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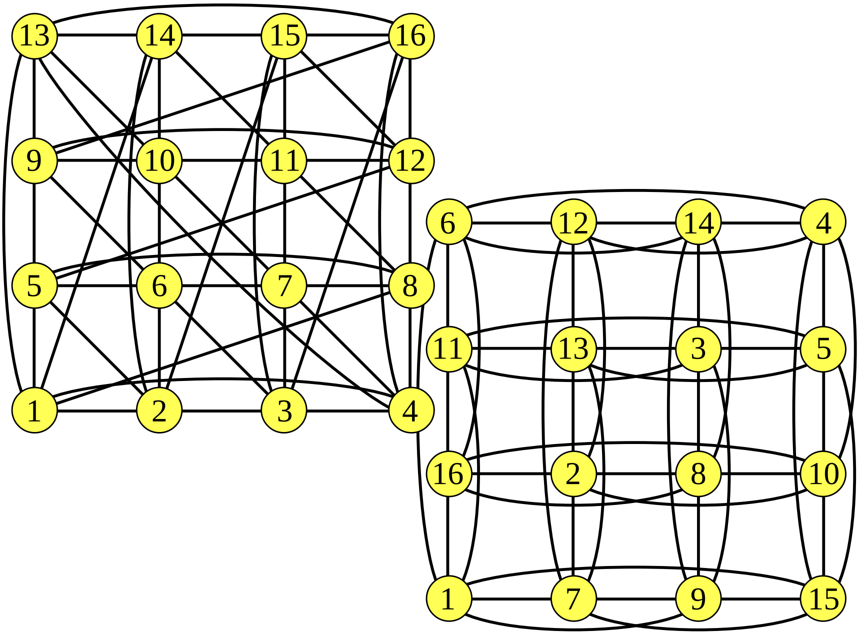
M500 327



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Editor – *Tony Forbes*

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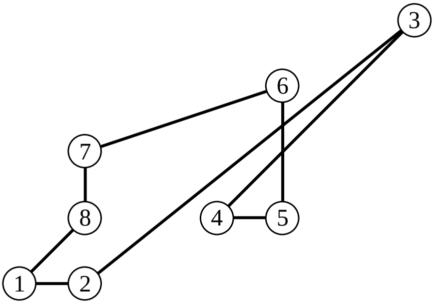
Solution 324.2 – Triangles

There are $n \geq 3$ distinct points in the plane, P_1, P_2, …, P_n, with even integer coordinates and not all collinear. Denote by $\triangle_{i,j}$ the signed area of the triangle with vertices P_i, P_j, $P_{(j \bmod n)+1}$, Let A be the matrix defined by $[A]_{i,j} = \triangle_{i,j}$. Show that A has rank 3. Or find a counter-example.



$$\begin{bmatrix} 0 & 8 & -12 & -2 & 16 & 10 & -2 & 0 \\ 0 & 0 & -6 & -2 & 12 & 12 & 0 & 2 \\ 8 & 0 & 0 & 6 & -8 & -2 & 10 & 4 \\ 2 & -6 & 0 & 0 & 4 & 10 & 4 & 4 \\ 2 & -14 & 6 & 0 & 0 & 12 & 6 & 6 \\ 6 & 6 & -6 & 4 & 0 & 0 & 6 & 2 \\ 4 & 20 & -18 & 2 & 12 & 0 & 0 & -2 \\ 2 & 10 & -12 & 0 & 12 & 6 & 0 & 0 \end{bmatrix}$$

The sign of the area of the triangle with vertices $S = (S_x, S_y)$, $T = (T_x, T_y)$, $U = (U_x, U_y)$ is the sign of the z component of the vector product

$$(T_x - S_x,\ T_y - S_y,\ 0) \times (U_x - S_x,\ U_y - S_y,\ 0).$$

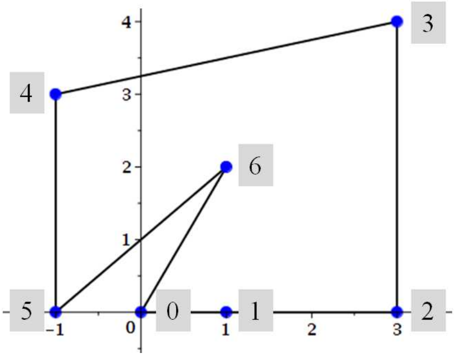
That is, the sign of the sine of the angle you have to rotate vector ST anticlockwise about S to align it with vector SU.

Robin Whitty

Problem 324.2 can be viewed as a continuation of Problem 317.3: Eight triangles, solved in M500 issue 319, pp. 7–9. Indeed, the latter is a lemma in the proof of the former, and seems to me unavoidable in confirming the required linear dependence among any four rows of the matrix in the current problem; but it would be nice to see a purely algebraic solution and perhaps some other reader of M500 may supply this.

To slightly restate the problem: we are given an n-vertex plane polygon P, closed, not necessarily simple, but non-degenerate (not all points collinear). Its vertices will be listed from 0 to $n-1$ in counterclockwise order. The edges of P may be written $[i, i+1]$, counting modulo n and using square brackets to avoid any confusion with plane coordinate pairs. We are interested in triangles formed on P by letting a vertex i subtend a non-incident edge $[j, j+1]$. The area of this triangle will be denoted by $\Delta_{i,j}$.

For completeness, vertex i subtends triangles of zero area on the edges preceding it and following it. Then the totality of triangle areas forms an $n \times n$ matrix $\Delta_P = [\Delta_{i,j}]$ which has zeros on the main diagonal and first lower diagonal. Entries elsewhere may be positive, for triangles oriented counterclockwise, negative, for triangles oriented clockwise, or zero, for triangles having collinear vertices. See Figure 1, for an example.



$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{pmatrix}
0 & 0 & 6 & \frac{13}{2} & \frac{3}{2} & -1 & 0 \\
0 & 0 & 4 & 7 & 3 & -2 & 1 \\
0 & 0 & 0 & 8 & 6 & -4 & 3 \\
2 & 4 & 0 & 0 & 6 & 0 & 1 \\
\frac{3}{2} & 3 & 8 & 0 & 0 & 3 & -\frac{5}{2} \\
0 & 0 & 8 & 6 & 0 & 0 & -1 \\
1 & 2 & 4 & 3 & 3 & 0 & 0
\end{pmatrix}
\end{array}$$

Figure 1: Polygon P on seven vertices $(0,0)$, $(1,0)$, $(3,0)$, $(3,4)$, $(-1,3)$, $(-1,0)$, $(1,2)$ and its triangle area matrix Δ_P

Now the problem is to show that the matrix Δ_P has rank 3. We will do this in two steps, firstly showing that $\operatorname{rank}(\Delta_P) \geq 3$ and secondly showing that $\operatorname{rank}(\Delta_P) \leq 3$.

The rank of a matrix is the size of the largest nonsingular square submatrix formed by intersecting an equal number of rows and columns. Trace the polygon counterclockwise from vertex 0 until a vertex, say u, is reached whose two incident edges are not collinear (this would be vertex 2 in figure 1). Such a vertex exists because our polygon is non-degenerate. Continue until the next vertex, say v, is reached having non-collinear incident edges. This vertex must exist because the polygon is closed. Intersecting the rows and columns of Δ_P corresponding to $0, u, v$ will give the following submatrix:

$$\begin{array}{cc}
 & \begin{array}{ccc} 0 & u & v \end{array} \\
\begin{array}{c} 0 \\ u \\ v \end{array} &
\begin{pmatrix}
0 & \Delta_{0,u} & \Delta_{0,v} \\
\Delta_{u,0} & 0 & \Delta_{u,v} \\
\Delta_{v,0} & \Delta_{v,u} & 0
\end{pmatrix}
\end{array},$$

with determinant $\Delta_{0,u}\Delta_{u,v}\Delta_{v,0} + \Delta_{0,v}\Delta_{u,0}\Delta_{v,u}$. This determinant is nonzero because the factors in the first term are all nonzero while the last two factors in the second term are zero.

Now for the upper bound. The Eight triangles problem (M500 Problem 317.3) was to prove the following.

Lemma 1 Suppose six points, A, B, C, D, X, Y are taken, counterclockwise, in the plane. Form four triangles with base XY and opposite vertices A, B, C and D, respectively, and denote their areas by Δ_A, Δ_B, Δ_C and Δ_D. Four further triangles are formed by taking each triple of points from A, B, C, D. Denote their areas by $|ABC|, |ABD|, |ACD|$ and $|BCD|$. Then

$$|BCD|\,\Delta_A - |ACD|\,\Delta_B + |ABD|\,\Delta_C - |ABC|\,\Delta_D \;=\; 0. \tag{1}$$

We shall refer at the end to the following immediate corollary.

Lemma 2 Let A, B and C be three collinear points in the plane, with the straight line distances between them denoted by $|AB|$, $|AC|$ and $|BC|$. Let X and Y be two further points. Form three triangles with base XY and opposite vertices A, B and C, respectively, and denote their areas by Δ_A, Δ_B and Δ_C. Then

$$|BC|\,\Delta_A - |AC|\,\Delta_B + |AB|\,\Delta_C \;=\; 0. \tag{2}$$

This follows from Lemma 1 by adding a point D anywhere not collinear with A, B, and C.

We can see Lemmas 1 and 2 in action for the polygon in Figure 1. Take the case where the six points of Lemma 1 are the last six vertices of the polygon taken in order. Identify the entries in equation (1) from the matrix Δ_P and check that the alternating sum of products in the last row is zero:

$$\begin{array}{cccccccc}
|BCD| & \Delta_A & |ACD| & \Delta_B & |ABD| & \Delta_C & |ABC| & \Delta_D \\
\Delta_{2,3} & \Delta_{1,5} & \Delta_{1,3} & \Delta_{2,5} & \Delta_{4,1} & \Delta_{3,5} & \Delta_{1,2} & \Delta_{4,5} \\
8 & -2 & 7 & -4 & 3 & 0 & 4 & 3
\end{array}$$

Meanwhile, observe that points $0, 1$ and 2 in Figure 1 are collinear, so in Lemma 2 take these to be A, B and C. Take X and Y to be points 3 and 4. As expected:

$$|BC|\,\Delta_A - |AC|\,\Delta_B + |AB|\,\Delta_C \;=\; 2 \times 13/2 - 3 \times 7 + 1 \times 8 \;=\; 0.$$

The final ingredients needed to prove that $\mathrm{rank}(\Delta_P) \leq 3$ are some basic facts about Δ_P collected in Lemma 3.

Lemma 3 For $i = 0, 1, \ldots, n-1$, in matrix Δ_P, we have the following.

1. The second lower diagonal is identical to the first upper diagonal, that is, $\Delta_{i+2,i} = \Delta_{i,i+1}$. This follows since the same triangle areas are being subtended from opposite directions.
2. $\Delta_{i,i+1} + \Delta_{i,i+2} = \Delta_{i+3,i} + \Delta_{i+3,i+1}$. The left-hand side specifies the first and second upper diagonal entries in row i which partition the area of the quadrilateral on vertices $i, i+1, i+2$ and $i+3$ into two triangles. The right-hand side specifies the first and second lower diagonal entries in row $i+3$ which partition the same quadrilateral along the other diagonal.
3. If vertices $i, i+1$ and $i+2$ are collinear then column $i+1$ of Δ_P is a multiple of column i. Specifically, suppose that h_j is the perpendicular distance to the line through vertices i, $i+1$ and $i+2$ from vertex j, not collinear with vertices i and $i+1$, then

$$\frac{\Delta_{j,i+1}}{\Delta_{j,i}} = \frac{\frac{1}{2}h_j|[i+1, i+2]|}{\frac{1}{2}h_j|[i, i+1]|}.$$

   If vertex j is collinear with vertices i and $i+1$, then $\Delta_{j,i+1} = \Delta_{j,i} = 0$.

Our strategy for proving that $\mathrm{rank}(\Delta_P) \leq 3$ will be to find sufficiently many linearly independent null space vectors: eigenvectors having corresponding eigenvalue zero. Specifically, we will show that the nullity of Δ_P, the dimension of its null space, is at least $n-3$. Since rank and nullity sum to n this will complete our proof.

We shall see that Lemmas 1 and 3 can be applied to produce a linear dependence between any four *consecutive* rows of Δ_P. This will produce a row vector v which multiplies Δ_P to give the zero vector, and we can put v into the null space. It will consist of zeros except for four consecutive entries which specify the linear dependence. Take the first $n-3$ such vectors, corresponding to the first $n-3$ diagonal entries of Δ_P. They will start and end with strings of zero entries which, respectively, increase and decrease in length as we go down the diagonal.

So suppose we take the first four rows. Lemma 3(1) determines the value $\Delta_{3,1} = \Delta_{1,2}$. Then Lemma 3(2) tells us that $\Delta_{3,0} + \Delta_{3,1} = \Delta_{0,1} + \Delta_{0,2}$, so that $\Delta_{3,0} = \Delta_{0,1} + \Delta_{0,2} - \Delta_{1,2}$. From this we can specify a linear dependency among the first four rows and three columns of Δ_P as shown:

$$\begin{array}{r|ccc} \Delta_{1,2}\,\times & 0 & \Delta_{0,1} & \Delta_{0,2} \\ -\Delta_{0,2}\,\times & 0 & 0 & \Delta_{1,2} \\ +(\Delta_{0,1}+\Delta_{0,2}-\Delta_{1,2})\,\times & \Delta_{0,1} & 0 & 0 \\ -\Delta_{0,1}\,\times & \Delta_{0,1}+\Delta_{0,2}-\Delta_{1,2} & \Delta_{1,2} & 0 \end{array} \quad = \quad 0 \quad 0 \quad 0.$$

We must show that this linear decadency extends to the whole of the first four rows, and this is precisely the raison d'être of Lemma 1: consider the first four entries in column k, namely, $\Delta_{0,k}, \Delta_{1,k}, \Delta_{2,k}$ and $\Delta_{3,k}$. In Lemma 1, the points X and Y are assigned the polygon edge $[k, k+1]$. The points A, B, C and D are assigned the polygon vertices $0, 1, 2$ and 3. Then Lemma 1 supplies the linear dependency that we require:

$$\Delta_{1,2}\Delta_{0,k} - \Delta_{0,2}\Delta_{1,k} + \Delta_{3,0}\Delta_{2,k} - \Delta_{0,1}\Delta_{3,k} \;=\; 0. \tag{3}$$

Suppose that polygon P has no consecutive collinear edges. Then $\Delta_{0,1}$ in equation (2) is nonzero. So the row vector

$$(\Delta_{1,2}, -\Delta_{0,2}, \Delta_{3,0}, -\Delta_{0,1}, 0, \ldots, 0)$$

has nonzero fourth entry followed by $n-4$ zeros. Repeating the above process $n-4$ times will produce a total of $n-3$ vectors with the ith vector having nonzero $(i+3)$-rd entry followed by all zeros. It is clear that these $n-3$ vectors must be linearly independent. So, in the absence of consecutive collinear edges, $\mathrm{rank}(\Delta_P) \le n - (n-3) = 3$.

It remains to observe that, given a polygon P with no consecutive collinear edges, subdividing an edge by inserting an extra vertex cannot increase rank. This follows from Lemma 3(3): if Δ_P has rank 3 then subdividing an edge produces an additional matrix column which is a multiple of an existing column and therefore does not change the rank of the matrix.

To end with a question: how might we write down a basis of vectors for the null space of Δ_P? Equation (3) gives us a null space vector except in the presence of *four* consecutive collinear vertices, in which case all coefficients in equation (2) are zero and we do not get a valid eigenvector. In this case, Lemma 2 comes to our rescue with an equation whose coefficients are all nonzero. Nevertheless the eigenvectors we assemble will not necessarily have nonzero entries placed, as above, so as to guarantee linear independence. For example, a polygon consisting of eight points equally placed around a square will, using equation (2), give a collection of eight null space vectors which has dimension 4 instead of the $8-3=5$ that we need. Where is the missing eigenvector in this example?

Tommy Moorhouse

The rank of a matrix $\mathbf{A}$ is the number of linearly independent rows (or columns) of $\mathbf{A}$. A deeper idea may underlie this problem but this is my brute force attempt at a solution. The strategy will be to find a family of four-by-four submatrices of $\mathbf{A}$ and show that they have vanishing determinant. We will deduce that any four consecutive rows of $\mathbf{A}$ are linearly dependent, but that three rows are linearly independent, from which the result follows.

The points P_i with coordinates (x_i, y_i) will be represented by complex numbers $z_i = x_i + iy_i$. Using the P_i cross products directly seemed to be too fiddly. The cross product can be expressed (using the name of the point as shorthand for its coordinates) as

$$P_i \times P_j \;=\; -\operatorname{Im} z_i \bar{z}_j .$$

To see this write $z_i = x_i + iy_i$ and $\bar{z}_j = x_j - iy_j$ and expand:

$$z_i \bar{z}_j \;=\; x_i x_j + y_i y_j - i(x_i y_j - x_j y_i).$$

The area of the triangle $P_i P_j P_{j+1}$ is then proportional to

$$P_j \times P_{j+1} + P_i \times (P_{j+1} - P_j) \;=\; -\operatorname{Im}(z_j \bar{z}_{j+1} + z_i \bar{z}_{j+1} + z_j \bar{z}_i),$$

where we have used $\operatorname{Im} z_i \bar{z}_j = -\operatorname{Im} z_j \bar{z}_i$. Note that, in particular, $\operatorname{Im} z_i \bar{z}_i = 0$ for any index i.

The expression for the area factorises into

$$2A_{i,j} \;=\; \operatorname{Im}((z_j - z_i)(\bar{z}_{j+1} - \bar{z}_j)).$$

For $n = 4$ the matrix is the imaginary part of

$$\mathbf{A} \;=\; \begin{pmatrix} 0 & a & b & 0 \\ 0 & 0 & c & d \\ a & 0 & 0 & b \\ d & c & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} a &= -\frac{1}{2}(z_2 - z_1)(\bar{z}_3 - \bar{z}_2) \;=\; -\frac{1}{2}(z_1 - z_3)(\bar{z}_2 - \bar{z}_1), \\ b &= -\frac{1}{2}(z_3 - z_1)(\bar{z}_4 - \bar{z}_3) \;=\; -\frac{1}{2}(z_4 - z_3)(\bar{z}_1 - \bar{z}_4), \end{aligned}$$

$$\begin{aligned} c &= -\frac{1}{2}(z_3 - z_2)(\bar{z}_4 - \bar{z}_3) = -\frac{1}{2}(z_2 - z_4)(\bar{z}_3 - \bar{z}_2), \\ d &= -\frac{1}{2}(z_4 - z_2)(\bar{z}_1 - \bar{z}_4) = -\frac{1}{2}(z_1 - z_4)(\bar{z}_2 - \bar{z}_1). \end{aligned}$$

The equality of the two expressions for each element can be checked using the identities above, remembering that we are taking the imaginary part in each case. The reader can check that the rows (or equivalently, the columns) of **A** are linearly dependent, and that **A** has rank 3.

In the general case of an $n \times n$ matrix **A** the entries $A_{k,k}$ and $A_{k+1,k}$ vanish for all relevant k. This follows from the definition in terms of areas of triangles with vertices P_k, P_k and P_{k+1}; and P_{k+1}, P_k and P_{k+1}, respectively. This means that along the diagonal there are four-by-four block submatrices of the general form

$$\mathbf{S}_k = \begin{pmatrix} 0 & a & b & u \\ 0 & 0 & c & v \\ a & 0 & 0 & w \\ d & c & 0 & 0 \end{pmatrix}.$$

The repeated entries occur in a regular pattern: $A_{k,k+1} = A_{k+2,k}$, again from the definition. $\mathbf{S}_k$ is effectively the truncated matrix arising from five points: we have isolated the first four rows and columns. The first three columns are linearly independent in general. The elements will differ between different $\mathbf{S}_k$ but to avoid clutter we leave the label k off a, b, c etc.

We will show that each $\mathbf{S}_k$ has vanishing determinant, and that rows $k, k{+}1, k{+}2$ and $k{+}3$ of **A** are linearly dependent for all relevant k. This will be done using the complex number representation, noting that the vector $\mathbf{c} = (-c, b, -d, a)$ is orthogonal to the first three columns of $\mathbf{S}_k$. In forming the dot product $\mathbf{c} \cdot \mathbf{u}$ for the last column $\mathbf{u}$ we insert $0 = z_{k+1} - z_{k+1}$ into the scalar product as follows, where we take the imaginary part as usual:

$$\begin{aligned} \mathbf{c} \cdot \mathbf{u} &= -c(z_{k+3} - z_k)(\bar{z}_{k+4} - \bar{z}_{k+3}) + b(z_{k+3} - z_{k+1})(\bar{z}_{k+4} - \bar{z}_{k+3}) \\ &\quad - d(z_{k+3} - z_{k+2})(\bar{z}_{k+4} - \bar{z}_{k+3}) \\ &\quad + a(z_{k+3} - z_{k+3})(\bar{z}_{k+4} - \bar{z}_{k+3}) \\ &= c((z_k - z_{k+1}) + (z_{k+1} - z_{k+3}))(\bar{z}_{k+4} - \bar{z}_{k+3}) \\ &\quad - b((z_{k+1} - z_{k+1}) + (z_{k+1} - z_{k+3}))(\bar{z}_{k+4} - \bar{z}_{k+3}) \\ &\quad + d((z_{k+2} - z_{k+1}) + (z_{k+1} - z_{k+3}))(\bar{z}_{k+4} - \bar{z}_{k+3}) \\ &\quad - a((z_{k+3} - z_{k+1}) + (z_{k+1} - z_{k+3}))(\bar{z}_{k+4} - \bar{z}_{k+3}) \\ &= 0. \end{aligned}$$

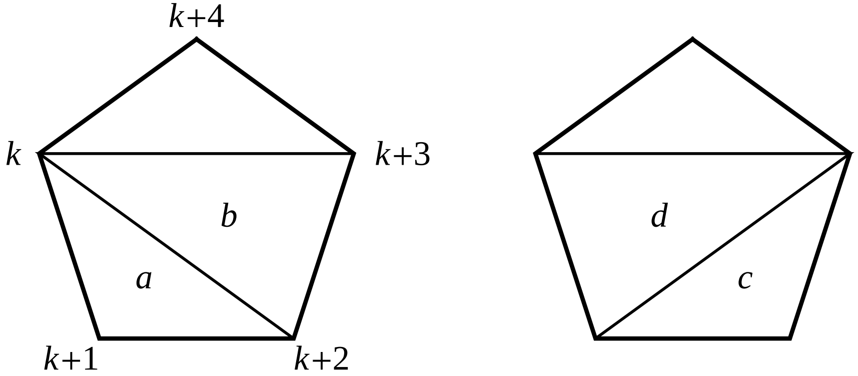


Figure 1: $a + b = c + d$

The sum of the first set of terms grouped in each bracket vanishes because the second column of $\mathbf{S}_k$ is orthogonal to $(-c, b, -d, a)$, and the sum of the second set of grouped terms vanishes since $a + b = c + d$, as shown in Figure 1. We could have inserted $z_k - z_k$ (for the first column) or $z_{k+2} - z_{k+2}$ (for the third column) to the same effect. The fourth column $\mathbf{u}$ is orthogonal to $\mathbf{c}$, so that each $\mathbf{S}_k$ has rank 3.

The vector $\mathbf{c}$ can be shown to be orthogonal to each four-element column segment of $\mathbf{A}$ starting on row k (i.e. $(A_{k,j}, A_{k+1,j}, A_{k+2,j}, A_{k+3,j})$) by the same technique, showing that the four consecutive rows are linearly dependent. We have now established that every set of four consecutive rows of $\mathbf{A}$ is linearly dependent. It is straightforward to show, for example by eliminating rows between these sets, that as a consequence any set of four or more rows of $\mathbf{A}$ is linearly dependent. The rank of $\mathbf{A}$ is thus at most 3, and the independence of rows 1 to 3 shows that the rank is exactly 3 for nontrivial sets of points.

Problem 327.1 – Jigsaw puzzle

A jigsaw puzzle has n edge pieces, n an even integer. How many pieces can it have altogether.

For example, if $n = 14$, then I think the answer is 14, 18 and 20.

A family of circles and inversion on the sphere

Tommy Moorhouse

Inversion in the unit circle We consider the standard plane with origin $\mathcal{O}$ and a unit circle $\mathcal{C}$ centred on $\mathcal{O}$. Inversion in $\mathcal{C}$ sends a point $\mathcal{P}$ in the plane to another point $i_1(\mathcal{P})$ defined as follows. Suppose $\mathcal{P}$ is a distance r from $\mathcal{O}$. Draw a straight line from $\mathcal{O}$ through $\mathcal{P}$ and extend it as far as required. $i_1(\mathcal{P})$ is the point on this line a distance $1/r$ from $\mathcal{O}$.

Part 1: A family of circles Fix the point $\mathcal{Q} = (0, -1)$ on $\mathcal{C}$. The image of the tangent line through $\mathcal{Q}$ under inversion is a circle of radius $1/2$ centred on $(0, -1/2)$. The family of lines through $\mathcal{Q}$ obtained by rotating the tangent line in a positive sense (so that it intersects $\mathcal{C}$ in $\mathcal{Q}$ and another point) gives rise under inversion in $\mathcal{C}$ to a family of circles passing through $\mathcal{O}$ (see Figure 1). What is the locus of the centres of these circles?

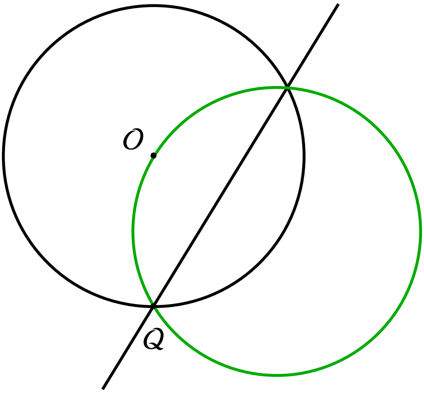


Figure 1: A line through $\mathcal{Q}$ and its associated circle

Part 2: Inversion on a sphere Now imagine a unit sphere $\mathcal{S}$ embedded in $\mathbb{R}^3$ with a great circle $\mathcal{C}$ (e.g. the equator) marked on its surface. Inversion in $\mathcal{C}$ can be described by a construction involving the projection of points of the sphere from the north and south poles. Projection of points from the north pole onto the plane through the equator maps the sphere to the plane, sending $\mathcal{P}$ to $\pi_N(\mathcal{P})$, say. Projecting instead from the south pole sends each point $\mathcal{P}$ to $\pi_S(\mathcal{P})$. The two projections of the same point on the sphere are related by inversion on the plane. Use a geometric, or other, argument to find a surprisingly simple description of inversion on the sphere. Is there an analogue of the result of Part 1 for the sphere?

No really, how far is the horizon?

J. M. Selig

Finally, I think I can say something about the distance to the apparent horizon. Recall, I promised this in M500 **320** and then again in M500 **326**. The apparent horizon is the position of the horizon taking into account refraction in the atmosphere of the Earth. To do this, Fermat's principle will be used in polar coordinates to produce a differential equation for the path of light. This is because we will assume that the refractive index of the atmosphere only varies with height above the surface of the Earth.

In polar coordinates the optical path length of a light ray is given by

$$\int L\,dt \;=\; \int n\sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}\,dt \;=\; \int n\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta.$$

Writing r' for the derivative $dr/d\theta$, the Euler–Lagrange equation for this functional is

$$\frac{d}{d\theta}\left(\frac{\partial L}{\partial r'}\right) - \frac{\partial L}{\partial r} \;=\; 0. \tag{1}$$

Assuming that n, the refractive index of the atmosphere, is a function of r only, we can compute,

$$\frac{\partial L}{\partial r} \;=\; \frac{\partial}{\partial r}\Big(n\big((r')^2 + r^2\big)^{1/2}\Big) \;=\; \frac{dn}{dr}\big((r')^2 + r^2\big)^{1/2} + nr\big((r')^2 + r^2\big)^{-1/2}$$

and

$$\frac{\partial L}{\partial r'} \;=\; nr'\big((r')^2 + r^2\big)^{-1/2}.$$

So that

$$\begin{aligned}\frac{d}{d\theta}\left(\frac{\partial L}{\partial r'}\right) \;=\; & \frac{dn}{dr}(r')^2\big((r')^2 + r^2\big)^{-1/2} \\ & + nr''\big((r')^2 + r^2\big)^{-1/2} - nr'(r'r'' + rr')\big((r')^2 + r^2\big)^{-3/2}.\end{aligned}$$

Substituting into the Euler–Lagrange equation (1), and multiplying through by $\big((r')^2 + r^2\big)^{3/2}$ gives

$$\begin{aligned}0 \;=\; & \frac{dn}{dr}(r')^2\big((r')^2 + r^2\big) + nr''\big((r')^2 + r^2\big) - nr'(r'r'' + rr') \\ & - \frac{dn}{dr}\big((r')^2 + r^2\big)^2 - nr\big((r')^2 + r^2\big).\end{aligned}$$

Expanding the brackets and simplifying this produces

$$nr^2r'' - \left(2nr + \frac{dn}{dr}r^2\right)(r')^2 \;=\; nr^3 + \frac{dn}{dr}r^4.$$

Next, we use a standard trick: replace the second derivative r'' by $r'dr'/dr$. Then we multiply the whole equation by $2/(n^3r^6)$ to produce

$$\frac{2}{n^2r^4}r'\frac{dr'}{dr} - \left(\frac{4}{n^2r^5} + \frac{2}{n^3r^4}\frac{dn}{dr}\right)(r')^2 \;=\; \frac{2}{n^2r^3} + \frac{2}{n^3r^2}\frac{dn}{dr}.$$

Both sides of this equation are now total derivatives,

$$\frac{d}{dr}\left(\frac{(r')^2}{n^2r^4}\right) \;=\; -\frac{d}{dr}\left(\frac{1}{n^2r^2}\right).$$

So, integrating and rearranging produces a first order differential equation,

$$\frac{dr}{d\theta} \;=\; \pm\, r\sqrt{Cn^2r^2 - 1},$$

where C is the constant of integration. Also, notice that this gives us an invariant of the system, since

$$\frac{d}{dr}\left(\frac{(r')^2 + r^2}{n^2r^4}\right) \;=\; \frac{dC}{dr} \;=\; 0;$$

thus this quantity must be constant on any solution. Hence we can use this to fix C. Suppose, when $\theta = 0$, we have that $r = R_\oplus$ the radius of the Earth, and at the Earth's surface the refractive index of air is n_0. We will also assume that the light ray at this initial point is tangential to the Earth's surface, that is, the initial point is at the apparent horizon. With these initial conditions get that $C = 1/n_0^2R_\oplus^2$. The first order equation for the path of a light ray from the apparent horizon is thus

$$\frac{dr}{d\theta} \;=\; \pm\, r\sqrt{\left(\frac{nr}{n_0R_\oplus}\right)^2 - 1}. \tag{2}$$

According to Walter Bislin's blog [1], close to the ground the refractive index varies exponentially with altitude. So we can approximate the function $n(r)$ as

$$n(r) \;=\; n_0e^{-k(r-R_\oplus)},$$

where, using the data given in [1], the refractive index at ground level is $n_0 = 1.00027$ and the radius of the Earth is $R_\oplus = 6371\,\text{km}$. The constant k can be fixed with another data point, again from [1], at 500 m the refractive index is approximately 1.00024. Hence

$$k \;=\; \frac{-1}{500}\ln\left(\frac{1.00024}{1.00027}\right) \;=\; 6.00\times 10^{-8}.$$

Now we make the substitution $h = (r - R_\oplus)/R_\oplus$. This transforms the differential equation to

$$\frac{dh}{d\theta} \;=\; \pm\,(h+1)\sqrt{(h+1)^2e^{-2\kappa h}-1}.$$

where the constant κ is $\kappa = R_\oplus k = 0.38$. The signs give the different possible directions to the light rays, so we just look at anti-clockwise travelling light. The distance to the apparent horizon d, is then given by the integral

$$d \;=\; R_\oplus\theta \;=\; R_\oplus\int_0^{e/R_\oplus}\frac{dh}{(h+1)\sqrt{(h+1)^2e^{-2\kappa h}-1}}. \qquad (3)$$

The quantity e here is the elevation of the observer, their height above sea-level.

This is one of those integrals that can't be done in terms of elementary functions. Worse, the integrand has a singularity at the bottom limit, substituting $h = 0$ into the square-root expression gives zero. This makes numerical integration difficult. Perhaps we can approximate the integrand as a finite part of an infinite series, we have to take account of the singularity though and write it as a Puiseux series, [3]. The term under the square-root can be expanded as a Taylor series,

$$\begin{aligned}&(h+1)^2e^{-2\kappa h}-1\\ &\quad= (1+2h+h^2)\Big(1-2\kappa h+\frac{4\kappa^2}{2!}h^2-\frac{8\kappa^3}{3!}h^3+\frac{16\kappa^4}{4!}h^4-\cdots\Big)-1,\\ &= 2(1-\kappa)h+(1-4\kappa+2\kappa^2)h^2-\frac{2}{3}(3-6\kappa+2\kappa^2)\kappa h^3+\frac{2}{3}(3-4\kappa+\kappa^2)\kappa^2h^4-\cdots.\end{aligned}$$

This vanishes when $h = 0$, hence the singularity. If we multiply the integrand in (3) above by $\sqrt{h}$ we get

$$\frac{\sqrt{h}}{(h+1)\sqrt{(h+1)^2e^{-2\kappa h}-1}} = \frac{1}{(h+1)\sqrt{2(1-\kappa)+(1-4\kappa+2\kappa^2)h-O(h^3)}}.$$

When $h \to 0$ this has a finite limit, $\sqrt{2(1-\kappa)}$. This removes the singularity and thus we can use a Taylor expansion of this function to write the integrand as

$$\begin{aligned}
&\frac{1}{(h+1)\sqrt{(h+1)^2 e^{-2\kappa h}-1}} \\
&\qquad = (h)^{-1/2}\Big(\frac{1}{\sqrt{2(1-\kappa)}} - \frac{(2\kappa^2-8\kappa+5)h}{4(1-\kappa)\sqrt{2(1-\kappa)}} \\
&\qquad\qquad + \frac{(4\kappa^4-64\kappa^3+276\kappa^2-336\kappa+129)h^2}{96(1-\kappa)^2\sqrt{2(1-\kappa)}} + \cdots\Big).
\end{aligned}$$

Assuming we can integrate term by term the integral can be expanded as

$$\begin{aligned}
d &= R_\oplus \int_0^{e/R_\oplus} \frac{dh}{(h+1)\sqrt{(h+1)^2 e^{-2\kappa h}-1}} \\
&= \sqrt{\frac{2e}{(1-\kappa)R_\oplus}}\left(R_\oplus + \frac{(2\kappa^2-8\kappa+5)e}{12(\kappa-1)}\right. \\
&\qquad \left. + \frac{(4\kappa^4-64\kappa^3+276\kappa^2-336\kappa+129)e}{480(\kappa-1)^2} + \cdots\right).
\end{aligned}$$

In this expansion we only need to consider the first term,

$$d \approx \sqrt{\frac{2eR_\oplus}{1-\kappa}} = 4,541\sqrt{e}.$$

Were the substitutions $R_\oplus = 6371\,\text{km}$ and $\kappa = 0.38$ have been made. The second term here, the approximate error, is then

$$\sqrt{\frac{2e}{(1-\kappa)R_\oplus}}\left(\frac{(2\kappa^2-8\kappa+5)e}{12(\kappa-1)}\right) = -0.000215e^{3/2}$$

at an elevation of $e = 200\,\text{m}$, this is $-0.6\,\text{m}$ while the first term is $64.2\,\text{km}$. These computation give very approximate answers anyway because conditions in the atmosphere are so variable. The exponential model for the refractive index only seems to be valid up to about $500\,\text{m}$ and the data it is based on was recorded on a clear day in Southern California, not a cold, damp day in the UK.

Walter Bislin [1] also suggest that a common way to account for refraction is to use the distance to the geometric horizon but with the radius of the Earth scaled by 1.6. As a comparison, consider the distance to the horizon from the White Cliffs of Dover at an elevation of $e = 110\,\mathrm{m}$. We have the following.

Geometric: $R_\oplus\sqrt{\dfrac{2e}{R_\oplus + e}} \approx 37.4\,\mathrm{km}$

Approximate exponential: $R_\oplus\sqrt{\dfrac{2e}{R_\oplus(1-\kappa)}} \approx 47.5\,\mathrm{km}$ (with $\kappa = 0.38$)

Enlarged Radius: $1.6R_\oplus\sqrt{\dfrac{2e}{1.6R_\oplus + e}} \approx 47.4\,\mathrm{km}$

The two approximations that include refraction are only about 100 m apart and give a substantially longer distance than the geometric model. The use of the enlarged radius might be justified as follows. Suppose we enlarge the Earth's radius by a factor m, that is we use $mR_\oplus$ rather than just $R_\oplus$ in our formula for the distance to the geometric horizon,

$$d \approx mR_\oplus\sqrt{\frac{2e}{mR_\oplus + e}} = R_\oplus\sqrt{\frac{2e}{\frac{R_\oplus}{m} + \frac{e}{m^2}}}.$$

Now, if we ignore the term e/m^2 since it is very small compared to the other term in the denominator, we can compare this to the formula that includes the effect of refraction and conclude that $(1-\kappa) = 1/m$ or $m = 1/(1-\kappa)$. If $\kappa = 0.38$ then

$$m = 1/(1-0.38) = 1.61.$$

Notice, we didn't actually find the path of the light ray. Clearly it must be some sort of spiral. The polar angle α, of a spiral curve is the angle between the tangent to the spiral at some point and the tangent to a circle through the same point with the same centre as the spiral. The polar slope is defined as the tan of this angle, $\tan\alpha = r'/r$, for our curve we have:

$$\tan\alpha = \sqrt{\left(\frac{nr}{n_0R_\oplus}\right)^2 - 1}, \quad \text{or} \quad \sec\alpha = \left(\frac{nr}{n_0R_\oplus}\right),$$

from equation (2). See, [5]. Unfortunately, this doesn't give us much help to determine the precise shape of the curve.

There are many other effects due to refraction in the atmosphere these include: looming, towering, stooping, and sinking, see [4]. Near sunset, the sun can appear flattened due to atmospheric refraction, [6]. Moreover, sunset can be delayed if the sun is past the geometric horizon but still visible due to refraction.

Perhaps the most spectacular effect is the "green flash", this can occur just after sunset and is due to the variation of the refractive index with wavelength. Generally, shorter wavelength are bent more than longer wavelengths. So, blue and green light from the sun is bent around the curve of the Earth just after sun has set. But the very short wavelengths, the blue light, is scattered more by the atmosphere and so a flash of green is seen in the direction the sun has set. Again, all these phenomena depend on particular atmospheric conditions; temperature, pressure and humidity.

References

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[2] Wikipedia contributors, "Fermat's principle", Wikipedia, The Free Encyclopedia, `https://en.wikipedia.org/wiki/Fermat's_principle` (accessed June 11, 2025).

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[4] Wikipedia contributors, "Looming and similar refraction phenomena", Wikipedia, The Free Encyclopedia, `https://en.wikipedia.org/wiki/Looming_and_similar_refraction_phenomena` (accessed August 7, 2025).

[5] Wikipedia contributors, "Spiral", Wikipedia, The Free Encyclopedia, `https://en.wikipedia.org/wiki/Spiral` (accessed August 14, 2025).

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Solution 261.2 – Trigonometric double integral

Show that

$$\int_0^{\pi/2}\int_0^{\pi/2}\frac{\sin^2\phi\;d\phi\;d\theta}{(2-\sin^2\theta)^{1/2}\;(2-\sin^2\phi)^{3/2}} \;=\; \frac{\pi}{4}.$$

Peter Fletcher

The given double integral can obviously be rewritten as

$$I \;=\; \int_0^{\pi/2}\frac{\mathrm{d}\theta}{\left(2-\sin^2(\theta)\right)^{1/2}}\cdot\int_0^{\pi/2}\frac{\sin^2(\phi)\;\mathrm{d}\phi}{\left(2-\sin^2(\phi)\right)^{3/2}}.$$

Consider the expressions $1/(2-x^2)^{1/2}$ and $y^2/(2-y^2)^{3/2}$. Using Claude (other large language models are freely available) or otherwise, we can find the Taylor expansions of these as

$$\frac{1}{\left(2-x^2\right)^{1/2}} \;=\; \frac{1}{\sqrt{2}}\cdot\sum_{n=0}^{\infty}\frac{(2n-1)!!}{2^{2n}\cdot n!}x^{2n}, \qquad x<\sqrt{2},$$

and

$$\frac{y^2}{\left(2-y^2\right)^{3/2}} \;=\; \frac{1}{2\sqrt{2}}\cdot\sum_{m=1}^{\infty}\frac{(2m-1)!!}{2^{2(m-1)}\cdot(m-1)!}y^{2m}, \qquad y<\sqrt{2}.$$

Now substituting $x=\sin(\theta)$ and $y=\sin(\phi)$ into these gives

$$\frac{1}{\left(2-\sin^2(\theta)\right)^{1/2}} \;=\; \frac{1}{\sqrt{2}}\cdot\sum_{n=0}^{\infty}\frac{(2n-1)!!}{2^{2n}\cdot n!}\sin^{2n}(\theta)$$

and

$$\frac{\sin^2(\phi)}{\left(2-\sin^2(\phi)\right)^{3/2}} \;=\; \frac{1}{2\sqrt{2}}\cdot\sum_{m=1}^{\infty}\frac{(2m-1)!!}{2^{2(m-1)}\cdot(m-1)!}\sin^{2m}(\phi).$$

We shall be wanting to integrate the sums term by term, for which we will need (thanks again to Claude)

$$\int_0^{\pi/2}\sin^{2p}(\psi)\mathrm{d}\psi \;=\; \frac{(2p-1)!!}{(2p)!!}\cdot\frac{\pi}{2}.$$

Therefore

$$\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\left(2-\sin^2(\theta)\right)^{1/2}} \;=\; \frac{\pi\sqrt{2}}{4}\cdot\sum_{n=0}^{\infty}\frac{\left((2n-1)!!\right)^2}{2^{2n}\cdot n!\cdot(2n)!!}$$

and

$$\int_0^{\pi/2} \frac{\sin^2(\phi)\mathrm{d}\phi}{\bigl(2-\sin^2(\phi)\bigr)^{3/2}} \;=\; \frac{\pi\sqrt{2}}{8}\cdot\sum_{m=1}^{\infty}\frac{\bigl((2m-1)!!\bigr)^2}{2^{2(m-1)}\cdot(m-1)!\cdot(2m)!!}.$$

If we find the sums of the first 200 terms of each of these in, for example, Maple, multiply them together and subtract $\pi/4$, the result is $\mathcal{O}(10^{-61})$, suggesting that these are correct.

We can now write the given double integral as

$$I \;=\; \frac{\pi^2}{16}\cdot\sum_{n=0}^{\infty}\frac{\bigl((2n-1)!!\bigr)^2}{2^{2n}\cdot n!\cdot(2n)!!}\cdot\sum_{m=1}^{\infty}\frac{\bigl((2m-1)!!\bigr)^2}{2^{2(m-1)}\cdot(m-1)!\cdot(2m)!!} \;=\; \frac{\pi}{4}.$$

Problem 327.2 – A sieve for integer powers

Here is something interesting that was presented by Robin Whitty during a lecture at the LSBU Maths Study Group meeting on 7 August 2025.

> Start with a positive integer k and $S = (1, 2, \dots)$.
>
> For $j = k,\ k-1,\ \dots,\ 2$:
>
> > form a new S from the cumulative sums of the sequence obtained by removing every j-th number from S.

Prove that the result is the sequence of k-th powers of the positive integers.

If you think this is a generalization of Problem 325.1 – A sieve for squares, you are probably correct. For example, here is what S looks like at various stages when $k = 4$.

Start	$(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, \dots)$
$j = 4$	$(1, 3, 6, 11, 17, 24, 33, 43, 54, 67, 81, 96, 113, 131, 150, 171, 193, \dots)$
$j = 3$	$(1, 4, 15, 32, 65, 108, 175, 256, 369, 500, 671, 864, 1105, 1372, \dots)$
$j = 2$	$(1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, 14641, 20736, \dots)$

Problem 327.3 – Integrals

Compute $\displaystyle\int_1^x \sin(\log t)dt$ and $\displaystyle\int_1^x \cos(\log t)dt$, $x \geq 1$.

Flattening a hemisphere

Tommy Moorhouse

Introduction. In M500 **322** I gave some results for the Weyl transformation (scaling) of two dimensional metrics, for the sphere and flat sheet. The metrics were scaled as $\hat{g}_{ab} = e^{2\omega(x^a)} g_{ab}$ and the curvature was shown to change by $-2\nabla^2\omega$. I thought it would be interesting to try and find ω explicitly for the flattening of part of a sphere to get a flat disk.

Flattening the northern hemisphere The usual metric of the unit sphere is expressed in terms of the latitude (θ) and longitude (ϕ) as

$$ds^2 \;=\; d\theta^2 + \sin^2\theta d\phi^2.$$

We restrict θ to $[0, \pi/2]$, the upper hemisphere. We want to bring the metric into the flat polar form for a disk

$$du^2 + u^2 d\phi^2$$

so we first apply a Weyl transformation. We don't want the change to depend on ϕ so ω is a function of θ only, and we find

$$d\hat{s}^2 \;=\; e^{2\omega(\theta)} d\theta^2 + e^{2\omega(\theta)} \sin^2\theta d\phi^2.$$

We now need to solve

$$\begin{aligned} du &= e^{\omega(\theta)} d\theta, \\ u &= e^{\omega(\theta)} \sin\theta. \end{aligned}$$

It isn't obvious that these equations are consistent, but the function $\omega(\theta)$ is so far undetermined. If we differentiate the second equation we find

$$du \;=\; e^{\omega(\theta)} \left(\frac{d\omega}{d\theta} \sin\theta + \cos\theta \right) d\theta$$

which gives, using the first equation for du,

$$\sin\theta \frac{d\omega}{d\theta} + \cos\theta \;=\; 1.$$

Using standard trigonometric identities (e.g. $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$), this is

$$\frac{d\omega}{d\theta} \;=\; \tan\frac{1}{2}\theta$$

leading to

$$\omega(\theta) \;=\; -\log\left(\cos^2\frac{1}{2}\theta\right).$$

Solving for u we find

$$u \;=\; 2\tan\frac{1}{2}\theta.$$

Differentiating we see that this gives the correct du. On the unit sphere the curvature scalar is $R = 2$ and it is an easy check that ω satisfies $\nabla^2\omega = 1$ so that $R - 2\nabla^2\omega = 0$ and the new metric is flat (as we aimed for). As a byproduct of the transformation we have found a solution to the equation $\nabla^2\omega = 1$. Other solutions can involve transformations of ϕ and so potentially give rise to other flat metrics.

In summary, to flatten the hemisphere we first scale the metric by the factor $e^{2\omega} = \cos^{-4}(\theta/2)$ then change coordinates to (u, ϕ) as above, keeping the same ϕ. The reverse transformation from a flat disk (in polar coordinates) to a hemisphere of constant curvature follows the same pattern and could make an interesting exercise.

Problem 327.4 – Three plus-or-minus 1s

Tony Forbes

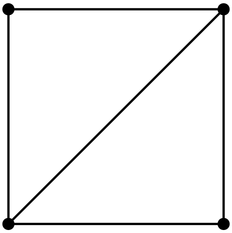
Solve

$$4a + 5b + 3c \;=\; 2 \tag{1}$$

for $a, b, c \in \{-1, 1\}$.

The solution should find useful application in graph theory. Suppose you have temporarily forgotten the signs of v, the number of vertices, e, the number of edges and f, the number of faces, in Euler's formula for connected planar graphs,

$$\pm v \pm e \pm f \;=\; 2. \tag{2}$$



No worries. All you need to do is observe that the coefficients in (1) apply to the graph consisting of a square with a diagonal. Then the solution of (1) will give the correct signs in (2).

The triangle is not suitable because $3a + 3b + 2c = 2$ has too many solutions. Nevertheless, you might find other small graphs that work. See Problem 327.5.

The Diophantine equation $1/u + 1/v = 1/w$

Victor Nicola

The following is an attempt at solving the Diophantine equation

$$\frac{1}{u} + \frac{1}{v} \;=\; \frac{1}{w}.$$

Multiply through by w:

$$\frac{w}{u} + \frac{w}{v} \;=\; 1.$$

Let $w/u = x$ and $w/v = y$. We get $x + y = 1$. In the (x, y)-plane let the line L_1 represent the equation $x + y = 1$. We require all the points on L_1 such that x and y $(= 1 - x)$ are rational numbers. Let L_2 represent the equation $y = (m/n)x$ (m and n are integers with $\gcd(m, n) = 1$). It is not difficult to see that every rational point on L_1 can be 'reached' by appropriate choice of m and n in line L_2 and, vice versa, every choice of m and n can make L_2 reach a rational point on L_1. The lines L_1 and L_2 meet at the point whose coordinates are $x = n/(m + n)$ and $y = m/(m + n)$.

Now, $x = w/u$ and $y = w/v$. It follows that $w/u = n/(m + n)$, or $u = (m+n)w/n$. Similarly, $v = (m+n)w/m$. By choosing $w = mn$, we get

$$u \;=\; m(m+n), \qquad v \;=\; n(m+n) \qquad \text{and} \qquad w = mn.$$

This, in my opinion, is a solution to the given Diophantine equation.

Problem 327.5 – Euler's formula for planar graphs

Tony Forbes

Determine all those connected planar graphs of v vertices, e edges and f faces where the formula

$$av + be + cf \;=\; 2 \tag{1}$$

does not have a unique solution for $a, b, c \in \{-1, 1\}$.

For example, The triangle qualifies. Since $v = e = 3$ and $f = 2$, equation (1) simplifies to $3a+3b+2c = 2$, which has two solutions, $(a, b, c) = (-1, 1, 1)$ or $(1, -1, 1)$.

On the other hand, the octahedron doesn't. With $v = 6$, $e = 12$, $f = 8$, the equation $6a+12b+8c = 2$ has a unique solution, which of course must be $(a, b, c) = (1, -1, 1)$ corresponding to Euler's formula for connected planar graphs, $v - e + f = 2$.

Solution 320.1 – Hexagonal numbers

The hexagonal numbers are defined by

$$H_1 \;=\; 1, \qquad H_n \;=\; H_{n-1} + 6(n-1), \quad n \ge 2.$$

Show that

$$H_1 + H_2 + \cdots + H_n \;=\; n^3.$$

Dave Wild

The plan is to use the recurrence relation for H_n to initially replace the last term in the sum, then H_{n-1}, etc. until we are left with an expression which only contains H_1. We will write the recurrence relation as

$$H_n \;=\; H_{n-1} + f(n), \qquad \text{where } f(n) = 6(n-1).$$

For example, for sufficiently large n, we have

$$H_{n-1} + H_n \;=\; H_{n-1} + (H_{n-1} + f(n)) \;=\; 2H_{n-1} + f(n)$$

and

$$H_{n-2}+(H_{n-1}+H_n) \;=\; H_{n-2}+(2H_{n-1}+f(n)) \;=\; 3H_{n-2}+2f(n-1)+f(n).$$

Therefore, when $n > 1$,

$$\sum_{m=1}^{n} H_m \;=\; nH_1 + \sum_{m=1}^{n-1} (n-m)f(m+1) \;=\; n + n\sum_{m=1}^{n-1} 6m - \sum_{m=1}^{n-1} 6m^2.$$

Using standard results for the sums,

$$\sum_{m=1}^{n} H_m \;=\; n + n(3(n-1)n) - (n-1)n(2n-1) \;=\; n^3.$$

Since $H_1 = 1^3$ the required result follows.

Problem 327.6 – Sum

Tony Forbes

Show that

$$\frac{1}{1^2 \cdot 3^2} + \frac{1}{5^2 \cdot 7^2} + \frac{1}{9^2 \cdot 11^2} + \ldots \;=\; \frac{\pi(\pi-2)}{32}.$$

Solution 324.1 – Rabbit

> A rabbit is in a ditch which stretches infinitely far to the east and to the west. Every minute the rabbit does one of three things:
>
> - it moves 1 metre to the east, with probability p;
> - it moves 1 metre to the west, with probability p;
> - it exits the ditch, with probability $1 - 2p$, at which point the rabbit is never seen again (clearly $0 < p < 1/2$).
>
> What is the probability the rabbit, at some point, exits the ditch from the same spot where it currently stands?
> A generalisation is to ask how the problem changes if the probability of going east is not the same as going west.

Reinhardt Messerschmidt

For the generalized problem, let p and q be the probabilities of moving east and west respectively, with $p+q < 1$. Let X_n be the random variable for the position of the rabbit after n minutes. This can be an integer, representing the number of metres from the starting point (positive for east and negative for west), or ∞, representing the outside of the ditch. We want to find

$$\mathbb{P}\Bigg[\bigcup_{n=0}^{\infty}\{X_{n+1} = \infty \text{ and } X_n = 0\}\Bigg] = \sum_{n=0}^{\infty}\mathbb{P}\big[X_{n+1} = \infty \text{ and } X_n = 0\big]$$
$$= \sum_{n=0}^{\infty}\mathbb{P}\big[X_{n+1} = \infty \,\big|\, X_n = 0\big]\mathbb{P}[X_n = 0] = (1 - p - q)\sum_{n=0}^{\infty}\mathbb{P}[X_n = 0].$$

If n is odd, then

$$\mathbb{P}[X_n = 0] = 0.$$

If $n = 2m$ for some nonnegative integer m, then $X_n = 0$ if and only if the rabbit has made m moves east and m moves west. There are

$$\binom{2m}{m}$$

such paths; therefore

$$\mathbb{P}[X_n = 0] = \binom{2m}{m}p^m q^m.$$

It follows that

$$\sum_{n=0}^{\infty} \mathbb{P}[X_n = 0] \;=\; \sum_{m=0}^{\infty} \frac{(2m)!}{(m!)(m!)}(pq)^m.$$

It only remains for us to evaluate the series. By the arithmetic and geometric mean inequality,

$$4pq \;\le\; 4\left(\frac{p+q}{2}\right)^2 \;=\; (p+q)^2 \;<\; 1.$$

It follows by the generalized binomial theorem (example 3 in section 5.4 of [1]) that

$$(1-4pq)^{-1/2} \;=\; \sum_{m=0}^{\infty} \frac{(-1/2)_m}{m!}(-4pq)^m,$$

where $(\cdot)_m$ is the falling factorial. We have

$$\begin{aligned}(-1/2)_m \;&=\; \left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\cdots\left(\frac{-1}{2}-(m-1)\right)\\ &=\; \left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\cdots\left(\frac{-(2m-1)}{2}\right)\\ &=\; \frac{(-1)^m(2m)!}{(2^m)(2^m)(m!)};\end{aligned}$$

therefore

$$\begin{aligned}(1-4pq)^{-1/2} \;&=\; \sum_{m=0}^{\infty} \frac{(-1)^m(2m)!}{(2^m)(2^m)(m!)(m!)}(-1)^m(4^m)(pq)^m\\ &=\; \sum_{m=0}^{\infty} \frac{(2m)!}{(m!)(m!)}(pq)^m.\end{aligned}$$

The final answer is therefore

$$\frac{1-p-q}{\sqrt{1-4pq}}.$$

References

[1] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 3rd ed., Cambridge University Press, Cambridge, 1920.

Solution 324.1 – Rabbit

David Sixsmith

1 Solution

How can the rabbit exit from where it currently stands? It has to move around in the ditch for some even period of minutes, ending up where it currently stands, and then exit the ditch. Note that the "even period of minutes" in this statement may be zero, and note that after an odd number of minutes in the ditch the rabbit cannot be back where it started. The probability of exiting the ditch is always $1 - 2p$.

To stay in the ditch for $2n$ minutes, the rabbit must make an equal number of choices of moving either east or west; in other words it has to move east exactly n times, and hence also west exactly n times. There are

$$\binom{2n}{n} = \frac{(2n)!}{n!n!}$$

of these possible choices, each with probability p^{2n}. Hence

$$P = (1-2p) \cdot \sum_{n=0}^{\infty} p^{2n} \frac{(2n)!}{n!n!}. \tag{1}$$

Note that if $4|x| < 1$, then

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} x^n \frac{(2n)!}{n!n!},$$

which can be checked using a binomial expansion. Hence

$$P = (1-2p) \cdot \frac{1}{\sqrt{1-4p^2}} = \sqrt{\frac{1-2p}{1+2p}}.$$

The problem also asked how the answer changes if the probability of going east is not the same as going west. In fact this is easy enough to do – if the probability of going east is p, and that of going west is q, then the same argument that lead to (1) gives

$$P = (1-p-q) \cdot \sum_{n=0}^{\infty} p^n q^n \frac{(2n)!}{n!n!},$$

in which case

$$P = \frac{1-p-q}{\sqrt{1-4pq}}.$$

2 A new problem

It is also interesting to ask what it the probability the rabbit eventually emerges n metres from where it currently stands – we have here calculated the $n=0$ case.

Problem 327.7 – Digits

Given an even integer $b \ge 6$, show that there exist b distinct integers

$$c_0,\, d_0,\, c_1,\, d_1,\, \ldots,\, c_{b/2-1},\, d_{b/2-1}$$

such that

$$\begin{aligned} 0 \ &\le\ c_0,\, d_0,\, c_1,\, d_1,\, \ldots,\, c_{b/2-2},\, d_{b/2-2}\ <\ b, \\ 1 \ &\le\ c_{b/2-1},\, d_{b/2-1}\ <\ b, \end{aligned}$$

and

$$\begin{aligned} (b-1)\left(c_0 + c_1 b + c_2 b^2 + \cdots + c_{b/2-1} b^{b/2-1}\right) \\ = \ d_0 + d_1 b + d_2 b^2 + \cdots + d_{b/2-1} b^{b/2-1}. \end{aligned}$$

With $b = 10$ this rather long-winded problem may be stated succinctly in the familiar setting of numbers represented by decimal digits. Find an integer n of five distinct digits such that $9n$ consists of the other five digits.

Thanks to Peter Cameron for the idea behind this problem. Here are some examples; they shouldn't be too difficult to decipher.

(6, 40, 200, 104_6, 532_6)

(8, 546, 3822, 1042_8, 7356_8)

(10, 10638, 95742, 10638_{10}, 95742_{10}) plus 2 more

(12, 259611, 2855721, $1062\text{a}3_{12}$, $\text{b}58749_{12}$) plus 2 more

(14, 7808502, 101510526, 1073942_{14}, $\text{d}6\text{a}58\text{bc}_{14}$) plus a few more

(16, 277013460, 4155201900, $1082\text{e}3\text{d}4_{16}$, $\text{f}7\text{ab}596\text{c}_{16}$) plus many more

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04.01

Problem 327.8 – Factorial and square

Tony Forbes

Find all solutions in positive integers n and s of

$$s^2 - n\sqrt{s} \ \le\ n! \ \le\ s^2 + n\sqrt{s}.$$

Front cover The Shrikhande graph and the cartesian product $K_4 \Box K_4$. Their union is isomorphic to the complete multipartite graph $K_{4,4,4,4}$, [`https://arxiv.org/abs/2505.00859`].