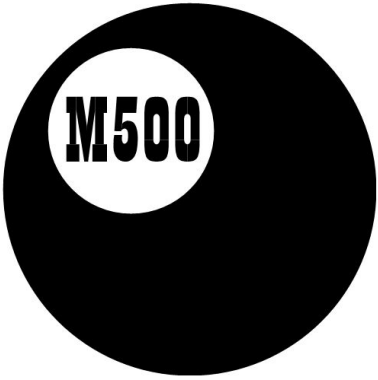
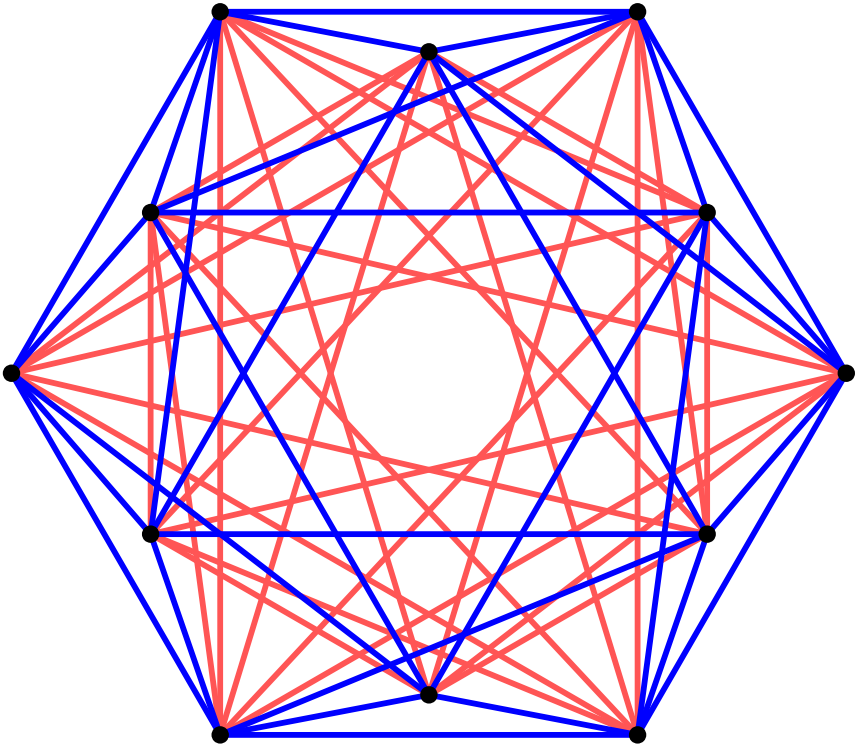
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On the misinterpretation of the Ramanujan summation: A critical perspective

Mako Sawin

Abstract

This short article critically examines the widely circulated identity

$$1+2+3+4+\cdots \;=\; -\frac{1}{12},$$

which is often referred to as a Ramanujan summation. While the formula has relevance within analytic number theory and quantum field theory via zeta function regularisation, I argue that its interpretation in common discourse often lacks necessary mathematical caution. By revisiting the classical divergent series involved and analyzing their manipulations, I propose that the identity arises from a formal trick rather than a factual summation in the conventional sense.

1 Introduction

The identity

$$1+2+3+4+\cdots \;=\; -\frac{1}{12}$$

has intrigued mathematicians and the public alike, especially following its popularisation in videos and physics lectures. Often attributed to Ramanujan, this formula is typically derived using a sequence of manipulations on divergent series. In this paper, I examine the classical derivation and offer a critical analysis, showing that its validity hinges on analytic continuation, not standard summation theory.

2 Classical Derivation and Its Problems

Step 1: Grandi's Series

$$S_\alpha \;=\; 1-1+1-1+\cdots.$$

By grouping terms:

$$S_\alpha \;=\; (1-1)+(1-1)+\cdots \;=\; 0+0+\cdots \;=\; 0.$$

Alternatively:

$$S_\alpha \;=\; 1+(-1+1)+(-1+1)+\cdots \;=\; 1+0+0+\cdots \;=\; 1.$$

These conflicting results indicate that the series is **divergent** in the standard sense. However, under *Cesàro summation*, it is assigned the value

$$S_\alpha \;=\; \frac{1}{2}.$$

This assignment is not a true sum but a regularisation.

Step 2: Alternating Series

$$S_\beta \;=\; 1-2+3-4+5-6+\cdots.$$

A trick is employed:

$$S_\beta \;=\; 1-2+3-4+\cdots$$

$$S_\beta \;=\; \qquad 1-2+3-4+\cdots.$$

Adding:

$$2S_\beta \;=\; 1-1+1-1+\cdots \;=\; S_\alpha \;=\; \frac{1}{2} \;\Rightarrow\; S_\beta \;=\; \frac{1}{4}.$$

Yet, the series diverges in the traditional sense, and the assignment $\frac{1}{4}$ again stems from regularisation.

Step 3: The Final Manipulation

Let $S = 1+2+3+4+\cdots$. Then:

$$\begin{aligned} S - S_\beta \;&=\; (1+2+3+4+\cdots) - (1-2+3-4+\cdots) \\ &=\; 4+8+12+\cdots \;=\; 4(1+2+3+\cdots) \;=\; 4S, \\ S - S_\beta \;&=\; 4S \;\Rightarrow\; S - \frac{1}{4} \;=\; 4S \;\Rightarrow\; -3S \;=\; \frac{1}{4} \;\Rightarrow\; S \;=\; -\frac{1}{12}. \end{aligned}$$

This result is **only valid** if all prior assignments (e.g. $S_\beta \;=\; \frac{1}{4}$) are valid in the summation sense, which they are not.

3 A Finite Counterexample

Let us consider finite partial sums. Let

$$S_K \;=\; 1+2+3+\cdots+K \;=\; \frac{K(K+1)}{2}.$$

Let

$$S_{\beta,K} \;=\; 1-2+3-4+\cdots+(-1)^{K+1}K.$$

This yields:

$$S_{\beta,K} \;=\; -\frac{K}{2} \quad \text{(when } K \text{ is even)}.$$

Then

$$S_K - S_{\beta,K} \;=\; 4\left(1+2+\cdots+\frac{K}{2}\right) \;=\; 4S_{K/2}.$$

Solving:

$$S_K \;=\; 4S_{K/2} - \frac{K}{2},$$

which contradicts the idea that $S = -\frac{1}{12}$ as $K \to \infty$, since it diverges positively.

4 Conclusion

The equation

$$1+2+3+4+\cdots \;=\; -\frac{1}{12}$$

is **not a fact** in standard analysis. Rather, it is a value derived via *analytic continuation of the Riemann zeta function*, where

$$\zeta(-1) \;=\; -\frac{1}{12}.$$

This method is valid in the context of string theory and quantum field theory, where regularized sums have meaning. However, these values should not be interpreted as literal summations of divergent series in the classical sense. The derivation, while elegant, is ultimately a formal trick that must be carefully contextualised.

Sources

1. Hardy, G. H., *Divergent Series*. Clarendon Press, 1949.
2. Apostol, T. M., *Mathematical Analysis*, 2nd Ed., Addison–Wesley, 1974.
3. Knopp, K., *Theory and Application of Infinite Series*, Dover, 1990.
4. Euler, L., *De seriebus divergentibus*, 1749.

5 Finite Term Analysis and Indexed Counterexamples

Finite Version of the Alternating Series S_β

Let

$$S_\beta \;=\; 1-2+3-4+5-6+\cdots+(-1)^{K+1}K.$$

For even K, say $K=14$, we compute

$$S_\beta \;=\; 1-2+3-4+\cdots+13-14 \;=\; -7.$$

But the analytic derivation suggests

$$S_\beta \;=\; \frac{1}{4}.$$

In general:

$$S_\beta \;=\; -\frac{K}{2}.$$

This value is clearly dependent on K and disagrees with the fixed regularised value. Therefore, we have

$$S_\beta \;\neq\; \frac{1}{4}.$$

Finite Counterexample to Ramanujan's Formula

Let

$$S \;=\; 1+2+3+\cdots+K, \quad S_\beta \;=\; 1-2+3-4+\cdots+(-1)^{K+1}K.$$

Assuming $K=14$, then

$$S \;=\; 105, \quad S_\beta \;=\; -7, \quad S-S_\beta \;=\; 112.$$

Now consider:

$$S-S_\beta \;=\; 4\left(1+2+3+\cdots+\frac{K}{2}\right) \;=\; 4\cdot\frac{(K/2)(K/2+1)}{2},$$

$$S-S_\beta \;=\; 4S_{K/2} \;\Rightarrow\; S \;=\; 4S_{K/2}-\frac{K}{2}.$$

Hence

$$\begin{aligned} S \;&=\; 4\left(\frac{K(K+2)}{8}\right)-\frac{K}{2} \;=\; \frac{K(K+2)}{2}-\frac{K}{2} \\ &=\; \frac{K(K+2)-K}{2} \;=\; \frac{K(K+1)}{2}. \end{aligned}$$

So $S = K(K+1)/2$. This grows positively as $K \to \infty$, which contradicts the Ramanujan assignment $S = -\frac{1}{12}$.

Conclusion of This Section

These finite examples and indexed expansions demonstrate that Ramanujan's summation cannot be interpreted as an ordinary arithmetic identity. The contradictions show that such formulas are meaningful only in the context of analytic continuation and regularisation, not in classical summation.

Problem 329.1 – Sum

Show that

$$\frac{1}{2\cdot 3^2} + \frac{1}{8\cdot 9^2} + \frac{1}{14\cdot 15^2} + \dots \;=\; \frac{2\sqrt{3}\pi - \pi^2 - 24\log 2 + 18\log 3}{72}.$$

Brief reviews

Lynda Goldenberg

Radio

I can recommend all of *John Finnemore's Souvenir Programs*, available on BBC Sounds, which often include mathsy content. But maybe less known is that he solved Cain's Jawbone and has written a similar book of his own—this time helpfully spiral bound. A project for another lockdown maybe?

Game

I'm a huge fan of the *iDventure* clue puzzles and clue boxes. The former are beautiful to display but both are fun to dismantle and reassemble alone or cooperatively. They are a bit of an extravagance so maybe you can add them to your birthday list.

Book

I've just read *What's the use? The Unreasonable Effectiveness of Mathematics* by Ian Stewart. This required a bit more brain engagement than I had expected but is an interesting approach to some odd bits of maths. I would recommend it for adults; but for children—the maths is too high a level for them, I feel. I bought the book to encourage students with studying maths.

A gravitationally bound atom

Tony Forbes

What is the radius of a negatively charged 'atom' with atomic number 0 that consists of a neutron and an electron bound to it by gravity?

In an attempt to answer this question, we shall assume (rather naïvely I suspect) that we have here the electron in a circular orbit around the neutron. Then we invoke quantum theoretical principles by insisting that the angular momentum of the electron is limited to positive integer multiples of $\hbar$, the reduced Planck's constant. We ignore spin. We also ignore the power dissipated by the accelerating electron ($< 10^{-243}\,\mathrm{W}$).

For an orbit of radius r, by equating forces whilst noting the absence of an electromagnetic component, we have

$$\frac{m_\mathrm{n} m_\mathrm{e} G}{r^2} \;=\; m_\mathrm{e} r \omega^2, \tag{1}$$

where ω is the electron's angular velocity. Denote the electron's angular momentum by L and its ordinary momentum in the direction tangential to the orbit by p. Then

$$L \;=\; m_\mathrm{e} r^2 \omega \;=\; n\hbar \quad \text{and} \quad p \;=\; \frac{L}{r}. \tag{2}$$

We can solve (1) and (2) to obtain

$$r \;=\; \frac{n^2\hbar^2}{m_\mathrm{n} m_\mathrm{e}^2 G}. \tag{3}$$

To verify that this is a sensible conclusion, we can compute the wavelength of the electron from the formula $\lambda = h/p$:

$$\lambda \;=\; \frac{h}{p} \;=\; \frac{hr}{L} \;=\; \frac{hr}{n\hbar} \;=\; \frac{2\pi r}{n},$$

which is indeed an integer fraction of the circumference of the orbit.

Substituting actual values in (3) for the constants,

$$m_\mathrm{n} \;=\; 1.67492750056 \times 10^{-27}\,\mathrm{kg}, \quad m_\mathrm{e} \;=\; 9.1093837139 \times 10^{-31}\,\mathrm{kg},$$
$$h \;=\; 2\pi\hbar \;=\; 6.62607015 \times 10^{-34}\,\mathrm{J\,s}, \quad G \;=\; 6.67430 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}},$$

for which we acknowledge help from *Wikipedia*, we obtain

$$r_1 \;=\; 1.19887 \times 10^{29}\,\mathrm{m} \;=\; 1.26721 \times 10^{13}\,\mathrm{ly}.$$

for the ground state, $n = 1$. Since r_1 is rather large we probably won't be able to detect gravitationally bound neutron–electron atoms until the Universe has thinned out a little. Moreover, we expect these atoms to decay to hydrogen-1 with the same half-life as that of a free neutron, about 10 minutes ($\mathrm{n} \to \mathrm{p} + \mathrm{e} + \overline{\nu}_\mathrm{e}$, occasionally $\mathrm{n} \to \mathrm{p} + \mathrm{e} + \overline{\nu}_\mathrm{e} + \gamma$ [*Wikipedia*]).

One is tempted to go on to study the atom's spectrum. However, I appreciate that some would regard this as a futile exercise. If the atom were created in the first excited state, $n = 2$, then a jump to the ground state, $n = 1$, would involve a change of orbit radius amounting to about 3.8×10^{13} ly. The neutron would almost certainly decay long before the completion of the transition.

The total energy (potential + kinetic) of the electron is

$$E_n \;=\; -\int_r^\infty \frac{m_\mathrm{n} m_\mathrm{e} G}{s^2} ds + \frac{m_\mathrm{e}(r\omega)^2}{2} \;=\; -\frac{m_\mathrm{n} m_\mathrm{e} G}{2r} \;=\; -\frac{m_\mathrm{n}^2 m_\mathrm{e}^3 G^2}{2n^2\hbar^2}.$$

Therefore we can calculate the energy difference between orbits n and n':

$$E_{n'\to n} \;=\; E_{n'} - E_n \;=\; -\frac{m_e}{2}\left(\frac{m_\mathrm{n} m_\mathrm{e} G}{\hbar}\right)^2 \left(\frac{1}{(n')^2} - \frac{1}{n^2}\right). \qquad (4)$$

The expression $m_\mathrm{n} m_\mathrm{e} G/\hbar \approx 9.7 \cdot 10^{-34}$ m/s is the electron's speed in the ground state. The energy difference when going from level 2 to level 1 is

$$E_{2\to 1} \;=\; E_2 - E_1 \;\approx\; 3.2 \times 10^{-97}\,\mathrm{J}.$$

If by some miracle the atom remains intact throughout the transition, the wavelength of the ultra-long-wave radio photon emitted would be

$$\lambda_{2\to 1} \;=\; \frac{hc}{E_{2\to 1}} \;=\; 6.6 \times 10^{55}\,\mathrm{ly}$$

approximately.

Finally, one might like to compare (4) with the corresponding formula—also provided by *Wikipedia*—for the hydrogen atom, one proton, one electron and no gravity:

$$H_{n'\to n} \;=\; -\frac{m_\mathrm{e}}{2}\left(\frac{e^2}{4\pi\varepsilon_0\hbar}\right)^2 \left(\frac{1}{(n')^2} - \frac{1}{n^2}\right).$$

Here, the wavelength for the transition $H_{2\to 1}$ is a mere 1.285×10^{-23} ly, somewhere in the ultraviolet.

Calculations of geometric quantities on S^3 by Cartan's method

Tommy Moorhouse

Introduction In this investigation we will derive in detail some geometric quantities on the 3-sphere S^3 using two methods, including the language of differential forms. Forms are linear maps from the tangent space of S^3 into the real numbers, which just means that they map vectors to real numbers. Cartan's method exploits the properties of forms to explore the geometry of manifolds, and we will apply the basic results to S^3.

Christoffel symbols from an action A simple metric on the 3-sphere can be written, using the summation convention (an index that occurs in an upper and a lower position is summed over)

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= g_{\chi\chi}d\chi^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \\ &= d\chi^2 + \sin^2\chi(d\vartheta^2 + \sin^2\vartheta d\varphi^2). \end{aligned}$$

Here the Greek indices are coordinate indices on S^3. If we consider the distance function to be the action of the system we can write

$$L = \dot{\chi}^2 + \sin^2\chi(\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2).$$

Here the dot represents a derivative w.r.t. a parameter t. Minimising the action with respect to each of the fields we find

$$\begin{aligned} \ddot{\chi} - \sin\chi\cos\chi(\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2) &= 0, \\ \frac{d}{dt}(\dot{\varphi}\sin^2\chi\sin^2\vartheta) &= 0, \\ \frac{d}{dt}(\dot{\vartheta}\sin^2\chi) - \sin\vartheta\cos\vartheta\sin^2\chi\dot{\varphi}^2 &= 0. \end{aligned}$$

Expanding the derivatives we can read off the Christoffel symbols from the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0.$$

For example

$$\Gamma^\chi_{\vartheta\vartheta} = -\sin\chi\cos\chi.$$

While it is possible to use the Christoffel symbols to calculate the components of the curvature tensor there is actually an easier way using Cartan's method involving forms on S^3.

Cartan's method We define a set of three one-forms $e^a, a = 1, 2, 3$, with

$$ds^2 \;=\; g_{\mu\nu}dx^\mu dx^\nu \;=\; (e^1)^2 + (e^2)^2 + (e^3)^2 \;=\; \eta_{ab}e^a e^b$$

where η_{ab} is the flat metric in the tangent space. For the metric above we have

$$\begin{aligned} e^1 &= d\chi, \\ e^2 &= \sin\chi d\vartheta, \\ e^3 &= \sin\chi \sin\vartheta d\varphi. \end{aligned}$$

Cartan's first structural equation (without torsion) is

$$de^a \;=\; -\,\omega^a{}_b \wedge e^b.$$

We readily find

$$\begin{aligned} \omega^3{}_1 &= \cot\chi e^3, \\ \omega^2{}_1 &= \cot\chi e^2, \\ \omega^3{}_2 &= \frac{\cot\vartheta}{\sin\chi} e^3, \end{aligned}$$

with the other components obtained using $\omega^a{}_b = -\omega^b{}_a$. This defines the spin connection one-forms $\omega^a{}_b$. The curvature two-form is defined to be

$$\rho^a{}_b \;=\; d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b.$$

This leads to the curvature components as set out above. For example

$$\begin{aligned} \rho^3{}_1 &= d\omega^3{}_1 + \omega^3{}_2 \wedge \omega^2{}_1 \\ &= e^3 \wedge e^1. \end{aligned}$$

In fact in this metric $\rho^a{}_b = e^a \wedge e^b$.

Changing to a coordinate basis The one-forms $e^a(x)$ form a basis for the cotangent space to S^3 at x. We can expand them in terms of the coordinate one-forms dx^μ as $e^a = e^a_\mu dx^\mu$:

$$\begin{aligned} e^1_\chi &= 1, \\ e^2_\vartheta &= \sin\chi, \\ e^3_\varphi &= \sin\chi\sin\vartheta. \end{aligned}$$

All other components vanish since the metrics are diagonal. Now we also need the dual vectors e_a such that

$$e^a(e_b) \;=\; \delta^a_b.$$

To express these in components we lower the tangent vector index with η_{ab} and raise the coordinate index with $g^{\mu\nu}$:

$$\begin{aligned} e_1^\chi &= 1, \\ e_2^\vartheta &= \frac{1}{\sin\chi}, \\ e_3^\varphi &= \frac{1}{\sin\chi\sin\vartheta}. \end{aligned}$$

To find the coordinate form of the curvature we first express the curvature two-form in coordinates:

$$\rho^a{}_b \;=\; e^a_\mu e^\nu_b \rho^\mu_\nu.$$

The calculation is straightforward and we find

$$\begin{aligned} \rho^\chi_\varphi &= \sin^2\chi\sin^2\vartheta d\chi\wedge d\varphi, \\ \rho^\vartheta_\varphi &= \sin^2\chi\sin^2\vartheta d\vartheta\wedge d\varphi, \\ \rho^\vartheta_\chi &= -d\chi\wedge d\vartheta, \\ \rho^\varphi_\chi &= -d\chi\wedge d\varphi, \\ \rho^\varphi_\vartheta &= -\sin^2\chi d\vartheta\wedge d\varphi, \\ \rho^\chi_\vartheta &= \sin^2\chi d\chi\wedge d\varphi. \end{aligned}$$

It remains to read off the six independent coordinate curvature components from

$$R_{\mu\nu\rho}{}^\sigma dx^\mu\wedge dx^\nu \;=\; \rho^\sigma_\rho.$$

We find

$$\begin{aligned} R_{\chi\varphi\varphi}{}^\chi &= \sin^2\chi\sin^2\vartheta, \\ R_{\vartheta\varphi\varphi}{}^\vartheta &= \sin^2\chi\sin^2\vartheta, \\ R_{\chi\varphi\chi}{}^\varphi &= -1, \\ R_{\chi\vartheta\vartheta}{}^\chi &= \sin^2\chi, \\ R_{\chi\vartheta\chi}{}^\vartheta &= -1, \\ R_{\vartheta\varphi\vartheta}{}^\varphi &= \sin^2\chi. \end{aligned}$$

We can now easily obtain the Ricci tensor $R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho}$:

$$\begin{aligned} R_{\varphi\varphi} &= 2\sin^2\chi\sin^2\theta, \\ R_{\vartheta\vartheta} &= 2\sin^2\chi, \\ R_{\chi\chi} &= 2. \end{aligned}$$

Comparing with the metric $g_{\mu\nu}$ we see that this standard three-sphere metric is an Einstein metric, meaning that the Ricci tensor is proportional to the metric: $R_{\mu\nu} = (R/3)g_{\mu\nu}$ with constant curvature scalar $R = 6$.

Conclusion The use of Cartan's method has allowed us to calculate the curvature tensors of S^3 without resorting to the potentially messy use of Christoffel symbols and their derivatives.

Problem 329.2 – Colouring pairs

Robin Whitty

In chapter 7 of Joseph G. Rosenstein's book *Linear Orderings*, Academic Press, 1982, he poses a nice problem which may be stated thus: suppose we list the unordered pairs of natural numbers in the order

$$\{i, m-i\}, \quad 0 \le i \le \lfloor (m-1)/2 \rfloor, \quad m = 1, 2, \ldots.$$

Colour all entries at odd positions in this listing Red and colour Blue the entries in even positions. So the first six Red pairs are

$$\{0,1\},\ \{0,3\},\ \{0,4\},\ \{0,5\},\ \{2,3\},\ \{1,5\},$$

and these alternate with the first six Blue pairs

$$\{0,2\},\ \{1,2\},\ \{1,3\},\ \{1,4\},\ \{0,6\},\ \{2,4\}.$$

Now Rosenstein's problem is to specify an infinite subset of the natural numbers for which every unordered pair is the same colour. In fact, it isn't very difficult to solve this for the colour Red. But it is not just difficult but impossible to solve for the colour Blue, and this suggests your M500 problem: show that if X is a set of natural numbers all of whose unordered pairs are Blue then X has at most four elements.

Problem 329.3 – Kilograms per year

For which positive integers n is it possible for an n-year-old human to weigh exactly n kilograms? What about other units of mass: grams, tonnes, pounds, stones, ... ?

Geometric quantities on S^3 Part 2: Geodesic deviation

Tommy Moorhouse

Introduction We have previously (M500 **316**) described geodesics in S^3 in terms of pairs of orthonormal vectors $(\mathbf{p}, \mathbf{v})$, say, in $\mathbb{R}^4$. This description facilitates some explicit calculations, and here we will find some Jacobi fields. Jacobi fields are vector fields that represent the separation of nearby geodesics and give information about the way the geodesics diverge or converge. There are many good differential geometry texts covering the ideas used here, [Crampin & Pirani] being one that may be familiar to OU students.

We choose the parameterisation of S^3, with suitable ranges for the angular variables,

$$\mathbf{x}(\chi, \vartheta, \varphi) \;=\; (\cos\chi,\ \sin\chi\cos\vartheta,\ \sin\chi\sin\vartheta\cos\varphi,\ \sin\chi\sin\vartheta\sin\varphi).$$

Simply represented geodesics We will take advantage of our representation of geodesics with parameter t as

$$\gamma(t) \;=\; \mathbf{p}\cos t + \mathbf{v}\sin t$$

to better understand the equation of geodesic deviation. We will fix $\mathbf{p}$ and $\mathbf{v}$ to get an explicit and quite simple picture of what is happening. $\mathbf{p}$ and $\mathbf{v}$ are unit orthogonal vectors in $\mathbb{R}^4$ with $\mathbf{p}$ being the origin of our geodesics and $\mathbf{v}$ determining their initial direction. Thus we choose our family of geodesics to be those passing through $\mathbf{p} = (1, 0, 0, 0)$ at $t = 0$ with θ and ϕ fixed. Using angular coordinates adapted to this point we can take $\mathbf{v} = \mathbf{x}_\chi|_{\chi=0}$. This gives

$$\gamma(t) \;=\; (\cos t,\ \sin t\cos\vartheta,\ \sin t\sin\vartheta\cos\varphi,\ \sin t\sin\vartheta\sin\varphi).$$

The path of the geodesic is just $\mathbf{x}(t, \theta, \phi)$. Now consider a geodesic $\gamma_\epsilon(t)$ that has a slightly different initial direction, say $\mathbf{x}_\chi(0, \vartheta+\epsilon, \varphi)$, where ϵ is a small parameter. Then you can check that two points 'abreast' of each other, say $\gamma_\epsilon(t)$ and $\gamma(t)$, are linked by the vector $\epsilon\mathbf{x}_\vartheta$ based at $\gamma(t)$. It seems obvious that the field of such vectors along the original geodesic is a Jacobi field, and we will show that this connecting vector indeed satisfies the equation of geodesic deviation with the curvature tensor found in Part 1, 'Calculations of geometric quantities on S^3 by Cartan's method', pages 8–11 of this M500.

The geodesic deviation equations Although it is possible to derive the equation of geodesic deviation quite simply (e.g. [Hughston & Tod] essentially develops the picture above, while [Crampin & Pirani] uses a more abstract method) we will start from the result for a Jacobi field $\mathbf{w}(\gamma(t))$:

$$\nabla_\chi^2 \mathbf{w} + R(\mathbf{w}, \mathbf{x}_\chi)\mathbf{x}_\chi \;=\; 0.$$

Here ∇_χ is the covariant derivative along the geodesic $\gamma(t)$. Now $R(\mathbf{w}, \mathbf{x}_\chi)$ is the curvature tensor evaluated on the two given vectors. This latter expression acts linearly on vectors to give a vector. Evidently the equation expresses a sort of acceleration of $\mathbf{w}$ in terms of the local curvature.

The treatment above involves vectors on $\mathbb{R}^4$, and we have to ensure that the results work in S^3 (by using the projection P in the covariant derivatives, for example, as used in the article in M500 **317**). For later calculations we will need the Christoffel symbols

$$\Gamma^\sigma_{\mu\nu}\mathbf{x}_\sigma \;=\; \nabla_\mu \mathbf{x}_\nu.$$

See M500 **317** for one way to calculate these. The non-vanishing ones (symmetric in the lower indices) are

$$\begin{aligned}
\Gamma^\chi_{\vartheta\vartheta} &= -\sin\chi\cos\chi,\\
\Gamma^\chi_{\varphi\varphi} &= -\sin\chi\cos\chi\sin^2\vartheta,\\
\Gamma^\vartheta_{\chi\vartheta} &= \cot\chi,\\
\Gamma^\vartheta_{\varphi\varphi} &= -\sin\vartheta\cos\vartheta,\\
\Gamma^\varphi_{\chi\varphi} &= \cot\chi,\\
\Gamma^\varphi_{\varphi\vartheta} &= \cot\vartheta.
\end{aligned}$$

We will show that $\mathbf{x}_\vartheta$ is a Jacobi field using the expression for $R(\mathbf{x}_\vartheta, \mathbf{x}_\chi)\mathbf{x}_\chi$

$$R(\mathbf{x}_\vartheta, \mathbf{x}_\chi)\mathbf{x}_\chi \;=\; \nabla_{\mathbf{x}_\vartheta}(\nabla_{\mathbf{x}_\chi}\mathbf{x}_\chi) - \nabla_{\mathbf{x}_\chi}(\nabla_{\mathbf{x}_\vartheta}\mathbf{x}_\chi) - \nabla_{[\mathbf{x}_\vartheta,\mathbf{x}_\chi]}\mathbf{x}_\chi.$$

The last term vanishes since the bracketed vectors are just $\partial/\partial\chi$ and $\partial/\partial\vartheta$ respectively, and these commute. Since $\Gamma^\mu_{\chi\chi} = 0$ for all indices μ we are left with just one term, $-\nabla_{\mathbf{x}_\chi}(\nabla_{\mathbf{x}_\vartheta}\mathbf{x}_\chi)$. This is easily worked out from the Christoffel symbols and we find

$$R(\mathbf{x}_\vartheta, \mathbf{x}_\chi)\mathbf{x}_\chi \;=\; \mathbf{x}_\vartheta.$$

Now, $\nabla_\chi^2 \mathbf{x}_\vartheta = -\mathbf{x}_\vartheta$ so $\mathbf{x}_\vartheta$ satisfies the equation of geodesic deviation. By the same argument $\mathbf{x}_\varphi$, and thus any linear combination of $\mathbf{x}_\vartheta$ and $\mathbf{x}_\varphi$ with constant coefficients, also satisfies the equation.

Readers might like to work out the corresponding solution using intrinsic quantities such as the forms e^a_μ described in Part 1.

References

[Crampin & Pirani] M. Crampin and F. A. E. Pirani, *Applicable Differential Geometry*, Cambridge University Press 1986 (reprinted 1994).

[Hughston & Tod] L. P. Hughston and K. P. Tod, *An Introduction to General Relativity*, Cambridge University Press 1990, chapter 10.

Solution 326.5 – Dot product

Suppose $\mathbf{u}$ and $\mathbf{v}$ are vectors in a real vector space of dimension d. Show that

$$(\mathbf{u}\cdot\mathbf{v})^2 \;=\; \text{trace}\left((\mathbf{u}\mathbf{u}^{\mathrm{T}})(\mathbf{v}\mathbf{v}^{\mathrm{T}})\right).$$

Tommy Moorhouse

We will use the properties of the trace to simplify

$$\text{trace}((\mathbf{u}\mathbf{u}^{\mathrm{T}})(\mathbf{v}\mathbf{v}^{\mathrm{T}})).$$

Since the trace of a matrix is just the sum of its diagonal entries if λ is a real number and A is a real square matrix then $\text{trace}(\lambda A) = \lambda\,\text{trace}(A)$. We also have

$$\begin{aligned}(\mathbf{u}\mathbf{u}^{\mathrm{T}})(\mathbf{v}\mathbf{v}^{\mathrm{T}}) &= \mathbf{u}(\mathbf{u}^{\mathrm{T}}\mathbf{v})\mathbf{v}^{\mathrm{T}}\\ &= (\mathbf{u}\cdot\mathbf{v})\mathbf{u}\mathbf{v}^{\mathrm{T}}.\end{aligned}$$

Here we have moved the scalar product (just a real number) $\mathbf{u}\cdot\mathbf{v}$ in front of $\mathbf{u}$ and so in front of the matrix $\mathbf{u}\mathbf{v}^{\mathrm{T}}$. We now have

$$\text{trace}((\mathbf{u}\mathbf{u}^{\mathrm{T}})(\mathbf{v}\mathbf{v}^{\mathrm{T}})) = \text{trace}((\mathbf{u}\cdot\mathbf{v})\mathbf{u}\mathbf{v}^{\mathrm{T}}) = (\mathbf{u}\cdot\mathbf{v})\text{trace}(\mathbf{u}\mathbf{v}^{\mathrm{T}}).$$

Note that $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$ since $\mathbf{u}^{\mathrm{T}}\mathbf{v} = (\mathbf{u}^{\mathrm{T}}\mathbf{v})^{\mathrm{T}} = \mathbf{v}^{\mathrm{T}}\mathbf{u}$ (the transpose of a real number is itself). Let $L^i_j = u^i v_j$ be the component form of the matrix $\mathbf{u}\mathbf{v}^{\mathrm{T}}$ (writing the column index on v_j in the lower position, with numerical value $v_j = v^j$). Then $\text{trace}(L) = \sum_i u^i v_i = \sum_i u_i v^i = \mathbf{u}^{\mathrm{T}}\mathbf{v} = \mathbf{u}\cdot\mathbf{v}$ and by the result above,

$$\text{trace}((\mathbf{u}\mathbf{u}^{\mathrm{T}})(\mathbf{v}\mathbf{v}^{\mathrm{T}})) = (\mathbf{u}\cdot\mathbf{v})^2.$$

Solution 327.1 – Jigsaw puzzle

A jigsaw puzzle has n edge pieces, n an even integer. How many pieces can it have altogether.

David Sixsmith

Problem 327.1 asked: if a jigsaw puzzle has n edge pieces, n an even integer, how many pieces can it have altogether?

Suppose the puzzle is size p by q, where we can assume that $p \leq q$. If $p = 1$, then $q = n$, and there are n pieces.

From now on, suppose that $2 \leq p \leq q$. It follows that there are $2(p + q) - 4$ edge pieces, and so we have

$$p + q \;=\; n/2 + 2.$$

If $p = 2$, then $q = n/2$, and once again there are exactly n pieces. So we can assume that $p \geq 3$. Hence p can take any integer value from 3 to $\lambda = \text{int}(n/4+1)$, and the total number of pieces is $p(n/2+2-p)$. One can check that these values are all distinct. So the number of possible pieces are:

$$n,\ 3(n/2-1),\ 4(n/2-2),\ 5(n/2-3),\ \ldots,\ \lambda(n/2+2-\lambda).$$

In the case $n = 14$, we have that $\lambda = 4$, and the possibilities are $14, 18, 20$, as suggested in the original problem.

Problem 329.4 – 42-card trick

Take a deck of 42 playing cards, all face down. Do the following.

(1) Turn over the top two cards and place them back on top of the deck. Then cut the deck.
(2) Repeat step (1) many times.
(3) Deal the cards, inverting the 2nd, 4th, 6th, ..., 42nd.
(4) Observe, possibly with a modicum of astonishment, that precisely 21 cards are inverted.

Explain.

Is Math complete? No, it is missing the letters ematics.

Solution 326.1 – Binomial coefficient products

Let $P(n)$ be the product of the elements of row n of Pascal's triangle; i.e.

$$P(n) \;=\; \prod_{m=0}^{n} \binom{n}{m}.$$

Show that for given integer k,

$$\frac{P(n-k)P(n+k)}{P(n)^2} \;\to\; \exp(k^2) \quad \text{as } n \to \infty.$$

For large k the convergence can be quite slow. For example, with $k = 10$ we have

$$\frac{P(90)\;P(110)}{P(100)^2\;e^{100}} \;=\; 0.608047, \qquad \frac{P(990)\;P(1010)}{P(1000)^2\;e^{100}} \;=\; 0.95126.$$

Dave Wild

Since the value of $P(n-k)P(n+k)/P(n)^2$ is unchanged if k is replaced by $-k$ then we can assume that k is a non-negative integer. The proof will be by induction.

When $k = 0$

$$\lim_{n\to\infty} \frac{P(n-0)P(n+0)}{P(n)^2} \;=\; 1 \;=\; \exp(0^2).$$

Therefore the required result is true when $k = 0$. Also

$$\lim_{n\to\infty} \frac{P(n-(k+1))P(n+(k+1))}{P(n)^2}$$
$$= \lim_{n\to\infty} \frac{P(n-k)P(n+k)}{P(n)^2} \lim_{n\to\infty} \left(\frac{P(n-(k+1))}{P(n-k)} \frac{P(n+(k+1))}{P(n+k)} \right).$$

Since we are assuming that

$$\lim_{n\to\infty} \frac{P(n-k)P(n+k)}{P(n)^2} \;=\; \exp(k^2),$$

we have

$$\lim_{n\to\infty} \frac{P(n-(k+1))P(n+(k+1))}{P(n)^2}$$

$$= \exp(k^2) \lim_{n\to\infty} \left(\frac{P(n-(k+1))}{P(n-k)} \frac{P(n+(k+1))}{P(n+k)} \right).$$

The following result will be useful when we try to evaluate the remaining limit on the right hand side.

$$\frac{P(N+1)}{P(N)} = \left[\prod_{m=0}^{N} \frac{(N+1)!}{m!(N+1-m)!} \frac{m!(N-m)!}{N!} \right] \binom{N+1}{N+1} = \frac{(N+1)^{N+1}}{(N+1)!}.$$

Using the result for $P(N+1)/P(N)$ twice,

$$\begin{aligned} &\lim_{n\to\infty} \frac{P(n-(k+1))}{P(n-k)} \frac{P(n+(k+1))}{P(n+k)} \\ &= \lim_{n\to\infty} \frac{(n-k)!}{(n-k)^{n-k}} \frac{(n+k+1)^{n+k+1}}{(n+k+1)!} \\ &= \lim_{n\to\infty} \frac{(n-k)!(n+k+1)^{2k+1}}{(n+k+1)!} \lim_{n\to\infty} \left(1+\frac{2k+1}{n-k}\right)^{n-k}. \end{aligned}$$

Hence

$$\lim_{n\to\infty} \frac{(n-k)!(n+k+1)^{2k+1}}{(n+k+1)!} = \lim_{n\to\infty} \frac{n^{2k+1}\left(1+\frac{k+1}{n}\right)^{2k+1}}{n^{2k+1}\left(1+\frac{1-k}{n}\right)\ldots\left(1+\frac{k+1}{n}\right)} = 1,$$

$$\lim_{n\to\infty} \left(1+\frac{2k+1}{n-k}\right)^{n-k} = \lim_{m\to\infty} \left(1+\frac{2k+1}{m}\right)^{m} = \exp(2k+1).$$

Here we have put $m = n-k$ and then used a standard result which may be found in the *Wikipedia* article 'Exponential function'. Therefore

$$\lim_{n\to\infty} \left(\frac{P(n-(k+1))}{P(n-k)} \frac{P(n+(k+1))}{P(n+k)} \right) = \exp(2k+1)$$

and

$$\lim_{n\to\infty} \frac{P(n-(k+1))P(n+(k+1))}{P(n)^2} = \exp(k^2)\exp(2k+1) = \exp((k+1)^2).$$

Since the result is true for $k+1$ when it is true for k and it is also true when $k=0$, then it holds for all integers k.

Using projection from S^3 to $\mathbb{R}^3$ to find the Euler number

Tommy Moorhouse

Introduction There is a close analogy between the stereographic projection of S^2 onto the real plane and that of S^3 onto the hyperplane $\mathbb{R}^3$. In both cases we picture the sphere as embedded in the ambient space ($\mathbb{R}^3$ and $\mathbb{R}^4$ respectively) and in both cases the projection is 'almost' one to one. Here we use the projection to calculate the Euler number of S^3. The projections will be considered as lying in the extended plane or hyperplane by adding a point at infinity.

Stereographic projection We take a point $\mathbf{p}_0 = (-1, 0, 0, 0)$ of S^3 embedded in $\mathbb{R}^4$ as base point and define the projection of other points of S^3 onto the hyperplane $x^1 = 0$ as follows. Take the point

$$\mathbf{p} \;=\; (p^1, p^2, p^3, p^4) \quad \text{with} \quad \sum_{i=1}^{4} (p^i)^2 \;=\; 1$$

and extend the line $(1-t)\mathbf{p}_0 + t\mathbf{p}$ until it meets the said hyperplane. This happens when $tp^1 + (-1)(1-t) = 0$ so that $t = 1/(1+p^1)$. The coordinates of the projected point are then

$$\left(q^1, q^2, q^3\right) \;=\; \left(\frac{p^2}{1+p^1}, \frac{p^3}{1+p^1}, \frac{p^4}{1+p^1}\right).$$

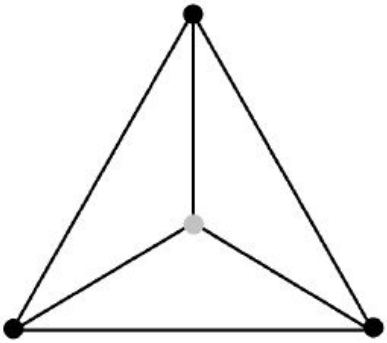


Figure 1: Projection of triangulation of S^2.

Euler characteristic of S^3 If we imagine a sphere S^2 with four vertices joined in pairs by segments of geodesics to form a tetrahedral triangulation, we can project from the south pole to get the graph in Figure 1. This shows the four triangular faces of the tetrahedron, one triangle being the whole (extended) plane outside the other three. It also shows that each face shares an edge with each of the other three faces. The boundary of the outer face consists of three edges as expected. Since the projection does not involve cutting or tearing the topology is unchanged.

We can imagine a decomposition of S^3 into identical geodesic tetrahedra. Projecting this 'triangulation' of S^3 onto $\mathbb{R}^3$ we have a collection of tetrahedra sharing triangular faces and with a tetrahedral boundary. The space outside the boundary, including a point 'at infinity' is the body of a tetrahedron extending to infinity but with no edges or vertices other than those in the finite part of the 3-space. Deduce the details of the projected 'graph' using S^2 as a model (the number of tetrahedra is analogously constrained by the geometry) and use it to calculate the Euler number

$$V - E + F - T \;=\; 0$$

of S^3, where T is the number of tetrahedra (including the one external to the others), F is the number of triangular faces, E is the number of edges and V is the number of vertices.

Two sudoku puzzles

Tony Forbes

Fill in the empty cells such that each row, column and 3×3 box contains all of the symbols 1, 2, . . . , 9.

		3		8				
						1		
					6		7	5
					8			
2					5			6
6						2	4	9
4					9			
5								2
					1	7	9	

*..3.8.....1.....6.75....8..2....5..66....2494....9...5.....2.....179./

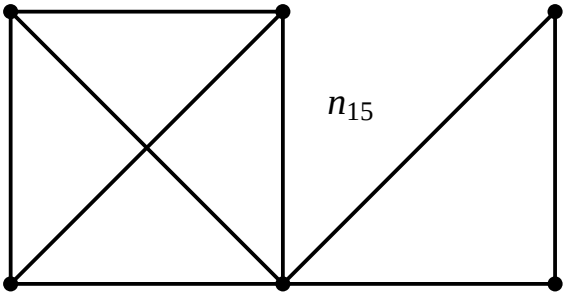
			4					
			3	9				
	2						6	5
							8	
6	8						5	2
7	4							
1								9
		4			6			7
9				2	8	6		

*..4.....39....2....65.....8.68....5274.....1.....9..4..6..79...286../

A tetrahedron and a triangle

Tony Forbes

Some years ago Terry Griggs and I worked on completing the solution of the design spectrum problem for graphs having 6 vertices and 9 edges. All rather straightforward except for one particular graph, which we called n_{15} (because it was the fifteenth in our list),



a tetrahedron and a triangle sharing a common vertex. The major difficulty was the following.

Conjecture 1. *There exists an edgewise decomposition of the complete graph K_{18} into 17 copies of graph n_{15}.*

After I spent a number of GHz years looking for this decomposition by a combination of random processes and brute force, we gave up and submitted our paper, [1], with Conjecture 1 undecided. The decomposition of K_{18} into n_{15} was stated as a possible exception.

Unknown to us at the time, someone else had also considered the same problem. In a paper under review he/she offered a proof by computer that Conjecture 1 is FALSE. If I was aware of that paper, I would have done no further work—with the possible consequence of the conjecture remaining false for the rest of eternity.

Instead, while [1] was under review I realised I could settle the conjecture by a computation involving a handful of computer cores and several days of real time. In view of an almost certain negative result, my biggest headache by far was how to persuade readers that I had looked everywhere. This is of course an essential part of any proof that requires substantial help from an electronic computer. Indeed, one is reminded of the controversy regarding the 1977 proof by Appel and Haken of the four-colour conjecture.

I was therefore surprised and delighted when the computer actually found a decomposition of K_{18} into n_{15}. Conjecture 1 is TRUE. I sent this email to my co-author.

```
Subject:  HOLD THE FRONT PAGE!
Date:  5 July 2017
I have found an n_15 design of order 18!
I will rewrite the paper.
T.
Needless to say I am very, very surprised.  I was
about half-way through what I believe to be an
exhaustive search, totally convinced that I was
going report a negative result.  Then the computer
found it.  The method is quite interesting.  Very
briefly, take a PSTS(11) with 10, 11 or 12 blocks,
extend it to a suitable PSTS(18) with 17 blocks, and
use the T-K method to extend it to 17 graphs (K_4
not-necessarily-disjoint-union K_3) by adding 17 K_4s.
Then marry the girls and boys.
```

I rewrote [1], this time with no possible exceptions, and resubmitted it.

If this story has to have a moral, I suppose it must be something like this. *Be ignorant of what other people are doing if you want to solve a problem concerning the existence of an interesting mathematical object.*

Reference

[1] A. D. Forbes and T. S. Griggs, Designs for graphs with six vertices and nine edges, *Australas. J. Combin.* **70** (2018), 52–74.

Problem 329.5 – Supermarket receipts

You regularly and often go to your local supermarket to buy things. Estimate the probability that the amount on your receipt is an exact number of pounds.

If you think the answer is 0.01, please give a convincing argument.

If you want to copy and paste something from a PDF file to a text file, then you might find that 'ff' causes trouble because of some bizarre convention of printed English. This suggests an interesting activity. Write a sensible sentence that contains a long sequence of words all of which contain 'ff'.

Bomb delivery

Tony Forbes

Suppose you wish to drop a bomb from an aircraft flying at great height. If the bomb is powerful, like the 50 Mt Tsar Bomba, you and the aircraft do not want to be too near the thing when it explodes.

One obvious escape strategy is to turn the aircraft through 180 degrees and fly directly away from the target. Surprisingly to me, but probably not to bomber pilots, this simple procedure is not optimal. In what follows we offer a better escape path based on a few simplifying assumptions.

Until it is far from the target, assume the aircraft's speed is a constant, v, and that its height above the ground is also constant.

Assume the bomb is released from $(0,0)$ in the (x,y)-plane and detonates after time t at (x,y)-location

$$D \;=\; (0,\ \alpha v t),$$

where $\alpha \le 1$ is a factor to allow for air resistance slowing the bomb down from the horizontal speed v which it acquires from the aircraft.

After releasing the bomb the aircraft follows a circular path of radius r for angle θ, $\pi/2 \le \theta \le \pi$ to get to point

$$A \;=\; (r - r\cos\theta,\ r\sin\theta).$$

It then flies straight for a distance

$$f \;=\; vt - r\theta, \qquad vt > r\theta,$$

to arrive at

$$B \;=\; A + (f\sin\theta,\ f\cos\theta),$$

when the bomb explodes. Look at the diagram on the next page, and it might help to recall that

$$\cos(\pi-\theta) \;=\; -\cos\theta \quad \text{and} \quad \sin(\pi-\theta) \;=\; \sin\theta.$$

We wish to maximize the distance from the explosion. Equivalently, we maximize $|B-D|^2$ as a function of θ. We have

$$B - D \;=\; (r - r\cos\theta + f\sin\theta,\ r\sin\theta + f\cos\theta - \alpha v t)$$

and hence

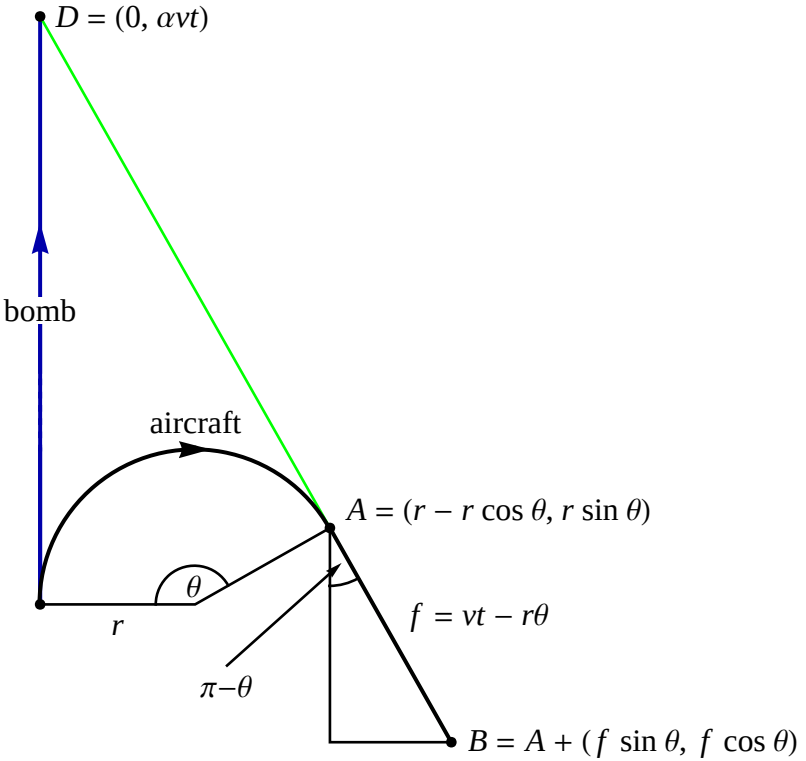
$$|B-D|^2 \;=\; (r - r\cos\theta + f\sin\theta)^2 + (r\sin\theta + f\cos\theta - \alpha v t)^2\,.$$

Differentiating and simplifying gives

$$\frac{d|B-D|^2}{d\theta} \;=\; 4(r\theta - tv)\sin\left(\frac{\theta}{2}\right)\left(r\sin\left(\frac{\theta}{2}\right) - \alpha tv\cos\left(\frac{\theta}{2}\right)\right).$$

The only point consistent with the constraints on θ where this conveniently factorized expression vanishes occurs at

$$\theta_0 \;=\; 2\arccos\left(\frac{r}{\sqrt{r^2+\alpha^2t^2v^2}}\right). \tag{1}$$



Having solved the problem according to the traditional method of locating the maximum of the distance function by differentiating and equating to zero, we now see that there has to be an easier way. The diagram clearly shows that D, A, and B are collinear. So it turns out that (1) can be determined by elementary trigonometry.

Problem 329.6 – Floater card trick

Donald Bell

Shuffle a full deck of cards and then divide it into three (nearly) equal packs face down. Choose any pack and turn it over, so that its bottom card becomes visible. This will be our *Control Card*. Its name can have as few as 10 letters (like S-I-X-O-F-C-L-U-B-S) or as many as 15 (like Q-U-E-E-N-O-F-D-I-A-M-O-N-D-S).

Choose another pack, pick it up and look at its bottom card. This will be our *Floater Card*. We don't need the third pack.

Deal cards one by one on to the table from the pack in your hand as you spell out the name of the Control Card. When you have finished, just drop the cards still in your hand on top of the pack on the table. You can, if you like, verify that the Floater Card has not yet reached the top of the pack.

Do the deal one more time. Verify that the Floater Card has still not yet reached the top. Do the deal a third time. Now the Floater Card HAS reached the top.

Explain.

The 2026 Winter Weekend

Judith Furner

The Winter Weekend 2026 took place at Kents Hill Conference Centre over the 9th – 11th January. We had a total of 29 delegates, several of whom made excellent presentations. We were also most fortunate in having Brigitte Stenhouse and Elinor Favell presenting on The History of Maths—in books and letters. Donald Bell and Dave Clark, both Winter Weekend regulars, also made presentations which were much appreciated. There were games to play, books to read, and conversations to be had. Apart from the chill in the corridors and the prices at the bar, feedback was excellent and we look forward to meeting again next year.

Diary date: 15th – 17th January 2027 at Kents Hill Conference Centre, Milton Keynes MK7 6BZ.

If you have any questions, please e-mail the Winter Weekend Organizer, Judith Furner, at winter@m500.org.uk.

Things you can't buy in shops – VI

Following on from the lists that we printed in M500s **278**, **289**, **293**, **296** and **308**, here are some more useful items for you to enquire about if your peaceful browsing in a shop gets interrupted by an enthusiastic salesperson uttering the dreaded words "Can I help you?"

1. A meter for measuring entropy.
2. A retractable metal tape measure that has the metric units on the top when zero is on the left.
3. A sandwich that can be eaten messinesslessly without a plate, knife and fork.
4. A toy nuclear reactor, possibly in the form of a collection of small, colourful, high-quality plastic components, push-fitted together and held in place by friction, from which you can assemble the device with the help of an instruction booklet. The last time I (TF) enquired about this in the Lego section of a local department store I was met with a puzzled look.
5. Potato crisps with no salt or any other flavour—as in the 1950s but without that little blue salt sachet.
6. An X-ray torch.
7. An inexpensive computer printer that does not require human intervention when printing a document of more than 0 pages. My current device fails miserably, but it does offer two options for delivering the printed output: (i) in a neat pile, (ii) scattered all over the floor.

Ten tips for staying safe online

I (TF) once heard a woman on the radio offering ten very sensible suggestions to help you avoid problems when online. So, for the benefit of M500 readers, I offer here my ten tips for staying safe when you are interacting with social media, dating apps, chat rooms and the like.

1. Close your account.
2. Close your account.
3. Close your account.
4. Close your account.
5. Close your account.
6. Close your account.
7. Close your account.
8. Close your account.
9. Close your account.
10. Close your account.

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04.00

Front cover The cocktail party graph $K_{2,2,2,2,2,2}$ split into two icosahedra, the red one and the blue one. The challenge is to draw the diagram with a reasonable level of symmetry whilst showing all the edges as clearly distinct straight lines.