, K_n is the complete graph with n vertices, and l_n is the linear graph with n vertices. Note in particular that the derivative of E_0 is the empty set; $\nabla E_0 = \emptyset$.

The Leibniz rule

Given G and H, both strongly connected, show that the smash product and the derivative satisfy the Leibniz rule:

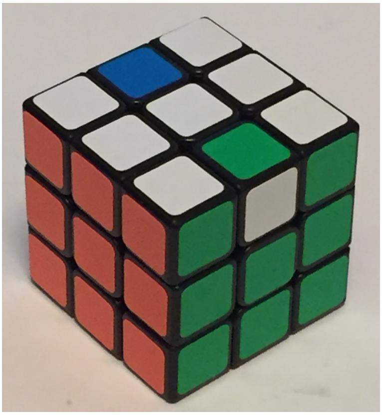
$$\nabla(G \vee H) = (\nabla G) \vee H + G \vee \nabla H.$$

There may be more to explore here, and (possibly!) fun to be had differentiating various graphs. The correspondence $K_n \sim x^n$ suggests itself, and perhaps other simple graphs can be interpreted as functions.

The restriction of the Leibniz rule to strongly connected graphs is needed because of the way we have defined the extension of $\vee$ to disconnected graphs. You could check where things go wrong when $G = l_3$ (not strongly connected), for example.

Rubik's cube edge processes

Tony Forbes



Introduction

This is a slightly rewritten and corrected version of the notes for a talk I gave at the LSBU Maths Study Group on 13 Feb 2025, [1]. I assume you are familiar with the basic properties of Rubik's cube—what it looks like, the way it works, and the main problem to be solved. You do not have to know how to solve the cube, and what I am not going to do here is offer a solution. Incidentally, I class myself as a beginner—yes, I can restore a scrambled Rubik cube, but not in less than 60 seconds.

Our main purpose is to examine in some detail certain types of Rubik cube edge processes. First, we introduce some notation, recalling much of which from David Singmaster's *Notes*, [2].

We assume the cube in its identity state has a fixed orientation in space.

Upper-case letters

$$U,\ D,\ F,\ B,\ L \text{ and } R$$

denote either the faces or the outer slices of the cube: up, down, front, back, left and right, respectively. This is David Singmaster's terminology [2].

Lower-case letters subscripted with stars

$$u_*,\ d_*,\ f_*,\ b_*,\ l_* \text{ and } r_*$$

denote the inner slices adjacent to outer slices U, D, F, B, L and R, respectively. There is some redundancy here since both u_* and d_* represent the same slice; similarly for $\{f_*, b_*\}$ and $\{l_*, r_*\}$.

A single piece of the cube is denoted by the letters of the outer slices that intersect it. For example: UFL, or any other permutation of these letters, is the piece in the up-front-left corner; UF or FU refers to the edge piece on the up-front edge; and U refers to the face centre piece of the up face. The same notation is also used for positions.

Edge pieces can only be permuted amongst themselves. However, an edge piece has two orientations. For example, if edge piece FU is in position FU, its correct place in the cube, it is either correctly orientated, in the sense that the F part of FU has the same colour as piece F, or it is flipped.

Corner pieces can only be permuted amongst themselves. A corner piece in its correct position can have three distinct orientations; it might be correctly orientated, or twisted 120° clockwise, or twisted 120° anticlockwise.

Face centre pieces can only be permuted amongst themselves. Although a face centre piece in its correct position has 4 orientations, these are ignored since they are indistinguishable on a standard Rubik cube.

The term *outer slice* refers to an entire assembly of 9 pieces: a face centre piece together with the 4 edge pieces and the 4 corner pieces adjacent to it. For example, outer slice F consists of {F, FU, FD, FL, FR, FUL, FUR, FDL, FDR}. The word *face* refers either to an actual face of the cube, or to an outer slice. The term *inner slice* refers to an entire assembly of 8 pieces sandwiched between two parallel outer slices. For example, inner slice f_* (or b_*) consists of {U, D, L, R, UL, UR, DL, DR}.

Each of the symbols $\{U, D, F, B, L, R, u_*, d_*, f_*, b_*, l_*, r_*\}$ is also used to denote a 90° clockwise turn of the cube slice of that name. We rely on the context for the correct interpretation. Multiples are indicated by exponents modulo 4, and we usually write X' for X^{-1}. For example, F^2 means turn the F outer slice through 180°, and u'_* means turn the u_* slice 90° anticlockwise. Note that $u'_* = d_*$, $f'_* = b_*$ and $l'_* = r_*$.

The inverse, $\mathcal{P}' = \mathcal{P}^{-1}$, of a sequence of moves $\mathcal{P}$ is the sequence of inverses of the moves of $\mathcal{P}$ but in the reverse order. Given two sequences of moves $\mathcal{P}$ and $\mathcal{Q}$, the *commutator* $[\mathcal{P}, \mathcal{Q}]$ is defined by

$$[\mathcal{P}, \mathcal{Q}] \; = \; \mathcal{P}\mathcal{Q}\mathcal{P}'\mathcal{Q}'.$$

Observe that $[\mathcal{P}, \mathcal{Q}]' = [\mathcal{Q}, \mathcal{P}]$.

The result of a sequence of moves is always computed relative to the cube's initial orientation. For example, the sequence $F^2 f_*^2 B^2$, which just turns the cube upside down, actually creates a permutation consisting of 4 corner piece 2-cycles, 6 edge piece 2-cycles and 2 face centre piece 2-cycles.

There is a case for banning inner slices turns. They are not needed, since anything you can do to the cube can be achieved with $\{U, D, F, B, L, R\}$. However, many sequences described in [2] and elsewhere can be rewritten more nicely if inner slice turns are allowed. For example, Thistlethwaite's 4-flip $[R^2B^2R^2U^2RL', B]BU$, [2, page 44], has this memorable alternative representation: $[(F^2l_*)^2l_*, U']$.

We measure sequences by the number of moves they contain, where we count any non-trivial power of a slice turn as a single move. For instance,

$$[BF^2U^2L'FD^2RDF, f_*d_*l_*^2],$$

which flips 6 edges, has 24 moves. Although some writers count inner slice turns with weight 2 because they do a move such as f_* in two stages, $(Ff_*)\ F'$, I think it is wise to treat inner and outer slices as equals. Using one hand, grip the F slice with the thumb and 4th finger and the B slice with the 1st and 3rd fingers. Now you can do f_* with the other hand.

Edge processes $[X_1X_2\dots X_p,\ y_1y_2\dots y_q]$

We are interested in Rubik cube processes of the form

$$\begin{aligned}
&[\mathcal{X}, \mathcal{Y}], \\
\mathcal{X} &= X_1X_2\dots X_p, \quad X_i \in \{F, B, L, R, U, D\}, \quad i = 1, 2, \dots, p, \qquad (1)\\
\mathcal{Y} &= y_1y_2\dots y_q, \quad y_j \in \{f_*, b_*, l_*, r_*, u_*, d_*\}, \quad j = 1, 2, \dots, q.
\end{aligned}$$

Thus $[\mathcal{X}, \mathcal{Y}]$ is a sequence of face turns commutated with a sequence of inner slice turns. Since $\mathcal{Y}$ does not move corners, the corners moved by $\mathcal{X}$ are undone by $\mathcal{X}'$. Similarly, $\mathcal{Y}'$ restores face centres moved by $\mathcal{Y}$ since $\mathcal{X}$ has no action on them. Therefore $[\mathcal{X}, \mathcal{Y}]$ moves only edges; it is an edge process.

We want to know what cycle structures can be created by these processes. For example, $[RDUB, l_*u_*^2]$ has cycle structure

(FU, UB, BD, DF) (LU, DR)
(FR, LD, RF, DL) (BL, RU, LB, UR) (FL, LF) (BR, RB),

where we have adopted Singmaster's representation, [2]. Moreover, we see that the last four cycles are *twisted*. If the cycle length is $2c$, the symbol at position $c + 1$ is just the first symbol repeated but reversed. In particular, the last two 2-cycles each act on a single edge piece by flipping it. Using the notation of [2], we represent a twisted edge cycle by truncating just before the repeated symbol and adding a plus superscript,

(FU, UB, BD, DF) (LU, DR) (FR, LD)$^+$ (BL, RU)$^+$ (FL)$^+$ (BR)$^+$.

When acting on edge pieces the cycle lengths are therefore 4, 2, 2, 2, 1, 1; ten pieces move and two pieces are just flipped.

We wish to express these cycle structures compactly. Suppose $\mathcal{A}$ is an edge process, i.e. a sequence of moves that leaves the corners and face centres fixed. We adopt the notation

$$(s;\, a_1, a_2, \ldots, a_r),$$

where s is the number of edge pieces moved or flipped by $\mathcal{A}$, and for $i = 1, 2, \ldots, r$, a_i, is a number representing the length of a cycle of $\mathcal{A}$, decorated with a hat if that cycle is twisted.

For instance, the cycle pattern of the permutation generated by the above example, $[RDUB, l_* u_*^2]$, is $(12;\, \tilde{2}, \tilde{2}, 2, \tilde{4}, \tilde{4}, 4)$. We can compute s by summing the a_i with weight 1/2 for twisted cycles and 1 for normal cycles. Here, $s = 2/2 + 2/2 + 2 + 4/2 + 4/2 + 4 = 12$, and we save the reader the trouble of doing the computation by including this number in the cycle pattern specification.

Let

$$\begin{aligned} A &= (a_1, a_2, \ldots, a_r), \quad a_1, a_2, \ldots, a_r \geq 2, \\ E &= (e_1, e_2, \ldots, e_r), \quad e_i \in \{1/2, 1\}\,, \quad i = 1, 2, \ldots, r, \\ s &= E \cdot A \;=\; e_1 a_1 + e_2 a_2 + \cdots + e_r a_r. \end{aligned} \tag{2}$$

We assume the elements of A are the lengths of the non-trivial cycles of an edge process $\mathcal{A}$ of the form $[\mathcal{X}, \mathcal{Y}]$ as in (1). We consider an element a_i of A as the length of a cycle, which is twisted iff $e_i = 1/2$, $i = 1, 2, \ldots, r$. Note that s is the number of edge pieces acted on nontrivially by $\mathcal{A}$. The following conditions must be satisfied.

(i) Since the cube has 12 edge pieces, each of which has 2 orientations, we must have $a_1, a_2, \ldots, a_r \leq 24$ and $s \leq 12$.

(ii) If a_i is odd, then $e_i = 1$. Twisted edge cycles must have even lengths.

(iii) The number of occurrences of $e_i = 1/2$ is even. For reasons clearly explained somewhere in [2], the number of twisted edge cycles must be even.

(iv) The number of occurrences of $e_i a_i$ even is even. This is an elementary group theoretic requirement. Observe that $e_i a_i$ even corresponds to an odd permutation of edge pieces. An outer slice turn creates a 4-cycle of edge pieces and a 4-cycle of corner pieces, resulting in an even permutation of cube pieces. Therefore, if the corners don't move, the permutation of the edge pieces must be even.

(v) The sum s has the same parity as r. The proof is straightforward. The sum $s = e_1a_1 + e_2a_2 + \cdots + e_ra_r$ has r terms. Remove all the even e_ia_i to leave r' terms and the parity of the sum unchanged. By (iv) the number of terms removed is even and therefore $r' \equiv r \pmod 2$. The remaining terms are the odd a_i and the $a_j/2$ with $a_j \equiv 2 \pmod 4$, all of which are odd. Therefore

$$s \;\equiv\; \sum_{i=1,\ a_i\ \text{odd}}^{r} a_i + \sum_{j=1,\ a_j\ \equiv\ 2\ (\text{mod}\ 4)}^{r} \frac{1}{2}a_j \;\equiv\; r' \;\equiv\; r \pmod 2.$$

Consider the cube as fixed in space and in its identity state. A non-trivial inner slice turn leaves the corner pieces fixed and moves only edge pieces and face centre pieces. Moreover the action of inner slice turns on edge pieces is commutative. For example, $[f_*, r_*]$ does the permutation (F L U)(B R D) to give the pretty 6-spot pattern. It fixes the edges. As a consequence, the result of $[\mathcal{X}, \mathcal{Y}]$ as defined in (1) does not depend on the order of the moves in $\mathcal{Y}$. Hence we may assume that $\mathcal{Y} = f_*^\alpha r_*^\beta u_*^\gamma$ for some $\alpha, \beta, \gamma \in \{0, 1, 2, 3\}$.

When we solve (2) subject to the conditions (i)–(v) we find 301 distinct (non-vacuous) cycle patterns, which we tabulate by number of non-fixed edge pieces.

s	2	3	4	5	6	7	8	9	10	11	12	
	1	1	4	3	9	10	22	27	49	65	110	301

The first few patterns are
$(2;\, \tilde 2, \tilde 2)$, $(3;\, 3)$,
$(4;\, 2, 2)$, $(4;\, \tilde 2, \tilde 6)$, $(4;\, \tilde 4, \tilde 4)$, $(4;\, \tilde 2, \tilde 2, \tilde 2, \tilde 2)$,
$(5;\, 5)$, $(5;\, \tilde 2, \tilde 2, 3)$, $(5;\, \tilde 2, 2, \tilde 4)$,
$(6;\, 2, 4)$, $(6;\, \tilde 2, \tilde{10})$, $(6;\, 3, 3)$, $(6;\, \tilde 4, \tilde 8)$, $(6;\, \tilde 6, \tilde 6)$,
$(6;\, \tilde 2, \tilde 2, 2, 2)$, $(6;\, \tilde 2, \tilde 2, \tilde 2, \tilde 6)$, $(6;\, \tilde 2, \tilde 2, \tilde 4, \tilde 4)$, $(6;\, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2)$,
$(7;\, 7)$, $(7;\, 2, 2, 3)$, $(7;\, \tilde 2, \tilde 2, 5)$, $(7;\, \tilde 2, 2, \tilde 8)$, $(7;\, \tilde 2, 3, \tilde 6)$, $(7;\, \tilde 2, \tilde 4, 4)$,
$(7;\, 2, \tilde 4, \tilde 6)$, $(7;\, 3, \tilde 4, \tilde 4)$, $(7;\, \tilde 2, \tilde 2, \tilde 2, \tilde 2, 3)$, $(7;\, \tilde 2, \tilde 2, \tilde 2, 2, \tilde 4)$, …,
and the last four are
$(12;\, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, 2, 2)$, $(12;\, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 6)$,
$(12;\, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 4, \tilde 4)$, $(12;\, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2, \tilde 2)$.

Odd numbers cannot have hats. The number of hats is even as is also the number of even unhatted numbers plus the number of hatted multiples of 4. Eight patterns occur as twins, which are identical except for the distribution of the hats:

$$\begin{array}{ll}(11;\ \tilde{2},\tilde{2},\tilde{2},\tilde{4},6), & (11;\ 2,2,2,\tilde{4},\tilde{6});\\ (12;\ \tilde{2},4,4,\tilde{6}), & (12;\ 2,\tilde{4},\tilde{4},6);\\ (12;\ \tilde{2},\tilde{2},\tilde{2},\tilde{2},2,6), & (12;\ \tilde{2},2,2,2,2,\tilde{6});\\ (12;\ \tilde{2},\tilde{2},\tilde{2},\tilde{2},4,4), & (12;\ 2,2,2,2,\tilde{4},\tilde{4}).\end{array}$$

It is of course highly desirable to find short edge processes of the form $[\mathcal{X},\mathcal{Y}]$ as in (1) to achieve each one of the 301 cycle patterns, the shorter the better. At present I can do only 300. If we restrict processes to my target of 18 moves, the number is reduced to 295. The six where I have been not totally successful are as follows.

(i) $(10;\ \tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2})$. A 10-flip. The best I can do at present is 34 moves,

$$[BRLBR^2DRL'FL'F'UR'L^2, b_*l_*u_*].$$

It is not the most efficient way to flip 10 edges. For example, with the 8-flip $(LRFBUD)^2$ followed by the 4-flip $(D'f_*)^4$ you can flip all edges except {FD, LD} in only 18 moves.

(ii) $(11;\ \tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},2,\tilde{4})$. Current best: 20 moves.

(iii) $(12;\ \tilde{4},\tilde{4},\tilde{4},\tilde{4},\tilde{4},\tilde{4})$. **NOT YET**. This one is a bit of a mystery. I can think of no obvious reason why the required edge process does not exist. However, all the procedures for obtaining the other 300 cycle patterns have so far failed to yield this one. So it remains a challenge. Find an edge process $[\mathcal{X},\mathcal{Y}]$ as in (1), or prove that no such process exists. The pattern can be obtained by other means; for example, $[LFL, f_*][FUF, u_*][ULU, l_*]$ (24 moves).

(iv) $(12;\ \tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{6})$. Current best: 20 moves.

(v) $(12;\ \tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{4},\tilde{4})$. Current best: 26 moves.

(vi) $(12;\ \tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2},\tilde{2})$. This is the 12-flip. The best I can do at present is 32 moves,

$$[R'U'DF'DBUD'FR^2U'L'D^2, f_*^2r_*^2u_*^2].$$

There are of course more efficient ways to achieve a 12-flip. A particularly simple sequence is $(Rf_*)^4(Dl_*)^4(Bu_*)^4$ (24 moves), which is even easier to remember if you are doing it on a real cube,

$$\big((Rf_*)^4 \text{ (rotate the cube 120° about the UFL–DBR diagonal)}\big)^3.$$

> However, what we want is a sequence that conforms to the commutator pattern $[\mathcal{X}, \mathcal{Y}]$. Recall that the 12-flip generates the centre of the Rubik cube group.

We finish by listing edge process for some of the more interesting cycle patterns. Go to [1] if you want to see the whole lot.

2 edge pieces, 2 cycles

$[FU'RF'U, l_*]$ (12 moves): $(\mathrm{FU})^+(\mathrm{BU})^+$

3 edge pieces, 1 cycle

$[U^2, f_*]$ (4 moves): (LU RU LD)

4 edge pieces, 2 cycles

$[U^2, f_*^2]$ (4 moves): (LU RU)(LD RD)

$[RFR'D, b_*]$ (10 moves): $(\mathrm{BD\ DL\ LU})^+(\mathrm{RD})^+$

$[L'B'L', b_*]$ (8 moves): $(\mathrm{BD\ DL})^+(\mathrm{LU\ UR})^+$

4 edge pieces, 4 cycles

$[D'R^2FB'U^2L, f_*^2]$ (14 moves): $(\mathrm{LU})^+(\mathrm{LD})^+(\mathrm{RU})^+(\mathrm{RD})^+$

5 edge pieces, 1 cycle

$[U, f_*]$ (4 moves): (FU UB UR LU DL)

6 edge pieces, 6 cycles

$[UL'U'F'R'D'R, u_*^2r_*^2]$ (18 moves): $(\mathrm{FL})^+(\mathrm{FR})^+(\mathrm{FD})^+(\mathrm{BU})^+(\mathrm{BR})^+(\mathrm{BL})^+$

8 edge pieces, 8 cycles

$[R'U'D'B'UD', f_*^2u_*^2]$ (16 moves):
$(\mathrm{FL})^+(\mathrm{FR})^+(\mathrm{BR})^+(\mathrm{BL})^+(\mathrm{LU})^+(\mathrm{LD})^+(\mathrm{RU})^+(\mathrm{RD})^+$

10 edge pieces, 10 cycles 10-flip – item (i) on page 15

11 edge pieces, 5 cycles

$[U'F'RD'F', u_*r_*f_*^2]$ (16 moves), normal 6-cycle:
$(\mathrm{FU\ BL\ DF\ RF\ LF\ DB})(\mathrm{BU})^+(\mathrm{BR})^+(\mathrm{LD\ DR})^+(\mathrm{RU})^+$

$[BD^2LF, r_*^2b_*]$ (12 moves), twisted 6-cycle:
$(\mathrm{FU\ BD})(\mathrm{FL\ UL\ BU})^+(\mathrm{FR\ BR})(\mathrm{FD\ RU})(\mathrm{LD\ RD})^+$

12 edge pieces, 4 cycles

$[RL'UF^2, r_*^2d_*]$ (12 moves), normal 4-cycles, twisted 6-cycle:
$(\mathrm{FU\ BD\ BU\ FD})(\mathrm{FL})^+(\mathrm{FR\ LB\ BR})^+(\mathrm{LU\ DR\ LD\ UR})$

$[D^2L'F'U'B, f_*^2d_*]$ (14 moves), twisted 4-cycles, normal 6-cycle:
$(\mathrm{FU\ DB\ FD\ DL\ UB\ UR})(\mathrm{FL\ LB})^+(\mathrm{FR\ BR})^+(\mathrm{LU\ RD})$

12 edge pieces, 6 cycles

$[U'D'F^2R, r_*^2f_*d_*]$ (14 moves), normal 6-cycle:
$(\mathrm{FU\ RB\ UB\ BL\ FL\ FR})(\mathrm{FD})^+(\mathrm{BD})^+(\mathrm{LU})^+(\mathrm{LD\ UR})(\mathrm{RD})^+$

$[FRD'BR, f_* r_*^2]$ (14 moves), twisted 6-cycle:
(FU BD)(FL BL)(FR LD)(FD BU)(BR UL UR)$^+$(RD)$^+$

$[F'D'R'F', u_*^2 f_*^2 r_*^2]$ (14 moves), normal 4-cycles:
(FU FD DR RF)(FL)$^+$(BU BD LB UL)(BR)$^+$(LD)$^+$(RU)$^+$

$[D'RLB'L, b_* d_* r_*]$ (16 moves), twisted 4-cycles:
(FU DB)(FL BR)(FR LU)$^+$(FD RD)(BU BL)$^+$(LD UR)

12 edge pieces, 12 cycles 12-flip – item (vi) on page 15

References

[1] Tony Forbes, Rubik's cube edge processes,
`https://www.theoremoftheday.org/MathsStudyGroup/index.html`.

[2] David Singmaster, *Notes on Rubik's 'Magic Cube'*, 5$^{\text{th}}$ edition, London, 1980.

Solution 323.7 – Rational polygon

> A regular polygon, P is drawn on the (x, y)-plane. Show that either P is a square, or P has a vertex with an irrational coordinate.

Ted Gore

We need to show that a regular polygon $P(n)$, other than a square, has at least one vertex that does not have two rational coordinates.

We centre $P(n)$ at $(0, 0)$ with a vertex at $(1, 0)$. The other vertices are at $(\cos(2k\pi/n), \sin(2k\pi/n))$ for $k \in [1, n-1]$. There are n vertices. We would like to know how many of them have two rational coordinates.

Niven's theorem says that the only rational values of $\sin(x)$ when x is a rational multiple of π are 0, $\pm 1/2$ and ± 1. It also says that $\sin(x)$ takes many other rational values but these require x/π to be irrational.

If $\sin(x) = \pm 1/2$, then $\cos(x) = \sqrt{3/4}$, which is irrational. If $\sin(x) = 0$, then $\cos(x) = \pm 1$. If $\sin(x) = \pm 1$, then $\cos(x) = 0$.

For any other vertices of $P(n)$, $\sin(x)$ is irrational.

$P(4m)$ has four vertices with two rational coordinates, these being $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$. Hence $P(4m + 2)$ has two vertices with two rational coordinates, these being $(1, 0)$, $(-1, 0)$.

$P(4m \pm 1)$ has one vertex with two rational coordinates $(1, 0)$.

For $n > 4$, there is always at least one other vertex with at least one irrational coordinate.

Solution 323.4 – Two dice

Here is a game involving two standard dice.

Throw the dice. Let s be the sum of the two numbers shown.

If $s = 7$ or 11, you win.

If $s \in \{2,\ 3,\ 12\}$, you lose.

Otherwise do the following until you win, lose or die.

Throw the dice. Let t be the sum of the two numbers shown.

If $t = s$, you win.

If $t = 7$, you lose.

Show that your probability of winning is 244/495 nearly. Of course the figure is exact if you happen to be immortal.

Ted Gore

The game consists of one or more rounds in which the result is win, lose or undecided.

Round 1. You throw two dice and sum the values. Let that sum be s. If s = 7 or 11 you win. The probability of that is

$$\frac{6}{36} + \frac{2}{36} \;=\; \frac{8}{36} \;=\; \frac{2}{9}.$$

If $s = 2, 3$ or 12, you lose with probability

$$\frac{1}{36} + \frac{2}{36} + \frac{1}{36} \;=\; \frac{4}{36} \;=\; \frac{1}{9}.$$

Otherwise $s \in \{4, 5, 6, 8, 9, 10\}$ with probability

$$\frac{3}{36} + \frac{4}{36} + \frac{5}{36} + \frac{5}{36} + \frac{4}{36} + \frac{3}{36} \;=\; \frac{24}{36}$$

and you proceed to round 2 with a constant k set to 3, 4 or 5 depending on the value of s. Here, k is the number of ways that s can be made by adding the values of two dice.

Round 2. You throw the dice again to get sum t.

If $t = s$, you win with relative probability $\dfrac{k}{36}$ and cumulative probability $\dfrac{24}{36}\dfrac{k}{36}$. If $t = 7$, you lose with relative probability $\dfrac{6}{36}$ and cumulative probability $\frac{24}{36}\frac{6}{36}$. If $t \neq s$ and $t \neq 7$, then $p = 36 - 6 - k$ and you proceed to round 3 with relative probability $\frac{p}{36}$ and cumulative probability $\frac{24}{36}\frac{p}{36}$.

Round 3. If $t = s$, you win with relative probability $\frac{k}{36}$ and cumulative probability $\frac{24}{36}\frac{p}{36}\frac{k}{36}$. If $t = 7$, you lose with relative probability $\frac{6}{36}$ and cumulative probability $\frac{24}{36}\frac{p}{36}\frac{6}{36}$. Otherwise (i.e. $t \neq s$ and $t \neq 7$) you proceed to round 4 with cumulative probability $\frac{24}{36}\left(\frac{p}{36}\right)^2$.

Round 4. If $t = s$, you win with relative probability $\frac{k}{36}$ and cumulative probability $\frac{24}{36}\left(\frac{p}{36}\right)^2\frac{k}{36}$. If $t = 7$, you lose with relative probability $\frac{6}{36}$ and cumulative probability $\frac{24}{36}\left(\frac{p}{36}\right)^2\frac{6}{36}$. Otherwise ($t \neq s$ and $t \neq 7$) you proceed to round 5 with cumulative probability $\frac{24}{36}\left(\frac{p}{36}\right)^3$.

A pattern emerges from the above. We have

$$\sum_{n=0}^{\infty}\left(\frac{p}{36}\right)^n = \frac{36}{k+6}.$$

For a given value of k the total probability of winning in rounds 2 to infinity is

$$\frac{24}{36}\left(\frac{36}{k+6}\right)\frac{k}{36} = \frac{2k}{3(k+6)} = \frac{2}{9},\ \frac{4}{15},\ \frac{10}{33}$$

for $k = 3, 4, 5$. The probability of k taking the value it has is $k/12$, so that the overall probability of winning over all rounds, for all values of k is

$$\frac{2}{9} + \sum_{k=3}^{5}\frac{k}{12}\frac{2k}{3(k+6)} = \frac{2}{9} + \frac{1}{18} + \frac{4}{45} + \frac{25}{198} = \frac{244}{495} = 0.49293.$$

Problem 326.2 – Integrals

Compute $\displaystyle\int_0^{\pi/2} \sin(\cos t)dt$ and $\displaystyle\int_0^{\pi/2} \cos(\sin t)dt$.

Solution 320.1 – Hexagonal numbers

The hexagonal numbers are defined by

$$H_1 = 1, \quad H_n = H_{n-1} + 6(n-1), \quad n \geq 2.$$

Show that

$$H_1 + H_2 + \cdots + H_n = n^3.$$

Peter Fletcher

We have $H_1 = 1^3$; so

$$\sum_{i=1}^{n} H_i = n^3$$

is true for $n = 1$.

If we now assume the sum is true for some value $n = k$ and add H_{k+1} to both sides, we find

$$\begin{aligned}
\left(\sum_{i=1}^{k} H_i\right) + H_{k+1} &= k^3 + H_k + 6k \\
&= k^3 + H_{k-1} + 6(k-1) + 6k \\
&= k^3 + H_{k-2} + 6(k-2) + 6(k-1) + 6k \\
&\ \vdots \\
&= k^3 + H_1 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6k \\
&= k^3 + 1 + 6\left(\frac{k(k+1)}{2}\right) \\
&= k^3 + 3k^2 + 3k + 1 \\
&= (k+1)^3.
\end{aligned}$$

Hence if the sum is true for $n = k$, it is true for $n = k + 1$. It is true for $n = 1$, so therefore it must be true for each positive integer n.

Problem 326.3 – Tiling a rectangle

A rectangle n metres by m metres is to be tiled by metre squares of c distinct colours, with no two colours sharing an edge.

How many ways?

Solution 320.5 – Twelve objects

> Twelve distinct objects are to be partitioned in to 4 sets of 3. How many ways?
> Obviously it generalizes. You can put $12 = n$ and $3 = m$ with the condition $n \equiv 0 \pmod{m}$.

Peter Fletcher

Suppose each set of three has three distinct 'slots' for three of the 12 objects. Over the four sets, there are then 12 distinct slots and hence 12! ways of allocating objects to slots.

Now suppose the ordering of the three slots in the first set is not actually fixed. The 3! possible distinct orderings are included in the 12!, which we hence need to divide by 3!.

If the ordering of the three slots in each of the other three sets is also not fixed, we need to divide the 12! by 3! for each of those as well.

By having 12 distinct slots and then four distinct sets of three, we have assumed that the ordering of the sets is fixed; but suppose it is not. The 4! possible distinct orderings of the four sets are also included in the 12!, which hence needs to be divided by 4! as well.

Therefore the number of ways of partitioning 12 distinct objects into four sets of three is

$$\frac{12!}{(3!)^4 4!} \;=\; 15\,400$$

and generalising, the number of ways of partitioning n distinct objects into n/m sets of size m is

$$\frac{n!}{(m!)^{n/m}(n/m)!}.$$

Problem 326.4 – Tilling a rectangle

A field n metres by m metres is to be ploughed by a device that can only do a single furrow 1 metre wide. What's the cheapest way of doing this, assuming you start and finish in the same corner?

By 'cheapest' we mean traversing the minimum number of metre squares subject to ploughing the entire field. For example with $n = m = 2$, go north to enter the field from the south at the southwest corner, then north, east, south, west; 5 square visitations.

M500 Revision Weekend May 2025

Judith Furner

The 49th Revision Weekend took place at Kents Hill Training and Conference Centre over 9th – 11th May 2025. We ran 12 undergraduate maths modules and 3 postgraduate maths modules. We also ran one Science module – SM380. We had 148 students, which was a disappointment after we had done so well the previous year. Of those, 133 were residential and 15 non-residential. Once again we shall be making a loss, for the fourth time since COVID: there were extra issues this year – two tutors were obliged to pull out for health reasons and we were unable to find alternatives. I was also unable to find tutors for two courses, despite assiduous searches. There were 17 tutors, who were mostly entirely satisfactory. All feedback was passed on to tutors and Kents Hill. There were six M500 'staff': Angela Allsopp, Paul Cooper, Lily Dibb–Fuller, Milena Dragic, Chris Furner and Judith Furner.

As ever, we had very good feedback – nearly all concerned clearly had an excellent and profitable weekend. We received 125 forms out of a potential 148 and Paul and Angela carried out their usual exemplary task of delivering feedback forms to tutorial classes, and retrieving them, along with the lanyards, from students on the Sunday afternoon. If you came to the Weekend, and find that you managed to take your lanyard home, I would be grateful if you would return it (the Revision Weekend in May 2026 is an acceptable time).

Feedback for the tutors is of primary importance, and once again it was very good. Many tutors have worked with us more than satisfactorily for many years, and new ones seem to pick up the demands very quickly indeed. Nearly all students seemed to agree with our views. Of the 114 comments, 103 were 'excellent', 9 'good' and 2 'average'. I followed up the tutors for these last two, and had satisfactory conversations. As usual, some tutors, year after year, are marked 'excellent' by 100 per cent of their tutees. Typical comments were 'excellent throughout', 'a very very well-prepared and structured course', 'brilliant tutor, explains everything so well', 'very patient and thorough', 'highly motivating', 'great ways to make us work faster and smarter', 'he delivered extra rigor into the topics and tied the disparate topic areas together', 'fantastic tuition as ever', 'kept the attention in the dreaded post-lunch sessions', 'fantastic tutor, very clear and enthusiastic'. I was sorry to see one less positive comment, which I am sure Jenny Oldroyd will bear in mind: 'not enough ducks'. Jenny and I have been in conversation about this.

We were delighted that of the 111 responses to Weekend Organization on the feedback form, 98 were 'excellent', 10 'good' and three 'average'. Although I am the Organiser, it would be completely impossible for me to work without the support of my invaluable team: the regulars Angela, Paul, Milena and Chris (Furner), and this year, Paul Chaston and Lily Dibb–Fuller. Heartfelt thanks from me to them, and you seemed to agree. Comments were 'I love the M500 Weekends, it's great to be able to talk maths with people', 'thanks to the team for this important and valued resource. It would be nice to have some social entertainment on Saturday', 'maybe some music on Saturday night – bring an instrument'. This has been noted, and will be considered next year. There was some criticism of the timing of lunch, and indeed the timetabling of lunch. I have taken this on board and intend to be clearer next year. There were two comments about the price increase. We do try to keep our prices down, and indeed to ensure that the increase is only £5 or £10 a year (which may be less than inflation).

People were also appreciative of the presence of the OU with literature and advice and the OUSA stand. Next year we hope to have representatives from the Institute of Mathematics.

Accommodation and refreshments did not rate so well, as ever. The general response was 'adequate', although that masks a number of 'excellent' and 'poor'. Many students considered their bedrooms clean and comfortable, and the Health Centre pool was appreciated. The variety on offer in the restaurant was appreciated. The bar once again led to the greatest number of complaints, with high prices and poor and slow service. However, Kents Hill staff were found to be constantly attentive and helpful.

Many students commented on our lack of publicity. Oh for the olden days, when every OU mathematics student received a little pink slip at frequent intervals, advertising the M500 Society and of course the Revision Weekend. We have great difficulty in making contact with students now that everything is online and we do not have access to the various groups. If you are a student, and can help us out with this, do please make contact with me; we should be most grateful.

DATES FOR YOUR DIARY. The Winter Weekend takes place over the 9 – 11th January 2026, and the Revision Weekend over the 8th – 10th May 2026. Both events take place at Kents Hill. The Winter Weekend is essentially a recreational weekend, where we enjoy mathematical puzzles and games, along with more cerebral demands. We have a variety of presentations, some more formal than others, and we find that most Weekenders become very engaged in puzzle-solving and discussion. Bookings are not yet open for these events, but keep an eye on the M500 website (which is not always updated as quickly as it might be (note use of the passive tense)). It is, however, useful to have the dates in your diary to ensure that you don't inadvertently book another event.

Revision Weekend May 2025 – Feedback forms from students

		Excellent	Good	Average	Avg–poor	Poor	Total
Tutors	Number	103	9	2	0	0	114
	Percent	90%	8%	2%	0	0	
Classrooms	Number	46	46	25	2	4	123
	Percent	37%	37%	20%	2%	3%	
Accommodation	Number	26	51	32	5	2	116
	Percent	22%	44%	28%	4%	2%	
Food and drink	Number	34	51	30	5	2	122
	Percent	28%	42%	25%	4%	2%	
Refreshments	Number	20	40	28	8	2	98
	Percent	20%	41%	29%	8%	2%	
Bar	Number	13	16	32	9	0	70
	Percent	19%	23%	46%	13%	0	
Weekend organisation	Number	98	10	3	0	0	111
	Percent	88%	9%	3%	0	0	

The temperature is getting warmer

'The temperature is getting warmer.' This was spotted in a popular science publication, but the phenomenon is widespread. I (TF) suppose it could be worse. Somewhere in M500 we quoted someone saying,

> 'The temperature of the ocean has risen by 5 degrees C (41 degrees F).'

Or, even better, why not combine the two?

> 'The temperature of the ocean has got warmer by 5 degrees C (278.15 kelvin).'

Presumably this is Global Warming at work, in which case you should take the last one seriously. Recall that the paper pages in books ignite at only 506 K. In my opinion it's nonsense. The weather gets warmer. Material things get warmer. Units of measurement don't; they increase.

To change the subject completely, I would intentionally try not to write a sentence like this one, probably lifted from some leaflet whose business it was to help preserve the Environment:

> 'We counted the number of trees we planted in last year and it came to 250000.'

The 250000 is surely false. It should be 1. The number of trees we planted in last year is a single entity, a number, and therefore if you try counting it, you won't get beyond 1. Of course, it is not difficult to guess what was meant. I can see that by counting the *trees*, one might have obtained a figure nearer to the stated 0.25 million.

The late Jeremy Humphries was often intrigued by the way the media talk about percentages. Indeed, he would have been amused to observe the widespread reporting that in the March 2024 Budget the Chancellor 'reduced the basic NI rate by 2 percent from 10 percent'. Presumably to 9.8 percent, not the 8 percent that most people were led to believe.

Something else I have difficulty understanding is why the verb *hoover* is almost always used as a synonym for *operating a domestic vacuum cleaner on.* I see no obvious connection with a certain past Director of the United States Federal Bureau of Investigation. One can speculate that possibly there is a reference to J. Edgar Hoover always insisting on personally cleaning his office floor. Perhaps someone can research this more fully.

The page is full. No more space and time for idle chatter. There is housework to be done. I have to dyson the carpets.

04.00

Problem 326.5 – Dot product

Suppose **u** and **v** are vectors in a real vector space of dimension d. Show that

$$(\mathbf{u}\cdot\mathbf{v})^2 \;=\; \operatorname{trace}\big((\mathbf{u}\mathbf{u}^{\mathrm{T}})(\mathbf{v}\mathbf{v}^{\mathrm{T}})\big)\,.$$

As usual, we consider **u** and **v** to be either ordered sets of d elements or $d\times 1$ matrices. In the latter case their transposes $\mathbf{u}^{\mathrm{T}}$ and $\mathbf{v}^{\mathrm{T}}$ are $1\times d$ matrices. Therefore when we compute $\mathbf{u}\mathbf{u}^{\mathrm{T}}$ and $\mathbf{v}\mathbf{v}^{\mathrm{T}}$ using matrix multiplication the results are $d\times d$ matrices. On the other hand, $\mathbf{u}\cdot\mathbf{v}$ is a number, the sole entry of the 1×1 matrix $\mathbf{u}^{\mathrm{T}}\mathbf{v}$.

Front cover Three cospectral 16-vertex 6-regular graphs.